

## **$f$ -FIXED POINTS OF ISOTONE $f$ -DERIVATIONS ON A LATTICE**

LEMNAOUAR ZEDAM, MOURAD YETTOU

AND

ABDELAZIZ AMROUNE

*Laboratory of Pure and Applied Mathematics*  
*Department of Mathematics*  
*Mohamed Boudiaf University*  
*Msila 28000, Algeria*

**e-mail:** l.zedam@gmail.com  
yettou28mourad28@gmail.com  
aamrounedz@yahoo.fr

### **Abstract**

In a recent paper, Çeven and Öztürk have generalized the notion of derivation on a lattice to  $f$ -derivation, where  $f$  is a given function of that lattice into itself. Under some conditions, they have characterized the distributive and modular lattices in terms of their isotone  $f$ -derivations. In this paper, we investigate the most important properties of isotone  $f$ -derivations on a lattice, paying particular attention to the lattice (resp. ideal) structures of isotone  $f$ -derivations and the sets of their  $f$ -fixed points. As applications, we provide characterizations of distributive lattices and principal ideals of a lattice in terms of principal  $f$ -derivations.

**Keywords:** lattice, isotone  $f$ -derivation, principal  $f$ -derivation,  $f$ -fixed points set.

**2010 Mathematics Subject Classification:** 03G10, 06B05, 06B10, 06B99.

### 1. INTRODUCTION

The notion of derivation appeared on the ring structures and it has many applications (see, e.g. [1]). Szász [15, 16] has extended the notion of derivation on a lattice structure  $L$  as a function  $d$  of  $L$  into itself satisfying the following two conditions:

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)) \text{ and } d(x \vee y) = d(x) \vee d(y),$$

for any  $x, y \in L$ . Ferrari [5] has investigated some properties of this notion and provided some interesting examples in particular classes of lattices. Xin *et al.* [19] have ameliorated the notion of derivation on a lattice by considering only the first condition, and they have showed that the second condition is obviously holds for the isotone derivations on a distributive lattice. In the same paper, they characterized also the distributive and modular lattices in terms of their isotone derivations. Later on, Xin [20] has focused his attention to the structure of the fixed sets of derivations on a lattice and showed some relationships between lattice ideals and these fixed sets.

In the same direction, Çeven and Öztürk [3] have generalized the notion of derivation on a lattice  $L$  to  $f$ -derivation on  $L$  by using a function  $f$  of  $L$  into itself. For a given function  $f$  of  $L$  into itself, an  $f$ -derivation on a lattice  $L$  is a function  $d$  of  $L$  into itself satisfying:

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)), \text{ for any } x, y \in L.$$

In this context, they also characterized the distributive and modular lattices by isotone  $f$ -derivations.

This notion of  $f$ -derivation on a lattice is witnessing increased attention. It studies, among others, in semi-lattices [21], in bounded hyperlattices [17], in quantales and residuated lattices [6, 18], in distributive lattices [12], and in several kinds of algebras [7, 9, 10]. Furthermore, it used in the definition of congruences and ideals in a distributive lattice [11].

The aim of the present paper is to investigate the most important properties of isotone and principal  $f$ -derivations on a lattice. We pay particular attention to the lattice structure of isotone  $f$ -derivations on a lattice, and to the ideal structure of their  $f$ -fixed points sets. More specifically, we show some cases that the set of principal  $f$ -derivations on a lattice has a lattice structure, and we provide a representation of any lattice in terms of its principal  $f$ -derivations. We give a relationship between a distributive lattice and its lattice of isotone  $f$ -derivations, and we show a characterization theorem of a distributive lattice in terms of its principal  $f$ -derivations. Furthermore, we investigate the structure of the set of  $f$ -fixed points of an isotone  $f$ -derivation on a lattice, and we show some cases that this set is an ideal (resp. a principal ideal). As applications, we provide a representation of any lattice in terms of its principal  $f$ -derivations, and we show characterization theorems of distributive lattices (resp. principal ideals of a lattice) in terms of principal  $f$ -derivations.

The remainder of the paper is structured as follows. In Section 2, we recall the necessary basic concepts and properties of lattices and  $f$ -derivations on lattices. In Section 3, we provide a representation (resp. a characterization theorem) of any lattice (resp. distributive lattice) in terms of its principal  $f$ -derivations.

In Section 4, we study the structure of the  $f$ -fixed points set of an isotone  $f$ -derivation on a lattice, and we provide a characterization theorem of principal ideals of a lattice in terms of its principal  $f$ -derivations. In Section 5, we show that the set of  $f$ -fixed points sets of isotone  $f$ -derivations on a distributive lattice has also a structure of a distributive lattice, and we provide a representation of any distributive lattice based on the  $f$ -fixed points sets of its principal  $f$ -derivations. Finally, we present some concluding remarks in Section 6.

## 2. BASIC CONCEPTS

In this section, we recall the necessary basic concepts and properties of lattices and  $f$ -derivations on lattices.

### 2.1. Lattices

In this subsection, we recall some definitions and properties of lattices that will be needed throughout this paper. Further information can be found in [2, 4, 8, 13, 14].

An *order relation*  $\leq$  on a set  $X$  is a binary relation on  $X$  that is *reflexive*, *antisymmetric* and *transitive*. A set  $X$  equipped with an order relation  $\leq$  is called a *partially ordered set* (*poset*, for short), denoted  $(X, \leq)$ . Let  $(X, \leq)$  be a poset and  $A$  be a subset of  $X$ . An element  $x_0 \in X$  is called a *lower bound* of  $A$  if  $x_0 \leq x$ , for any  $x \in A$ .  $x_0$  is called the *greatest lower bound* (or the *infimum*) of  $A$  if  $x_0$  is a lower bound and  $m \leq x_0$ , for any lower bound  $m$  of  $A$ . *Upper bound* and *least upper bound* (or *supremum*) are defined dually. Let  $(X, \leq_X)$  and  $(Y, \leq_Y)$  be two posets. A mapping  $\varphi : X \rightarrow Y$  is called an *order isomorphism* if it is surjective and satisfies the following condition:

$$x \leq_X y \text{ if and only if } \varphi(x) \leq_Y \varphi(y), \text{ for any } x, y \in X.$$

If  $X = Y$ , an order isomorphism  $\varphi : X \rightarrow X$  is called an order automorphism.

A poset  $(L, \leq)$  is called a  $\wedge$ -semi-lattice if any two elements  $x$  and  $y$  have a greatest lower bound, denoted by  $x \wedge y$  and called the *meet* (infimum) of  $x$  and  $y$ . Analogously, it is called a  $\vee$ -semi-lattice if any two elements  $x$  and  $y$  have a smallest upper bound, denoted by  $x \vee y$  and called the *join* (supremum) of  $x$  and  $y$ . A poset  $(L, \leq)$  is called a lattice if it is both a  $\wedge$ -semi-lattice and a  $\vee$ -semi-lattice. A lattice can also be defined as an algebraic structure: a set  $L$  equipped with two binary operations  $\wedge$  and  $\vee$  that are idempotent, commutative and associative, and satisfy the absorption laws (i.e.,  $x \wedge (x \vee y) = x$  and  $x \vee (x \wedge y) = x$ , for any  $x, y \in L$ ). The order relation and the meet and join operations are then related as follows:  $x \leq y$  if and only if  $x \wedge y = x$ ;  $x \leq y$  if and only if  $x \vee y = y$ . Usually, the notation  $(L, \leq, \wedge, \vee)$  is used for a lattice. A poset  $(L, \leq)$  is called bounded

if it has a least and a greatest element, respectively denoted by 0 and 1. Often, the notation  $(L, \leq, \wedge, \vee, 0, 1)$  is used to describe a bounded lattice. A non-empty subset  $M$  of a lattice  $(L, \leq, \wedge, \vee)$  is called a sublattice of  $L$  if, for any  $x, y \in M$ , it holds that  $x \wedge y \in M$  and  $x \vee y \in M$ . A poset  $(L, \leq)$  is called a complete lattice if every subset  $A$  of  $L$  has both a greatest lower bound, denoted by  $\bigwedge A$  and called the infimum of  $A$ , and a least upper bound, denoted by  $\bigvee A$  and called the supremum of  $A$ , in  $(L, \leq)$ . A lattice  $(L, \leq, \wedge, \vee)$  is called distributive if one of the following two equivalent conditions holds:

- (a)  $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ , for any  $x, y, z \in L$ ;
- (a<sup>δ</sup>)  $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ , for any  $x, y, z \in L$ .

Let  $(L, \leq, \wedge, \vee)$  and  $(M, \preceq, \frown, \smile)$  be two lattices. A mapping  $\varphi : L \rightarrow M$  is called a  $\wedge$ -homomorphism (resp.  $\vee$ -homomorphism), if it satisfies  $\varphi(x \wedge y) = \varphi(x) \frown \varphi(y)$  (resp.  $\varphi(x \vee y) = \varphi(x) \smile \varphi(y)$ ), for any  $x, y \in L$ . A  $\wedge$ -monomorphism is an injective  $\wedge$ -homomorphism. Also, a  $\wedge$ -epimorphism is a surjective  $\wedge$ -homomorphism.  $\vee$ -monomorphism and  $\vee$ -epimorphism are defined dually. A lattice homomorphism is both a  $\wedge$ -homomorphism and a  $\vee$ -homomorphism, a lattice isomorphism is a bijective lattice homomorphism. If  $L = M$ , a lattice isomorphism  $\varphi : L \rightarrow L$  is called a lattice automorphism.

**Proposition 2.1** [4]. *Let  $L$  and  $M$  be two lattices, and  $\varphi : L \rightarrow M$  be a mapping. The following statements are equivalent:*

- (i)  $\varphi$  is an order isomorphism;
- (ii)  $\varphi$  is a lattice isomorphism.

## 2.2. $f$ -derivations on a lattice

In this subsection, we recall the definition and some properties of  $f$ -derivation on a lattice. Further information can be found in [3, 19, 21].

**Definition 2.1** [19]. Let  $(L, \leq, \wedge, \vee)$  be a lattice. A function  $d : L \rightarrow L$  is called a *derivation* on  $L$  if it satisfies the following condition:

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)), \text{ for any } x, y \in L.$$

**Definition 2.2** [3]. Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \rightarrow L$  be a function. A function  $d : L \rightarrow L$  is called an  *$f$ -derivation* on  $L$  if it satisfies the following condition:

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)), \text{ for any } x, y \in L.$$

Throughout this paper, we shortly write  $dx$  instead of  $d(x)$  and  $fx$  instead of  $f(x)$ .

**Definition 2.3** [3]. Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $d$  be an  $f$ -derivation on  $L$ .  $d$  is called *isotone* if it satisfies the following condition:

$$x \leq y \text{ implies } dx \leq dy, \text{ for any } x, y \in L.$$

The following proposition gives some proprieties of  $f$ -derivations on a lattice.

**Proposition 2.2** [3]. Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $d$  be an  $f$ -derivation on  $L$ . Then the following holds.

- (i)  $dx \leq fx$ , for any  $x \in L$ ;
- (ii) If  $(L, \leq, \wedge, \vee)$  is distributive,  $f$  is a  $\vee$ -homomorphism and  $d$  is isotone, then  $d(x \vee y) = dx \vee dy$ .

**Proposition 2.3** [3]. Let  $(L, \leq, \wedge, \vee)$  be a lattice,  $\alpha \in L$  and  $f : L \rightarrow L$  be a function satisfies  $f(x \wedge y) = fx \wedge fy$ , for any  $x, y \in L$ . Then the function  $d_{(\alpha, f)} : L \rightarrow L$  defined by  $d_{(\alpha, f)}(x) = \alpha \wedge fx$ , for any  $x \in L$ , is an  $f$ -derivation on  $L$ . In addition, if  $f$  is an increasing function, then  $d_{(\alpha, f)}$  is an isotone  $f$ -derivation.

The following sets are the key notions of this paper.

**Notation 2.1.** Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \rightarrow L$  be a function. We denote by:

- (i)  $\mathfrak{I}_f(L)$  the set of isotone  $f$ -derivations on  $L$ ;
- (ii)  $\mathcal{P}_f(L) := \{d_{(\alpha, f)} \mid \alpha \in L\}$ .

The following result shows that the set of isotone  $f$ -derivations on a distributive lattice has also a structure of a distributive lattice.

**Theorem 1** [21]. Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $d_1, d_2$  be two isotone  $f$ -derivations on  $L$ . Define  $(d_1 \sqcap d_2)(x) = d_1x \wedge d_2x$  and  $(d_1 \sqcup d_2)(x) = d_1x \vee d_2x$ , for any  $x \in L$ . Then the structure  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$  is a distributive lattice, where the order relation  $\preceq$  is defined as:

$$d_1 \preceq d_2 \text{ if and only if } d_1 \sqcup d_2 = d_2, \text{ for any } d_1, d_2 \in \mathfrak{I}_f(L).$$

### 3. PRINCIPAL $f$ -DERIVATIONS ON A LATTICE

In this section, we show a necessary and sufficient condition that the functions  $d_{(\alpha, f)}$  on a lattice  $L$  are  $f$ -derivations (called principal  $f$ -derivations). Also, we show that their set has a lattice structure. Furthermore, we provide a representation (resp. a characterization) theorem of any lattice (resp. distributive lattice) in terms of its principal  $f$ -derivations.

### 3.1. Poset structure for the set of principal $f$ -derivations on a lattice

The following result shows a necessary and sufficient condition that the functions  $d_{(\alpha,f)}$  on a lattice  $L$  being  $f$ -derivations on  $L$ .

**Theorem 2.** *Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \rightarrow L$  be a function. Then it holds that  $f$  is a  $\wedge$ -homomorphism if and only if  $d_{(\alpha,f)}$  is an  $f$ -derivation on  $L$ , for any  $\alpha \in L$ .*

*Proof.* The direct implication follows from Proposition 2.3. For the converse implication, we assume that  $d_{(\alpha,f)}$  is an  $f$ -derivation on  $L$ , for any  $\alpha \in L$ . It follows that

$$\begin{aligned} d_{(\alpha,f)}(x \wedge y) &= \alpha \wedge f(x \wedge y) \\ &= (d_{(\alpha,f)}(x) \wedge fy) \vee (fx \wedge d_{(\alpha,f)}(y)) \\ &= (\alpha \wedge fx \wedge fy) \vee (fx \wedge \alpha \wedge fy) \\ &= \alpha \wedge fx \wedge fy, \text{ for any } \alpha, x, y \in L. \end{aligned}$$

Hence,  $\alpha \wedge f(x \wedge y) = \alpha \wedge fx \wedge fy$ , for any  $\alpha, x, y \in L$ . On the one hand, setting  $\alpha = f(x \wedge y)$ . Then it follows that  $f(x \wedge y) = f(x \wedge y) \wedge (fx \wedge fy)$ , for any  $x, y \in L$ . Hence,  $f(x \wedge y) \leq fx \wedge fy$ , for any  $x, y \in L$ . On the other hand, setting  $\alpha = fx \wedge fy$ . Then it follows that  $(fx \wedge fy) \wedge f(x \wedge y) = (fx \wedge fy) \wedge (fx \wedge fy) = fx \wedge fy$ , for any  $x, y \in L$ . Hence,  $fx \wedge fy \leq f(x \wedge y)$ , for any  $x, y \in L$ . Thus,  $f(x \wedge y) = fx \wedge fy$ , for any  $x, y \in L$ . Therefore,  $f$  is a  $\wedge$ -homomorphism. ■

The following corollary expresses the relationship between  $\mathcal{P}_f(L)$  and  $\mathfrak{I}_f(L)$ .

**Corollary 3.1.** *Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \rightarrow L$  be a function. Then it holds that  $f$  is a  $\wedge$ -homomorphism if and only if  $\mathcal{P}_f(L)$  is a subset of  $\mathfrak{I}_f(L)$ .*

*Proof.* The proof is directly from Theorem 2. ■

In what follows, for a given lattice  $L$ , the  $f$ -derivations  $d_{(\alpha,f)}$  will be called principal  $f$ -derivations on  $L$ , and  $\mathcal{P}_f(L)$  denotes their set. On  $\mathcal{P}_f(L)$ , we define a binary relation  $\leq'$  as follows:

$$d_{(\alpha,f)} \leq' d_{(\beta,f)} \text{ if and only if } d_{(\alpha,f)}(x) \leq d_{(\beta,f)}(x), \text{ for any } x \in L.$$

One easily verifies that  $\leq'$  is an order relation on  $\mathcal{P}_f(L)$ .

**Remark 3.1.** If  $(L, \leq, \wedge, \vee, 0, 1)$  is a bounded lattice, then the poset  $(\mathcal{P}_f(L), \leq')$  is also bounded, where  $0_{\mathcal{P}_f(L)} = d_{(0,f)}$  and  $1_{\mathcal{P}_f(L)} = d_{(1,f)}$  such that  $d_{(0,f)}(x) = 0$  and  $d_{(1,f)}(x) = fx$ , for any  $x \in L$ .

### 3.2. Lattice structure for the poset of principal $f$ -derivations on a lattice

In this subsection, we show some cases in which the poset  $(\mathcal{P}_f(L), \leq')$  of principal  $f$ -derivations on a lattice  $L$  has a lattice structure. Also, we provide a representation of a lattice in terms of its principal  $f$ -derivations. First, we show that  $(\mathcal{P}_f(L), \leq')$  is a  $\wedge$ -semi-lattice.

**Proposition 3.1.** *Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \rightarrow L$  be a  $\wedge$ -homomorphism. Then the poset  $(\mathcal{P}_f(L), \leq')$  is a  $\wedge$ -semi-lattice.*

*Proof.* Let  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ . It is easy to verify that  $d_{(\alpha \wedge \beta, f)}$  is the greatest lower bound of  $d_{(\alpha,f)}$  and  $d_{(\beta,f)}$ . Thus,  $(\mathcal{P}_f(L), \leq')$  is a  $\wedge$ -semi-lattice. ■

The following theorem shows that the set of principal  $f$ -derivations on a complete lattice is also a complete lattice.

**Theorem 3.** *Let  $(L, \leq, \wedge, \vee)$  be a complete lattice and  $f : L \rightarrow L$  be a  $\wedge$ -homomorphism. Then the poset  $(\mathcal{P}_f(L), \leq')$  is a complete lattice.*

*Proof.* Proposition 3.1 guarantees that  $(\mathcal{P}_f(L), \leq')$  is a  $\wedge$ -semi-lattice. Let  $A$  be a non-empty subset of  $\mathcal{P}_f(L)$  and  $P^u$  be the set of upper bounds of  $A$ . The fact that  $(L, \leq, \wedge, \vee)$  is a complete lattice implies that  $d_{(\beta,f)} = d_{(\bigwedge \alpha_i, f)}$  with  $d_{(\alpha_i, f)} \in P^u$  is the least upper bound of  $A$ . Thus,  $(\mathcal{P}_f(L), \leq')$  is a complete lattice. ■

The following corollary follows from the above theorem.

**Corollary 3.2.** *Let  $(L, \leq, \wedge, \vee)$  be a finite lattice and  $f : L \rightarrow L$  be a  $\wedge$ -homomorphism. Then  $(\mathcal{P}_f(L), \leq')$  is a finite lattice.*

The following theorem shows that the set of principal  $f$ -derivations on a distributive lattice is also a distributive lattice.

**Theorem 4.** *Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \rightarrow L$  be a  $\wedge$ -homomorphism. Then the poset  $(\mathcal{P}_f(L), \leq')$  is a distributive lattice.*

*Proof.* Let  $d_{(\alpha,f)}, d_{(\beta,f)} \in L$ . The fact that  $(L, \leq, \wedge, \vee)$  is distributive implies that  $d_{(\alpha \vee \beta, f)}$  is the least upper bound of  $d_{(\alpha,f)}$  and  $d_{(\beta,f)}$ . Thus,  $(\mathcal{P}_f(L), \leq')$  is a lattice. Moreover, its distributivity follows from that of  $(L, \leq, \wedge, \vee)$ . ■

Next, we provide a representation of a lattice  $L$  based on its principal  $f$ -derivations. This representation gives also another case where the poset  $(\mathcal{P}_f(L), \leq')$  is a lattice. First, we need to show the following lemma.

**Lemma 5.** *Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \rightarrow L$  be a function. If  $f$  is surjective, then for any  $\alpha, \beta \in L$ , the following equivalence holds:*

$$\alpha \leq \beta \text{ if and only if } d_{(\alpha,f)} \leq' d_{(\beta,f)}.$$

**Proof.** The direct implication is immediate. For the converse implication, assume that  $f$  is surjective. Let  $\alpha, \beta \in L$  such that  $d_{(\alpha,f)} \leq' d_{(\beta,f)}$ . Then  $\alpha \wedge fx \leq \beta \wedge fx$ , for any  $x \in L$ . Since  $f$  is surjective, it holds that there exists  $m \in L$  such that  $fm = \alpha$ . Hence,  $\alpha \wedge fm \leq \beta \wedge fm$ . Thus,  $\alpha \leq \beta$ . ■

Now, we are able to provide a representation of a lattice in terms of its principal  $f$ -derivations.

**Theorem 6.** *Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f$  be a  $\wedge$ -homomorphism. If  $f$  is a  $\wedge$ -epimorphism, then the poset  $(\mathcal{P}_f(L), \leq')$  is a lattice, where  $d_{(\alpha,f)} \wedge' d_{(\beta,f)} = d_{(\alpha \wedge \beta, f)}$  and  $d_{(\alpha,f)} \vee' d_{(\beta,f)} = d_{(\alpha \vee \beta, f)}$ , for any  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ . Moreover,  $(L, \leq, \wedge, \vee)$  and  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  are isomorphic.*

**Proof.** Assume that  $f : L \rightarrow L$  is a  $\wedge$ -epimorphism. Proposition 3.1 guarantees that  $(\mathcal{P}_f(L), \leq')$  is a  $\wedge$ -semi-lattice, where  $d_{(\alpha,f)} \wedge' d_{(\beta,f)} = d_{(\alpha \wedge \beta, f)}$  is the greatest lower bound of  $d_{(\alpha,f)}$  and  $d_{(\beta,f)}$ , for any  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ . Now, we show that  $(\mathcal{P}_f(L), \leq')$  is a  $\vee$ -semi-lattice. Let  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ , since  $f$  is surjective, it follows from Lemma 5 that  $d_{(\alpha,f)} \vee' d_{(\beta,f)} = d_{(\alpha \vee \beta, f)}$  is the least upper bound of  $d_{(\alpha,f)}$  and  $d_{(\beta,f)}$ . Hence,  $(\mathcal{P}_f(L), \leq')$  is a  $\vee$ -semi-lattice. Thus, the structure  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  is a lattice.

Next, let  $\psi : L \rightarrow \mathcal{P}_f(L)$  be a mapping defined as  $\psi(\alpha) = d_{(\alpha,f)}$ , for any  $\alpha \in L$ . It is obvious to verify that  $\psi$  is surjective. Furthermore, Lemma 5 guarantees that

$$\alpha \leq \beta \text{ if and only if } \psi(\alpha) \leq' \psi(\beta), \text{ for any } \alpha, \beta \in L.$$

Thus,  $\psi$  is an order isomorphism between  $L$  and  $\mathcal{P}_f(L)$ . Proposition 2.1 guarantees that  $\psi$  is a lattice isomorphism. Therefore,  $(L, \leq, \wedge, \vee)$  and  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  are isomorphic. ■

In the following, we present an illustrative example of Theorem 6.

**Example 3.1.** Let  $L = D(30)$  be the lattice of the positive divisors of 30 given by Hasse diagram in Figure 1, and  $f : D(30) \rightarrow D(30)$  be a function defined by the following table:

$x$	1	2	3	5	6	10	15	30
$fx$	1	2	5	3	10	6	15	30



The following table presents the elements of  $\mathcal{P}_f(D(30))$ .

$x$	1	2	3	5	6	10	15	30
$d_{(1,f)}(x)$	1	1	1	1	1	1	1	1
$d_{(2,f)}(x)$	1	2	1	1	2	2	1	2
$d_{(3,f)}(x)$	1	1	1	3	1	3	3	3
$d_{(5,f)}(x)$	1	1	5	1	5	1	5	5
$d_{(6,f)}(x)$	1	2	1	3	2	6	3	6
$d_{(10,f)}(x)$	1	2	5	1	10	2	5	10
$d_{(15,f)}(x)$	1	1	5	3	5	3	15	15
$d_{(30,f)}(x)$	1	2	5	3	10	6	15	30

One easily verifies that  $f$  is a lattice automorphism. Hence, Theorem 6 guarantees that  $(\mathcal{P}_f(D(30)), \leq', \wedge', \vee')$  is a lattice and isomorphic to  $(D(30), |, gcd, lcm)$ .

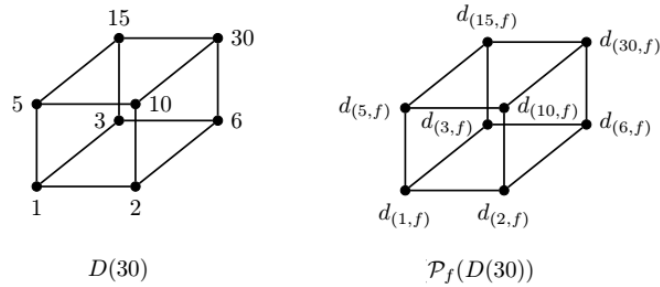


Figure 1. The Hasse diagrams of the lattices  $(D(30), |, gcd, lcm)$  and  $(\mathcal{P}_f(D(30)), \leq', \wedge', \vee')$ .

Note that the converse of the Theorem 6 does not necessarily hold, as can be seen in the following example.

**Example 3.2.** Let  $L = D(6)$  be the lattice of the positive divisors of 6 given by Hasse diagram in Figure 2, and  $f : D(6) \rightarrow D(6)$  be a function defined by the following table:

$x$	1	2	3	6
$fx$	1	1	6	6

The following table presents the elements of  $\mathcal{P}_f(D(6))$ .

$d_{(1,f)}(x)$	1	1	1	1
$d_{(2,f)}(x)$	1	1	2	2
$d_{(3,f)}(x)$	1	1	3	3
$d_{(6,f)}(x)$	1	1	6	6

One easily verifies that  $f$  is a  $\wedge$ -homomorphism. Moreover, since there not exists  $x \in D(6)$  such that  $fx = 2$ , it holds that  $f$  is not surjective. But, as can be seen in Figure 2 that the poset  $(\mathcal{P}_f(D(6)), \leq')$  is a lattice and isomorphic to  $(D(6), |, gcd, lcm)$ .

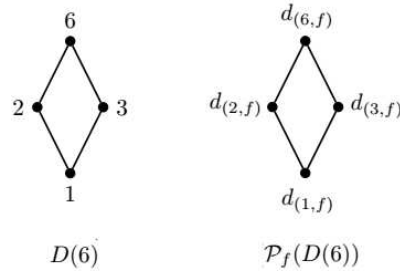


Figure 2. The Hasse diagrams of the lattices  $(D(6), |, gcd, lcm)$  and  $(\mathcal{P}_f(D(6)), \leq', \wedge', \vee')$ .

### 3.3. A relationship between a distributive lattice and its lattice of isotone $f$ -derivations

In this subsection, we give a relationship between a distributive lattice  $L$  and its lattice of isotone  $f$ -derivations  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ . Also, we show a characterization theorem of a distributive lattice in terms of its principal  $f$ -derivations. First, we need to recall the following result.

**Proposition 3.2** [21]. *Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \rightarrow L$  be a  $\wedge$ -homomorphism. Then the structure  $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$  is a sublattice of the distributive lattice  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ .*

In the case of  $(L, \leq, \wedge, \vee)$  is a distributive lattice, the following proposition shows that the lattice structures  $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$  and  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  coincide.

**Proposition 3.3.** *Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \rightarrow L$  be a  $\wedge$ -homomorphism. Then  $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$  coincides with  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$ .*

**Proof.** On the one hand,  $(d_{(\alpha,f)} \sqcap d_{(\beta,f)})(x) = (\alpha \wedge fx) \wedge (\beta \wedge fx) = (\alpha \wedge \beta) \wedge fx = d_{(\alpha \wedge \beta, f)}(x) = (d_{(\alpha,f)} \wedge' d_{(\beta,f)})(x)$ , for any  $\alpha, \beta, x \in L$ . Then  $\sqcap$  coincides with  $\wedge'$  on  $\mathcal{P}_f(L)$ , for any lattice  $(L, \leq, \wedge, \vee)$ . On the other hand, we assume that  $(L, \leq, \wedge, \vee)$  is a distributive lattice. Let  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ , then  $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(x) = (\alpha \wedge fx) \vee (\beta \wedge fx) = (\alpha \vee \beta) \wedge fx = d_{(\alpha \vee \beta, f)}(x) = (d_{(\alpha,f)} \vee' d_{(\beta,f)})(x)$ , for any  $x \in L$ . Thus,  $\sqcup$  coincides with  $\vee'$  on  $\mathcal{P}_f(L)$ . Therefore, the lattice structures  $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$  and  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  coincide. ■

Combining Propositions 3.2 and 3.3 leads to the following corollary.

**Corollary 3.3.** *Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \rightarrow L$  be a  $\wedge$ -homomorphism. Then  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  is a sublattice of  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ .*

The following theorem shows a relationship between a distributive lattice and its lattice of isotone  $f$ -derivations.

**Theorem 7.** *Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \rightarrow L$  be a  $\wedge$ -epimorphism. Then  $(L, \leq, \wedge, \vee)$  is isomorphic to a sublattice of  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ .*

**Proof.** Assume that  $(L, \leq, \wedge, \vee)$  is a distributive lattice and  $f : L \rightarrow L$  is a  $\wedge$ -epimorphism. On the one hand, Theorem 6 guarantees that  $(L, \leq, \wedge, \vee)$  is isomorphic to  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$ . On the other hand, Corollary 3.3 shows that  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  is a sublattice of  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ . Consequently,  $(L, \leq, \wedge, \vee)$  is isomorphic to a sublattice of  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ . ■

We conclude this subsection by a characterization theorem of a distributive lattice in terms of its principal  $f$ -derivations.

**Theorem 8.** *Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \rightarrow L$  be a  $\wedge$ -epimorphism. The following statements are equivalent:*

- (i)  $(L, \leq, \wedge, \vee)$  is distributive;
- (ii)  $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$  is a distributive lattice;
- (iii)  $\sqcup$  is a binary operation on  $\mathcal{P}_f(L)$ ;
- (iv)  $\sqcup$  coincides with  $\vee'$  on  $\mathcal{P}_f(L)$ .

**Proof.** (i) $\Rightarrow$ (ii): A straightforward application of Proposition 3.2.

(ii) $\Rightarrow$ (iii): The proof is immediate.

(iii) $\Rightarrow$ (iv): Let  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ . The fact that  $\sqcup$  is a binary operation on  $\mathcal{P}_f(L)$  implies that there exists  $d_{(\gamma,f)} \in \mathcal{P}_f(L)$  such that  $d_{(\alpha,f)} \sqcup d_{(\beta,f)} = d_{(\gamma,f)}$ , this equivalent to

$$(\alpha \wedge fx) \vee (\beta \wedge fx) = \gamma \wedge fx, \text{ for any } x \in L.$$

Since  $f$  is surjective, it follows that there exist  $a, b, c \in L$  such that  $fa = \alpha$ ,  $fb = \beta$  and  $fc = \gamma$ . Setting  $x = a$  (resp.  $x = b$ ), it holds that  $\alpha = \gamma \wedge \alpha$  (resp.  $\beta = \gamma \wedge \beta$ ). Moreover, setting  $x = c$ , we obtain that  $\gamma = \alpha \vee \beta$ . Hence,  $d_{(\alpha,f)} \sqcup d_{(\beta,f)} = d_{(\alpha \vee \beta, f)} = d_{(\alpha,f)} \vee' d_{(\beta,f)}$ . Thus,  $\sqcup$  coincides with the binary operation  $\vee'$  on  $\mathcal{P}_f(L)$ .

(iv) $\Rightarrow$ (i): Let  $\alpha, \beta, \gamma \in L$ . Since  $\sqcup$  coincides with  $\vee'$  on  $\mathcal{P}_f(L)$ , it holds that  $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(x) = (d_{(\alpha,f)} \vee' d_{(\beta,f)})(x) = d_{(\alpha \vee \beta, f)}(x)$  for any  $x \in L$ , this equivalent to

$$(\alpha \wedge fx) \vee (\beta \wedge fx) = (\alpha \vee \beta) \wedge fx, \text{ for any } x \in L.$$

The fact that  $f$  is surjective implies that there exists  $c \in L$  satisfying  $fc = \gamma$ . Setting  $x = c$ , we obtain that  $(\alpha \wedge \gamma) \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge \gamma$ . Thus,  $(L, \leq, \wedge, \vee)$  is distributive.  $\blacksquare$

**Remark 3.2.** From Theorem 8, we conclude that if  $(L, \leq, \wedge, \vee)$  is not distributive and  $f : L \rightarrow L$  is a  $\wedge$ -epimorphism, then  $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$  has not a lattice structure. It is only a  $\sqcap$ -semi-lattice, indeed, in this case  $\sqcup$  can not be a binary operation on  $\mathcal{P}_f(L)$ .

#### 4. IDEAL STRUCTURE OF $f$ -FIXED POINTS OF AN ISOTONE $f$ -DERIVATION ON A LATTICE

This section is devoted to study the structure of the set of  $f$ -fixed points of an isotone  $f$ -derivation on a lattice  $L$ . More specifically, we present some cases that this set is an ideal of  $L$ , and we provide a characterization theorem of principal ideals of  $L$  in terms of its principal  $f$ -derivations. Furthermore, we show a relationship between prime ideals of  $L$  and  $f$ -derivations on  $L$ . First, we recall the following definitions.

##### 4.1. Definitions

A non-empty subset  $I$  of a lattice  $L$  is called an *ideal*, if the following two conditions hold:

- (i) if  $x \in L$  and  $y \in I$  such that  $x \leq y$ , then  $x \in I$ ;
- (ii) if  $x, y \in I$ , then  $x \vee y \in I$ .

An ideal  $I$  is called *prime* if  $x \wedge y \in I$  implies that  $x \in I$  or  $y \in I$ , for any  $x, y \in L$ . An ideal is called *principal*, if it is generated by an element  $x \in L$ . It is the smallest ideal contains  $x$  and is given by the set  $\downarrow x = \{y \in L \mid y \leq x\}$ .

**Definition 4.1** [3]. Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $d$  be an  $f$ -derivation on  $L$ . The set of  $f$ -fixed points of  $d$  is given by:

$$Fix_{(d,f)}(L) = \{x \in L \mid dx = fx\}.$$

**Notation 4.1.** Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \rightarrow L$  be a function. We denote by:

- (i)  $\mathfrak{F}_f(L) := \{Fix_{(d,f)}(L) \mid d \in \mathfrak{I}_f(L)\};$
- (ii)  $\mathcal{F}_f(L) := \{Fix_{(d_{(\alpha,f)},f)}(L) \mid d_{(\alpha,f)} \in \mathcal{P}_f(L)\}.$

#### 4.2. Ideal structure of the set of $f$ -fixed points of an isotone $f$ -derivation on a lattice

In this subsection, we present some cases that the set of  $f$ -fixed points of an isotone (resp. a principal)  $f$ -derivations on a lattice  $L$  is an ideal of  $L$ .

**Theorem 9.** *Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $d$  be an isotone  $f$ -derivation on  $L$ . If  $f$  is a  $\vee$ -homomorphism and  $Fix_{(d,f)}(L)$  is a non-empty set, then  $Fix_{(d,f)}(L)$  is an ideal of  $L$ .*

**Proof.** Assume that  $f : L \rightarrow L$  is a  $\vee$ -homomorphism. Let  $d$  be an isotone  $f$ -derivation on  $L$  such that  $Fix_{(d,f)}(L)$  is a non-empty set. On the one hand, let  $x, y \in L$  such that  $x \in Fix_{(d,f)}(L)$  and  $y \leq x$ . The fact that  $d$  is an  $f$ -derivation on  $L$  implies from Proposition 2.2 that  $dy \leq fy$ . Since  $f$  is increasing,  $y \leq x$  and  $x \in Fix_{(d,f)}(L)$ , it follows that  $fy \leq fx = dx$ . Hence,  $dy = d(x \wedge y) = (dx \wedge fy) \vee (fx \wedge dy) = fy \vee dy$ , and this implies that  $fy \leq dy$ . Thus,  $dy = fy$ , i.e.,  $y \in Fix_{(d,f)}(L)$ . On the other hand, let  $x, y \in Fix_{(d,f)}(L)$ . This implies that  $dx = fx$  and  $dy = fy$ . Since  $(L, \leq, \wedge, \vee)$  is distributive,  $f$  is a  $\vee$ -homomorphism and  $d$  is an isotone  $f$ -derivation on  $L$ , it follows from Proposition 2.2 that  $d(x \vee y) = dx \vee dy = fx \vee fy = f(x \vee y)$ . Hence,  $x \vee y \in Fix_{(d,f)}(L)$ . Finally, we conclude that  $Fix_{(d,f)}(L)$  is an ideal of  $L$ . ■

**Remark 4.1.** In general, the set of  $f$ -fixed points of an  $f$ -derivation on a lattice  $L$  is a non-empty set. Indeed, if  $(L, \leq, \wedge, \vee)$  is a lattice has a least element  $0 \in L$  and  $f0 = 0$ , then  $0$  is an  $f$ -fixed point of any  $f$ -derivation on  $L$ .

**Theorem 10.** *Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $d_{(\alpha,f)}$  be a principal  $f$ -derivation on  $L$  such that  $Fix_{(d_{(\alpha,f)},f)}(L)$  is a non-empty set. If  $f$  is a lattice homomorphism, then  $Fix_{(d_{(\alpha,f)},f)}(L)$  is an ideal of  $L$ .*

**Proof.** Assume that  $f : L \rightarrow L$  is a lattice homomorphism. Let  $d_{(\alpha,f)} \in \mathcal{P}_f(L)$  such that  $Fix_{(d_{(\alpha,f)},f)}(L)$  is a non-empty set. On the one hand, let  $x, y \in L$  such that  $x \in Fix_{(d_{(\alpha,f)},f)}(L)$  and  $y \leq x$ . The fact that  $x \in Fix_{(d_{(\alpha,f)},f)}(L)$  implies

that  $d_{(\alpha,f)}(x) = \alpha \wedge fx = fx$ . Hence,  $fx \leq \alpha$ . Now, since  $f$  is increasing and  $y \leq x$ , it holds that  $fy \leq fx$ . Hence,  $fy \leq \alpha$ . Thus,  $d_{(\alpha,f)}(y) = \alpha \wedge fy = fy$ . Therefore,  $y \in \text{Fix}_{(d_{(\alpha,f)},f)}(L)$ . On the other hand, let  $x, y \in \text{Fix}_{(d_{(\alpha,f)},f)}(L)$ . Then  $d_{(\alpha,f)}(x) = \alpha \wedge fx = fx$  and  $d_{(\alpha,f)}(y) = \alpha \wedge fy = fy$ . This implies that  $fx \vee fy \leq \alpha$ . The fact that  $f$  is a  $\vee$ -homomorphism and  $x, y \in \text{Fix}_{(d_{(\alpha,f)},f)}(L)$  imply that  $f(x \vee y) = fx \vee fy \leq \alpha$ . Hence,  $d_{(\alpha,f)}(x \vee y) = \alpha \wedge f(x \vee y) = f(x \vee y)$ . Thus,  $x \vee y \in \text{Fix}_{(d_{(\alpha,f)},f)}(L)$ . Therefore,  $\text{Fix}_{(d_{(\alpha,f)},f)}(L)$  is an ideal of  $L$ . ■

### 4.3. Characterization of principal ideals in terms of principal $f$ -derivations on a lattice

In this subsection, we show a characterization theorem of principal ideals of a lattice in terms of its principal  $f$ -derivations. First, we show the following key results.

**Proposition 4.1.** *Let  $(L, \leq, \wedge, \vee)$  be a lattice,  $\downarrow x$  be a principal ideal of  $L$  and  $f : L \rightarrow L$  be a  $\wedge$ -monomorphism. Then there exists a principal  $f$ -derivation  $d_{(\alpha,f)} \in \mathcal{P}_f(L)$  such that  $\downarrow x = \text{Fix}_{(d_{(\alpha,f)},f)}(L)$ , where  $\alpha = f(x)$ .*

**Proof.** Let  $\downarrow x$  be a principal ideal of  $L$ . Since  $f$  is a  $\wedge$ -monomorphism, it follows that

$$\begin{aligned} \downarrow x &= \{y \in L \mid y \leq x\} \\ &= \{y \in L \mid x \wedge y = y\} \\ &= \{y \in L \mid f(x \wedge y) = f(y)\} \\ &= \{y \in L \mid f(x) \wedge f(y) = f(y)\} \\ &= \{y \in L \mid d_{(f(x),f)}(y) = f(y)\} \\ &= \text{Fix}_{(d_{(f(x),f)},f)}(L). \end{aligned}$$

Thus, there exists  $d_{(\alpha,f)} \in \mathcal{P}_f(L)$  such that  $\downarrow x = \text{Fix}_{(d_{(\alpha,f)},f)}(L)$ , where  $\alpha = f(x)$ . ■

**Proposition 4.2.** *Let  $(L, \leq, \wedge, \vee)$  be a lattice,  $f : L \rightarrow L$  be a lattice automorphism and  $d_{(\alpha,f)}$  be a principal  $f$ -derivation on  $L$ . Then  $\text{Fix}_{(d_{(\alpha,f)},f)}(L)$  is a principal ideal of  $L$  generated by  $f^{-1}(\alpha)$ .*

**Proof.** Let  $d_{(\alpha,f)} \in \mathcal{P}_f(L)$ . Since  $f$  is a lattice automorphism, it follows that

$$\begin{aligned} \text{Fix}_{(d_{(\alpha,f)},f)}(L) &= \{y \in L \mid d_{(\alpha,f)}(y) = fy\} \\ &= \{y \in L \mid \alpha \wedge fy = fy\} \\ &= \{y \in L \mid fy \leq \alpha\} \\ &= \{y \in L \mid y \leq f^{-1}(\alpha)\} \\ &= \downarrow f^{-1}(\alpha). \end{aligned}$$

■

Combining Propositions 4.1 and 4.2 leads to the following characterization theorem of principal ideals of a lattice  $L$  in terms of its principal  $f$ -derivations.

**Theorem 11.** *Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \rightarrow L$  be a lattice automorphism. Then  $\mathcal{F}_f(L) = \{Fix_{(d_{(\alpha,f)},f)}(L) \mid d_{(\alpha,f)} \in \mathcal{P}_f(L)\}$  is exactly the set of principal ideals of  $L$ .*

In the following, we present an illustrative example of the above Theorem 11.

**Example 4.1.** Let  $L = D(30)$  be the lattice of the positive divisors of 30 given by Hasse diagram in Figure 1, and  $f$  be the  $D(30)$ -automorphism given in Example 3.1. Then the following holds:

$$\left\{ \begin{array}{l} Fix_{(d_{(1,f)},f)}(D(30)) = \{1\} = \downarrow 1 = \downarrow f^{-1}(1); \\ Fix_{(d_{(2,f)},f)}(D(30)) = \{1, 2\} = \downarrow 2 = \downarrow f^{-1}(2); \\ Fix_{(d_{(3,f)},f)}(D(30)) = \{1, 5\} = \downarrow 5 = \downarrow f^{-1}(3); \\ Fix_{(d_{(5,f)},f)}(D(30)) = \{1, 3\} = \downarrow 3 = \downarrow f^{-1}(5); \\ Fix_{(d_{(6,f)},f)}(D(30)) = \{1, 2, 5, 10\} = \downarrow 10 = \downarrow f^{-1}(6); \\ Fix_{(d_{(10,f)},f)}(D(30)) = \{1, 2, 3, 6\} = \downarrow 6 = \downarrow f^{-1}(10); \\ Fix_{(d_{(15,f)},f)}(D(30)) = \{1, 3, 5, 15\} = \downarrow 15 = \downarrow f^{-1}(15); \\ Fix_{(d_{(30,f)},f)}(D(30)) = D(30) = \downarrow 30 = \downarrow f^{-1}(30). \end{array} \right.$$

Thus,  $\mathcal{F}_f(D(30))$  is the set of principal ideals of  $D(30)$ .

#### 4.4. A relationship between prime ideals and $f$ -derivations on a lattice

In this subsection, we show a relationship between prime ideals of a lattice  $L$  and  $f$ -derivations on  $L$ . This relationship is a generalization of the result of Theorem 4.13 given by Xin in [20].

**Theorem 12.** *Let  $(L, \leq, \wedge, \vee)$  be a lattice,  $f : L \rightarrow L$  be a function and  $I$  be a prime ideal of  $L$ . The following implications hold:*

- (i) *if  $f$  is a  $\wedge$ -homomorphism, then there exists an  $f$ -derivation  $d$  on  $L$  such that  $I \subseteq Fix_{(d,f)}(L)$ ;*
- (ii) *if  $f$  is a  $\wedge$ -monomorphism, then there exists an  $f$ -derivation  $d$  on  $L$  such that  $I = Fix_{(d,f)}(L)$ .*

**Proof.** Let  $\alpha \in I$  and  $d : L \rightarrow L$  be a function defined as

$$dx = \begin{cases} fx, & \text{if } x \in I; \\ f(\alpha \wedge x), & \text{otherwise.} \end{cases}$$

(i) The fact that  $f$  is a  $\wedge$ -homomorphism and  $I$  is a prime ideal of  $L$  imply that  $d(x \wedge y) = (dx \wedge fy) \vee (fx \wedge dy)$ , for any  $x, y \in L$ . Thus,  $d$  is an  $f$ -derivation on  $L$ . The proof of  $I \subseteq \text{Fix}_{(d,f)}(L)$  is straightforward.

(ii) Assume that  $f$  is a  $\wedge$ -monomorphism. On the one hand, (i) guarantees that  $d$  is an  $f$ -derivation on  $L$  and  $I \subseteq \text{Fix}_{(d,f)}(L)$ . On the other hand, let  $x \in \text{Fix}_{(d,f)}(L)$ . Here, we distinguish two possible cases, which are  $x \in I$  or  $x \notin I$ . Now, we prove that the case of  $x \notin I$  is an impossible case. Suppose that  $x \notin I$ , then  $dx = f(\alpha \wedge x)$ . The fact that  $x \in \text{Fix}_{(d,f)}(L)$  implies that  $dx = fx$ . Hence,  $f(\alpha \wedge x) = fx$ . Since  $f$  is injective, it holds that  $\alpha \wedge x = x$ , i.e.,  $x \leq \alpha$ . Since  $\alpha \in I$  and  $I$  is an ideal, it holds that  $x \in I$ , which contradicts the hypothesis that  $x \notin I$ . Hence, necessarily  $x \in I$ . Thus,  $\text{Fix}_{(d,f)}(L) \subseteq I$ . Finally, we conclude that  $I = \text{Fix}_{(d,f)}(L)$ . ■

## 5. STRUCTURE OF THE SET OF $f$ -FIXED POINTS SETS OF ISOTONE $f$ -DERIVATIONS ON A DISTRIBUTIVE LATTICE

In this section, for a given distributive lattice  $L$ , we show that the set of  $f$ -fixed points sets  $\mathfrak{F}_f(L)$  of its isotone  $f$ -derivations has also a structure of a distributive lattice. Moreover, we prove that the set of  $f$ -fixed points sets  $\mathcal{F}_f(L)$  of principal  $f$ -derivations on  $L$  is a sublattice of  $\mathfrak{F}_f(L)$ . Finally, we provide a representation of any distributive lattice based on the  $f$ -fixed points of its principal  $f$ -derivations. First, we prove the following key result.

**Proposition 5.1.** *Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice. For any  $\text{Fix}_{(d_1,f)}(L), \text{Fix}_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$ , we define:*

$$\text{Fix}_{(d_1,f)}(L) \sqcap' \text{Fix}_{(d_2,f)}(L) = \text{Fix}_{(d_1 \sqcap d_2, f)}(L),$$

and

$$\text{Fix}_{(d_1,f)}(L) \sqcup' \text{Fix}_{(d_2,f)}(L) = \text{Fix}_{(d_1 \sqcup d_2, f)}(L).$$

Then  $\sqcap'$  and  $\sqcup'$  are idempotent, commutative and associative binary operations on  $\mathfrak{F}_f(L)$ , and they satisfy the absorption laws.

**Proof.** Let  $\text{Fix}_{(d_1,f)}(L), \text{Fix}_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$ . Then  $d_1, d_2 \in \mathfrak{I}_f(L)$ , i.e.,  $d_1$  and  $d_2$  are two isotone  $f$ -derivations on  $L$ . Since  $(L, \leq, \wedge, \vee)$  is distributive, it follows from Theorem 1 that  $d_1 \sqcap d_2$  and  $d_1 \sqcup d_2$  are also isotone  $f$ -derivations on  $L$ , i.e.,  $d_1 \sqcap d_2, d_1 \sqcup d_2 \in \mathfrak{I}_f(L)$ . Hence,  $\text{Fix}_{(d_1,f)}(L) \sqcap' \text{Fix}_{(d_2,f)}(L), \text{Fix}_{(d_1,f)}(L) \sqcup' \text{Fix}_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$ . Thus,  $\sqcap'$  and  $\sqcup'$  are binary operations on  $\mathfrak{F}_f(L)$ . Furthermore, the fact that  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$  is a lattice implies that  $\sqcap$  and  $\sqcup$  are idempotent, commutative and associative binary operations on  $\mathfrak{I}_f(L)$ , and they satisfy the absorption laws. These imply that  $\sqcap'$  and  $\sqcup'$  are also idempotent,



commutative and associative binary operations on  $\mathfrak{F}_f(L)$ , and they satisfy the absorption laws.  $\blacksquare$

**Theorem 13.** *Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \rightarrow L$  be a function. Then the structure  $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$  is a distributive lattice, where the order relation  $\preceq'$  is defined as  $Fix_{(d_1, f)}(L) \preceq' Fix_{(d_2, f)}(L)$  if and only if  $Fix_{(d_1, f)}(L) \sqcup' Fix_{(d_2, f)}(L) = Fix_{(d_2, f)}(L)$ , for any  $Fix_{(d_1, f)}(L), Fix_{(d_2, f)}(L) \in \mathfrak{F}_f(L)$ .*

**Proof.** Proposition 5.1 guarantees that  $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$  is a lattice. Moreover, from the distributivity of  $(\mathfrak{F}_f(L), \preceq, \sqcap, \sqcup)$ , we easily verify that  $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$  is also distributive.  $\blacksquare$

The following Proposition lists some proprieties of the sets of  $f$ -fixed points of principal  $f$ -derivations on a lattice.

**Proposition 5.2.** *Let  $(L, \leq, \wedge, \vee)$  be a lattice,  $f : L \rightarrow L$  be a  $\wedge$ -homomorphism and  $d_{(\alpha, f)}, d_{(\beta, f)}$  be two principal  $f$ -derivations on  $L$ . Then it holds that*

- (i)  $Fix_{(d_{(\alpha, f)}, f)}(L) \cap Fix_{(d_{(\beta, f)}, f)}(L) = Fix_{(d_{(\alpha, f)} \sqcap d_{(\beta, f)}, f)}(L) = Fix_{(d_{(\alpha \wedge \beta, f)}, f)}(L)$ ;
- (ii)  $Fix_{(d_{(\alpha, f)}, f)}(L) \cup Fix_{(d_{(\beta, f)}, f)}(L) \subseteq Fix_{(d_{(\alpha, f)} \sqcup d_{(\beta, f)}, f)}(L) \subseteq Fix_{(d_{(\alpha \vee \beta, f)}, f)}(L)$ ;
- (iii) *If  $(L, \leq, \wedge, \vee)$  is distributive, then  $Fix_{(d_{(\alpha, f)} \sqcup d_{(\beta, f)}, f)}(L) = Fix_{(d_{(\alpha \vee \beta, f)}, f)}(L)$ .*

**Proof.** (i) Let  $d_{(\alpha, f)}, d_{(\beta, f)} \in \mathcal{P}_f(L)$ . We only prove that  $Fix_{(d_{(\alpha, f)}, f)}(L) \cap Fix_{(d_{(\beta, f)}, f)}(L) = Fix_{(d_{(\alpha \wedge \beta, f)}, f)}(L)$ , as the fact that  $d_{(\alpha, f)} \sqcap d_{(\beta, f)} = d_{(\alpha \wedge \beta, f)}$  implies that  $Fix_{(d_{(\alpha, f)} \sqcap d_{(\beta, f)}, f)}(L) = Fix_{(d_{(\alpha \wedge \beta, f)}, f)}(L)$ . Then

$$\begin{aligned} Fix_{(d_{(\alpha, f)}, f)}(L) \cap Fix_{(d_{(\beta, f)}, f)}(L) &= \{x \in L \mid d_{(\alpha, f)}(x) = d_{(\beta, f)}(x) = fx\} \\ &= \{x \in L \mid \alpha \wedge fx = \beta \wedge fx = fx\} \\ &= \{x \in L \mid fx \leq \alpha \wedge \beta\} \\ &= \{x \in L \mid (\alpha \wedge \beta) \wedge fx = fx\} \\ &= \{x \in L \mid d_{(\alpha \wedge \beta, f)}(x) = fx\} \\ &= Fix_{(d_{(\alpha \wedge \beta, f)}, f)}(L). \end{aligned}$$

Thus,  $Fix_{(d_{(\alpha, f)}, f)}(L) \cap Fix_{(d_{(\beta, f)}, f)}(L) = Fix_{(d_{(\alpha, f)} \sqcap d_{(\beta, f)}, f)}(L) = Fix_{(d_{(\alpha \wedge \beta, f)}, f)}(L)$ .

(ii) On the one hand, let  $x \in Fix_{(d_{(\alpha, f)}, f)}(L) \cup Fix_{(d_{(\beta, f)}, f)}(L)$ . Then  $x \in Fix_{(d_{(\alpha, f)}, f)}(L)$  or  $x \in Fix_{(d_{(\beta, f)}, f)}(L)$ . Assume that  $x \in Fix_{(d_{(\alpha, f)}, f)}(L)$ , it holds that  $d_{(\alpha, f)}(x) = \alpha \wedge fx = fx$ . Then  $(d_{(\alpha, f)} \sqcup d_{(\beta, f)})(x) = (\alpha \wedge fx) \vee (\beta \wedge fx) = fx \vee (\beta \wedge fx) = fx$ . Thus,  $x \in Fix_{(d_{(\alpha, f)} \sqcup d_{(\beta, f)}, f)}(L)$ . The case of  $x \in Fix_{(d_{(\beta, f)}, f)}(L)$  can be proved similarly. Therefore,

$$Fix_{(d_{(\alpha, f)}, f)}(L) \cup Fix_{(d_{(\beta, f)}, f)}(L) \subseteq Fix_{(d_{(\alpha, f)} \sqcup d_{(\beta, f)}, f)}(L).$$

On the other hand, let  $y \in \text{Fix}_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)}, f)}(L)$ . Then  $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(y) = (\alpha \wedge fy) \vee (\beta \wedge fy) = fy$ . This implies that  $d_{(\alpha \vee \beta, f)}(y) = (\alpha \vee \beta) \wedge fy = (\alpha \vee \beta) \wedge [(\alpha \wedge fy) \vee (\beta \wedge fy)] = (\alpha \wedge fy) \vee (\beta \wedge fy) = fy$ . Hence,  $y \in \text{Fix}_{(d_{(\alpha \vee \beta, f)}, f)}(L)$ . Thus,

$$\text{Fix}_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)}, f)}(L) \subseteq \text{Fix}_{(d_{(\alpha \vee \beta, f)}, f)}(L).$$

Therefore,

$$\text{Fix}_{(d_{(\alpha,f)}, f)}(L) \cup \text{Fix}_{(d_{(\beta,f)}, f)}(L) \subseteq \text{Fix}_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)}, f)}(L) \subseteq \text{Fix}_{(d_{(\alpha \vee \beta, f)}, f)}(L).$$

(iii) Since  $(L, \leq, \wedge, \vee)$  is distributive, it follows from Proposition 3.3 that  $d_{(\alpha,f)} \sqcup d_{(\beta,f)} = d_{(\alpha,f)} \vee' d_{(\beta,f)} = d_{(\alpha \vee \beta, f)}$ . Thus,

$$\text{Fix}_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)}, f)}(L) = \text{Fix}_{(d_{(\alpha \vee \beta, f)}, f)}(L). \quad \blacksquare$$

The following result shows that the set of  $f$ -fixed points sets  $\mathcal{F}_f(L)$  of principal  $f$ -derivations on  $L$  is a sublattice of  $\mathfrak{F}_f(L)$ .

**Theorem 14.** *Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \rightarrow L$  be a  $\wedge$ -homomorphism. Then the structure  $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$  is a sublattice of  $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$ .*

**Proof.** Since  $(L, \leq, \wedge, \vee)$  is a distributive lattice, it holds from Theorem 13 that  $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$  is a distributive lattice. The fact that  $f$  is a  $\wedge$ -homomorphism implies that  $\mathcal{F}_f(L)$  is a subset of  $\mathfrak{F}_f(L)$ . Furthermore, Proposition 5.2 guarantees that

$$\text{Fix}_{(d_{(\alpha,f)}, f)}(L) \sqcap' \text{Fix}_{(d_{(\beta,f)}, f)}(L) = \text{Fix}_{(d_{(\alpha,f)} \sqcap d_{(\beta,f)}, f)}(L) = \text{Fix}_{(d_{(\alpha \wedge \beta, f)}, f)}(L)$$

and

$$\text{Fix}_{(d_{(\alpha,f)}, f)}(L) \sqcup' \text{Fix}_{(d_{(\beta,f)}, f)}(L) = \text{Fix}_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)}, f)}(L) = \text{Fix}_{(d_{(\alpha \vee \beta, f)}, f)}(L),$$

for any  $\text{Fix}_{(d_{(\alpha,f)}, f)}(L), \text{Fix}_{(d_{(\beta,f)}, f)}(L) \in \mathcal{F}_f(L)$ . Thus,  $\mathcal{F}_f(L)$  is closed under  $\sqcap'$  and  $\sqcup'$ . Therefore,  $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$  is a sublattice of  $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$ .  $\blacksquare$

Combining Theorems 13 and 14 leads to the following corollary.

**Corollary 5.1.** *Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \rightarrow L$  be a  $\wedge$ -homomorphism. Then the structure  $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$  is a distributive lattice.*

Next, we provide a representation of any distributive lattice based on the  $f$ -fixed points of its principal  $f$ -derivations.

**Theorem 15.** *Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \rightarrow L$  be a lattice automorphism. Then  $(L, \leq, \wedge, \vee)$  is isomorphic to  $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ .*

**Proof.** Assume that  $(L, \leq, \wedge, \vee)$  is a distributive lattice and  $f : L \rightarrow L$  is a lattice automorphism. Corollary 5.1 guarantees that  $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$  is a distributive lattice. Moreover, let  $\psi : L \rightarrow \mathcal{F}_f(L)$  be a mapping defined as:

$$\psi(\alpha) = \text{Fix}_{(d_{(f(\alpha),f)},f)}(L), \text{ for any } \alpha \in L.$$

Now, we show that  $\psi$  is surjective. Let  $\text{Fix}_{(d_{(f(\beta),f)},f)}(L) \in \mathcal{F}_f(L)$ . Since  $f$  is a lattice automorphism, it holds that there exists  $\alpha \in L$  such that  $f(\alpha) = \beta$ . Then  $\psi(\alpha) = \text{Fix}_{(d_{(f(\alpha),f)},f)}(L) = \text{Fix}_{(d_{(f(\beta),f)},f)}(L)$ . Hence,  $\psi$  is surjective.

Next, we prove that

$$\alpha \leq \beta \text{ if and only if } \psi(\alpha) \preceq' \psi(\beta), \text{ for any } \alpha, \beta \in L.$$

Since  $(L, \leq, \wedge, \vee)$  is a distributive lattice and  $f$  is a lattice automorphism, it follows from Propositions 4.1 and 5.2 that

$$\begin{aligned} \alpha \leq \beta &\iff \alpha \vee \beta = \beta \\ &\iff \downarrow(\alpha \vee \beta) = \downarrow \beta \\ &\iff \text{Fix}_{(d_{(f(\alpha \vee \beta),f)},f)}(L) = \text{Fix}_{(d_{(f(\beta),f)},f)}(L) \\ &\iff \text{Fix}_{(d_{(f(\alpha) \vee f(\beta),f)},f)}(L) = \text{Fix}_{(d_{(f(\beta),f)},f)}(L) \\ &\iff \text{Fix}_{(d_{(f(\alpha),f)} \sqcup d_{(f(\beta),f)},f)}(L) = \text{Fix}_{(d_{(f(\beta),f)},f)}(L) \\ &\iff \text{Fix}_{(d_{(\alpha,f)},f)} \sqcup' \text{Fix}_{(d_{(\beta,f)},f)}(L) = \text{Fix}_{(d_{(f(\beta),f)},f)}(L) \\ &\iff \psi(\alpha) \sqcup' \psi(\beta) = \psi(\beta) \\ &\iff \psi(\alpha) \preceq' \psi(\beta), \end{aligned}$$

for any  $\alpha, \beta \in L$ . Hence,  $\psi$  is an order isomorphism between  $L$  and  $\mathcal{F}_f(L)$ . Proposition 2.1 guarantees that  $\psi$  is a lattice isomorphism. Thus, the distributive lattices  $(L, \leq, \wedge, \vee)$  and  $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$  are isomorphic. ■

Combining Theorems 6 and 15 leads to the following corollary.

**Corollary 5.2.** *Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \rightarrow L$  be a lattice automorphism. Then the three distributive lattices  $(L, \leq, \wedge, \vee)$ ,  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  and  $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$  are isomorphic.*

## 6. CONCLUSION

In this work, we have investigated the most important properties of isotone (resp. principal)  $f$ -derivations on a lattice. In particular, we have focused on the lattice (resp. ideal) structures of isotone  $f$ -derivations and their  $f$ -fixed points sets. These properties and structures lead to some interesting results, such as the characterizations of principal ideals and distributive lattices in terms of principal

$f$ -derivations. Also, a representation of a lattice (resp. a distributive lattice) in terms of its principal  $f$ -derivations (resp. the  $f$ -fixed points sets of its principal  $f$ -derivations).

## REFERENCES

- [1] M. Ashraf, S. Ali and C. Haetinger, *On derivations in rings and their applications*, Aligarh Bull. Math. **25** (2006) 79–107.
- [2] G. Birkhoff, *Lattice Theory*, 3rd edition, Amer. Math. Soc. (Providence, RI, 1967).
- [3] Y. Çeven and M. Öztürk, *On  $f$ -derivations of lattices*, Bull. Korean Math. Soc. **45** (2008) 701–707.
- [4] B.A. Davey and H.A. Priestley, *Introduction to Lattices and Order*, 2nd edition (Cambridge University Press, 2002).
- [5] L. Ferrari, *On derivations of lattices*, Pure Math. and Appl. **12** (2001) 365–382.
- [6] P. He, X. Xin and J. Zhan, *On derivations and their fixed point sets in residuated lattices*, Fuzzy Sets and Systems **303** (2016) 97–113.  
doi:10.1016/j.fss.2016.01.006
- [7] K.H. Kim and B. Davvaz, *On  $f$ -derivations of BE-algebras*, J. Chungcheong Math. Soc. **28** (2015) 127–138.  
doi:10.14403/jcms.2015.28.1.127
- [8] B. Kolman, R.C. Busby and S.C. Ross, *Discrete Mathematical Structures*, 4th edition (Prentice Hall PTR, 2000).
- [9] S.M. Lee and K.H. Kim, *A note on  $f$ -derivations of BCC-algebras*, Pure Math. Sci. **1** (2012) 87–93.
- [10] Ş.A. Özbal and A. Firat, *On  $f$ -derivations of incline algebras*, Int. Electronic J. Pure and Appl. Math. **3** (2011) 83–90.
- [11] M.S. Rao, *Congruences and ideals in a distributive lattice with respect to a derivation*, Bulletin of the Section of Logic **42** (2013) 1–10.
- [12] G.C. Rao and K.R. Babu, *The theory of derivations in almost distributive lattice*, Bulletin of the International Mathematical Virtual Institute **7** (2017) 203–216.
- [13] S. Roman, *Lattices and Ordered Sets* (Springer Science+Business Media, New York, 2008).
- [14] B.S. Schröder, *Ordered Sets* (Birkhauser, Boston, 2003).
- [15] G. Szász, *Translationen der verbände*, Acta Fac. Rer. Nat. Univ. Comenianae **5** (1961) 53–57.
- [16] G. Szász, *Derivations of lattices*, Acta Sci. Math. **37** (1975) 149–154.
- [17] J. Wang, Y. Jun, X.L. Xin, T.Y. Li and Y. Zou, *On derivations of bounded hyperlattices*, J. Math. Res. Appl. **36** (2016) 151–161.  
doi:10.3770/j.issn:2095-2651.2016.02.003

- [18] Q. Xiao and W. Liu, *On derivations of quantales*, Open Mathematics **14** (2016) 338–346.  
doi:10.1515/math-2016-0030
- [19] X.L. Xin, T.Y. Li and J.H. Lu, *On derivations of lattices*, Information Sciences **178** (2008) 307–316.  
doi:10.1016/j.ins.2007.08.018
- [20] X.L. Xin, *The fixed set of a derivation in lattices*, Fixed Point Theory and Applications **218** (2012) 1–12.  
doi:10.1186/1687-1812-2012-218
- [21] Y.H. Yon and K.H. Kim, *On  $f$ -derivations from semilattices to lattices*, Commun. Korean Math. Soc. **29** (2014) 27–36.  
doi:10.4134/CKMS.2014.29.1.027

Received 11 November 2018

Revised 10 January 2019

Accepted 13 January 2019