

## ISOMORPHISM THEOREMS ON QUASI MODULE

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### Abstract

A quasimodel is an algebraic axiomatisation of the hyperspace structure based on a module. We initiated this structure in our paper [2]. It is a generalisation of the module structure in the sense that every module can be embedded into a quasi module and every quasi module contains a module. The structure a quasimodel is a conglomeration of a commutative semigroup with an external ring multiplication and a compatible partial order. In the entire structure partial order has an intrinsic effect and plays a key role in any development of the theory of quasi module. In the present paper we have discussed order-morphism which is a morphism like concept. Also with the help of the quotient structure of a quasi module by means of a suitable compatible congruence, we have proved order-isomorphism theorem.

**Keywords:** module, quasi module, order-morphism, congruence, quotient.

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### 1. INTRODUCTION

Quasi module is an algebraic axiomatisation of the hyperspace structure based on a module. We proposed this structure in our paper [2], while we were studying

the family  $\mathcal{C}(M)$  of all nonempty compact subsets of a Hausdorff topological module  $M$  over some topological unitary ring  $R$ . This family, commonly known as *hyperspace*, is closed under usual addition of two sets and the ring multiplication of a set defined by:  $A + B := \{a + b : a \in A, b \in B\}$  and  $rA := \{ra : a \in A\}$ , for any  $A, B \in \mathcal{C}(M)$  and  $r \in R$ . Moreover, in the semigroup  $\mathcal{C}(M)$  singletons are the only invertible elements,  $\{\theta\}$  acting as the identity ( $\theta$  being the identity in  $M$ ). Considering these singletons as the minimal elements of  $\mathcal{C}(M)$  with respect to the usual set-inclusion as partial order, we can identify the collection  $\{\{m\} : m \in M\}$  of all minimal elements of  $\mathcal{C}(M)$  with the module  $M$  through the isomorphism  $\{m\} \mapsto m$  ( $m \in M$ ). Again for any two  $r, s \in R$  and  $A, B \in \mathcal{C}(M)$  we have  $(r + s)A \subseteq rA + sA$  and  $rA \subseteq rB$ , whenever  $A \subseteq B$ . We have axiomatised these properties of the hyperspace  $\mathcal{C}(M)$  and introduced the concept of *quasi module* whose definition is as follows:

**Definition 1.1** [2]. Let  $(X, \leq)$  be a partially ordered set, ‘+’ be a binary operation on  $X$  [called *addition*] and ‘ $\cdot$ ’:  $R \times X \rightarrow X$  be another composition [called *ring multiplication*,  $R$  being a unitary ring]. If the operations and partial order satisfy the following axioms then  $(X, +, \cdot, \leq)$  is called a *quasi module* (in short *qmod*) over  $R$ .

$A_1$  :  $(X, +)$  is a commutative semigroup with identity  $\theta$ .

$A_2$  :  $x \leq y$  ( $x, y \in X$ )  $\Rightarrow x + z \leq y + z$ ,  $r \cdot x \leq r \cdot y$ ,  $\forall z \in X, \forall r \in R$ .

$A_3$  : (i)  $r \cdot (x + y) = r \cdot x + r \cdot y$ ,

(ii)  $r \cdot (s \cdot x) = (rs) \cdot x$ ,

(iii)  $(r + s) \cdot x \leq r \cdot x + s \cdot x$ ,

(iv)  $1 \cdot x = x$ , ‘1’ being the multiplicative identity of  $R$ ,

(v)  $0 \cdot x = \theta$  and  $r \cdot \theta = \theta \forall x, y \in X, \forall r, s \in R$ .

$A_4$  :  $x + (-1) \cdot x = \theta$  if and only if  $x \in X_0 := \{z \in X : y \not\leq z, \forall y \in X \setminus \{z\}\}$ .

$A_5$  : For each  $x \in X, \exists y \in X_0$  such that  $y \leq x$ .

The elements of the set  $X_0$  are the minimal elements of  $X$  with respect to the defined partial order of  $X$ . These elements of  $X_0$  are called ‘*one order*’ elements of  $X$ . In [2] we have shown that this  $X_0$  becomes a module over the same unitary ring  $R$ . In the same paper [2] it has also been shown that every module can be embedded into a quasi module in the following sense: “*Given any module  $M$  over some unitary ring  $R$ , there exists a quasi module  $X$  over  $R$  such that  $M$  is isomorphic with  $X_0$  as a module.*” For this reason we call ‘quasi module’ a generalisation of the module structure.

**Example 1.2.** Let  $\mathbb{Z}$  be the ring of integers and  $\mathbb{Z}^+ := \{n \in \mathbb{Z} : n \geq 0\}$ . Then under the usual addition,  $\mathbb{Z}^+$  is a commutative semigroup with the identity 0.

Also it is a partially ordered set with respect to the usual order ( $\leq$ ) of integers. If we define the ring multiplication ‘ $\cdot$ ’ :  $\mathbb{Z} \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$  by  $(m, n) \mapsto |m|n$ , then it is a routine work to verify that  $(\mathbb{Z}^+, +, \cdot, \leq)$  is a quasi module over  $\mathbb{Z}$ . Here the set of all one order elements is given by  $[\mathbb{Z}^+]_0 = \{0\}$ .

To prove some isomorphism theorem we need first some morphism-like concept between two quasi modules over a common unitary ring. So we start with the concept of ‘order-morphism’ which is capable enough to have some adequate theory on isomorphisms. We shall also discuss with the help of suitable examples some properties of order-morphisms.

**Definition 1.3** [2]. A mapping  $f : X \longrightarrow Y$  ( $X, Y$  being two quasi modules over a unitary ring  $R$ ) is called an *order-morphism* if

- (i)  $f(x + y) = f(x) + f(y), \forall x, y \in X$
- (ii)  $f(rx) = rf(x), \forall r \in R, \forall x \in X$
- (iii)  $x \leq y (x, y \in X) \Rightarrow f(x) \leq f(y)$
- (iv)  $p \leq q (p, q \in f(X)) \Rightarrow f^{-1}(p) \subseteq \downarrow f^{-1}(q)$  and  $f^{-1}(q) \subseteq \uparrow f^{-1}(p)$ , where  $\uparrow A := \{x \in X : x \geq a \text{ for some } a \in A\}$  and  $\downarrow A := \{x \in X : x \leq a \text{ for some } a \in A\}$  for any  $A \subseteq X$ .

A surjective (injective, bijective) order-morphism is called an *order-epimorphism* (*order-monomorphism*, *order-isomorphism* respectively).

If  $f : X \longrightarrow Y$  is an order-morphism and  $\theta, \theta'$  be the identity elements of  $X, Y$  respectively then  $f(\theta) = f(0.\theta) = 0.f(\theta) = \theta'$ .

**Example 1.4.** Let  $f : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$  be defined by  $f(n) := 2n, \forall n \in \mathbb{Z}^+$ . Then for any  $n, m \in \mathbb{Z}^+$  we have  $f(n + m) = 2(n + m) = 2n + 2m = f(n) + f(m)$ . Also  $f(r \cdot n) = f(|r|n) = 2|r|n = |r|f(n) = r \cdot f(n)$ , for any  $r \in \mathbb{Z}$  [Note that the ring multiplication ‘ $\cdot$ ’ in  $\mathbb{Z}^+$  is defined by  $r \cdot n := |r|n, \forall r \in \mathbb{Z}, \forall n \in \mathbb{Z}^+$ , see Example 1.2]. Now for  $n, m \in \mathbb{Z}^+, n \leq m \Leftrightarrow 2n \leq 2m \Leftrightarrow f(n) \leq f(m)$ . This justifies that  $f$  is an order-monomorphism, since  $f^{-1}(2n) = \{n\}$ , for all  $n \in \mathbb{Z}^+$ . This is not onto, since  $f^{-1}(3) = \emptyset$ .

**Example 1.5.** Let us consider the ring of integers  $\mathbb{Z}$  which can be thought of as a topological module over the ring  $\mathbb{Z}$  with respect to the discrete topology on  $\mathbb{Z}$ . Then the set  $\mathcal{C}(\mathbb{Z})$  of all nonempty compact subsets of  $\mathbb{Z}$  form a quasi module over  $\mathbb{Z}$  with respect to the operations defined as:

- (i)  $A + B := \{a + b : a \in A, b \in B\}$ ,
- (ii)  $n.A := \{na : a \in A\}$ ,

where  $A, B \in \mathcal{C}(\mathbb{Z}), n \in \mathbb{Z}$  and usual set-inclusion as the partial order of  $\mathcal{C}(\mathbb{Z})$ .

Let  $f : \mathbb{Z}^+ \rightarrow \mathcal{C}(\mathbb{Z})$  be defined by  $f(n) := [-n, n]$ ,  $\forall n \in \mathbb{N}$ , where

$$[-n, n] := \{-n, -n+1, \dots, -1, 0, 1, \dots, n-1, n\}, \quad \text{for } n \in \mathbb{N}.$$

and  $f(0) := \{0\}$ . Since  $[-n, n]$  is a finite subset of  $\mathbb{Z}$  it follows that  $[-n, n]$  is compact in  $\mathbb{Z}$  and hence  $[-n, n] \in \mathcal{C}(\mathbb{Z})$ ,  $\forall n \in \mathbb{N}$ . This justifies that  $f$  is well-defined. We now show that  $f$  is an order-morphism.

Let  $m, n \in \mathbb{Z}^+$ . Then  $[-m, m] + [-n, n] = [-m-n, m+n] \Rightarrow f(m) + f(n) = f(m+n)$ . Here by the set  $[-0, 0]$  we mean  $\{0\}$ . Again for any  $r \in \mathbb{Z}$  we have  $f(r \cdot n) = f(|r|n) = [-|r|n, |r|n] = r[-n, n] = rf(n)$ . Now let  $n \leq m$  in  $\mathbb{Z}^+$ . Then  $[-n, n] \subseteq [-m, m] \Rightarrow f(n) \subseteq f(m)$ .

To complete our justification that  $f$  is an order-morphism let  $A, B \in f(\mathbb{Z}^+)$  with  $A \subseteq B$ . Then  $\exists n, m \in \mathbb{Z}^+$  such that  $A = f(n) = [-n, n]$  and  $B = f(m) = [-m, m]$ . So we can say that  $n \leq m$ . Since  $f$  is injective we have  $f^{-1}(A) = \{n\}$  and  $f^{-1}(B) = \{m\}$ . So  $n \leq m$  implies  $f^{-1}(A) \subseteq \downarrow f^{-1}(B)$  and  $f^{-1}(B) \subseteq \uparrow f^{-1}(A)$ . Thus  $f$  is an order-monomorphism which is not surjective. In fact, for  $C = \{1\} \in \mathcal{C}(\mathbb{Z})$ ,  $f^{-1}(C) = \emptyset$ .

**Example 1.6.** Let  $\mathcal{C}_s(\mathbb{Z}) := \{A \in \mathcal{C}(\mathbb{Z}) : 0 \in A, A \text{ is symmetric about } 0\}$ . Then  $\mathcal{C}_s(\mathbb{Z})$  is a quasi module over  $\mathbb{Z}$  with respect to the operations and partial order as defined in above Example 1.5.

Let  $f : \mathcal{C}_s(\mathbb{Z}) \rightarrow \mathbb{Z}^+$  be defined by  $f(A) := \max A$ ,  $\forall A \in \mathcal{C}_s(\mathbb{Z})$ . Since each  $A \in \mathcal{C}_s(\mathbb{Z})$  is compact and hence finite so  $\max A$  exists. Also  $A$  being symmetric about 0 it follows that  $\max A \in \mathbb{Z}^+$ . This justifies that  $f$  is well-defined. We now show that  $f$  is **not** an order-morphism although it satisfies *almost all* the axioms of an order-morphism.

We first show that  $f$  preserves the addition and ring multiplication on  $\mathcal{C}_s(\mathbb{Z})$ . For this let  $A, B \in \mathcal{C}_s(\mathbb{Z})$ . Then  $\max(A+B) = \max A + \max B \Rightarrow f(A+B) = f(A) + f(B)$ . Again for any  $r \in \mathbb{Z}$  and  $A \in \mathcal{C}_s(\mathbb{Z})$  we have

$$\begin{aligned} \max(rA) &= \begin{cases} r \cdot \max A, & \text{if } r \geq 0 \\ r \cdot \min A, & \text{if } r < 0 \end{cases} \\ &= \begin{cases} |r| \cdot \max A, & \text{if } r \geq 0 \\ -|r| \cdot \min A, & \text{if } r < 0 \end{cases} \\ &= \begin{cases} |r| \cdot \max A, & \text{if } r \geq 0 \\ |r| \cdot \max(-A), & \text{if } r < 0 \end{cases} \\ &= |r| \cdot \max A \quad [ \because A = -A \text{ for, } A \text{ is symmetric} ]. \end{aligned}$$

The above calculation implies that  $f(rA) = |r|f(A) = r \cdot f(A)$ .

If  $A, B \in \mathcal{C}_s(\mathbb{Z})$  such that  $A \subseteq B$  then  $\max A \leq \max B \Rightarrow f(A) \leq f(B)$ .

Before verifying the remaining axiom for  $f$  to be an order-morphism let us observe that  $f$  is surjective. In fact, for any  $n \in \mathbb{Z}^+$  the set  $[-n, n] \in \mathcal{C}_s(\mathbb{Z})$ , where  $[-n, n] := \{-n, -n+1, \dots, -1, 0, 1, \dots, n-1, n\}$ , for  $n \in \mathbb{N}$  and  $[-0, 0] \equiv \{0\}$ , as is explained in the above Example 1.5. Then  $f([-n, n]) = \max[-n, n] = n$ .

Now let  $n \leq m$  in  $\mathbb{Z}^+$  and  $A \in f^{-1}(n)$ . Then  $f(A) = \max A = n$ . Put  $B := A \cup \{m, -m\}$ . Then  $B$  is compact in  $\mathbb{Z}$  and symmetric about 0 so that  $B \in \mathcal{C}_s(\mathbb{Z})$ . Since  $\max A = n \leq m$  it follows that  $f(B) = \max B = m$ . Clearly  $A \subseteq B$ . Thus we have  $f^{-1}(n) \subseteq\downarrow f^{-1}(m)$ .

Next let  $D \in f^{-1}(m)$ . Then  $f(D) = \max D = m$ .

*Case I.* If  $n \in D$  then put  $C := \{x \in D : |x| \leq n\}$ . Then  $C$  being a finite subset of  $D$  is compact, symmetric (by construction) and hence  $C \in \mathcal{C}_s(\mathbb{Z})$ . Now  $f(C) = \max C = n$  and  $C \subseteq D \Rightarrow D \in\uparrow f^{-1}(n)$ .

*Case II.* If  $n \notin D$  then  $D$  cannot contain any symmetric proper subset  $E$  such that  $\max E = n$ , since  $\max E \in E$ . So such a  $D \in f^{-1}(m)$  cannot lie in  $\uparrow f^{-1}(n)$ . Consequently,  $f^{-1}(m) \not\subseteq\uparrow f^{-1}(n)$ .

Thus although  $f$  satisfies *almost all* the axioms of an order-morphism, it fails to do so the last axiom.

**Definition 1.7** [2]. Let  $f : X \rightarrow Y$  ( $X, Y$  being two qmods over the same unitary ring  $R$ ) be an order-morphism. We define  $\ker f := \{(x, y) \in X \times X : f(x) = f(y)\}$  and call it the ‘kernel of  $f$ ’.

It is immediate from definition that  $(x, x) \in \ker f, \forall x \in X$  and thus if we write  $\Delta := \{(x, x) : x \in X\}$  then  $\Delta \subseteq \ker f$ , equality holds iff  $f$  is injective.

We now discuss some concepts which will be necessary for the further development of the theory in this paper.

**Definition 1.8** [2]. Let  $\{X_\mu : \mu \in \Lambda\}$  be an arbitrary family of quasi modules over the unitary ring  $R$ . Let  $X := \prod_{\mu \in \Lambda} X_\mu$  be the Cartesian product of these quasi modules defined as:  $x \in X$  if and only if  $x : \Lambda \rightarrow \bigcup_{\mu \in \Lambda} X_\mu$  is a map such that  $x(\mu) \in X_\mu, \forall \mu \in \Lambda$ . Then by the axiom of choice we know that  $X$  is nonempty, since  $\Lambda$  is nonempty and each  $X_\mu$  contains at least the additive identity  $\theta_\mu$  (say).

Let us denote  $x_\mu := x(\mu), \forall \mu \in \Lambda$ . Also we write each  $x \in X$  as  $x = (x_\mu)$ , where  $x_\mu = p_\mu(x), p_\mu : X \rightarrow X_\mu$  being the projection map,  $\forall \mu \in \Lambda$ . Now we define addition, ring multiplication and partial order as follows: for  $x = (x_\mu), y = (y_\mu) \in X$  and  $r \in R$

- (i)  $x + y = (x_\mu + y_\mu)$ ; (ii)  $r.x = (rx_\mu)$ ; (iii)  $x \leq y$  if  $x_\mu \leq y_\mu, \forall \mu \in \Lambda$ .

**Definition 1.9** [4]. Let  $E$  be an equivalence relation on a qmod  $X$  over an unitary ring  $R$ . Then  $E$  is said to be a *congruence* on  $X$  if it satisfies the following:

- (i)  $(a, b) \in E \implies (x + a, x + b) \in E, \forall x \in X,$

- (ii)  $(a, b) \in E \implies (ra, rb) \in E, \forall r \in R,$
- (iii)  $x \leq y \leq z \ \& \ (x, z) \in E \implies (x, y) \in E$  [and hence  $(y, z) \in E$ ],
- (iv)  $a \leq x \leq b \ \& \ (x, y) \in E \implies \exists c, d \in X$  such that  $c \leq y \leq d$  and  $(a, c) \in E, (b, d) \in E.$

Any congruence  $E$  on a qmod  $X$  (over a unitary ring  $R$ ) produces the quotient set  $X/E := \{[x] : x \in X\}$ , where  $[x]$  denotes the equivalence class containing  $x$  (with respect to  $E$ ) i.e.,  $[x] := \{y \in X : (x, y) \in E\}$ . We now make this quotient set a quasi module by defining operations and partial order suitably.

**Theorem 1.10** [4]. *For any congruence  $E$  on a qmod  $X$  over a unitary ring  $R$ ,  $X/E$  becomes a qmod over  $R$  with respect to the following operations and partial order.*

- (i)  $[x] + [y] := [x + y], \forall [x], [y] \in X/E,$
- (ii)  $r[x] := [rx], \forall [x] \in X/E, \forall r \in R,$
- (iii)  $[x] \preceq [y] \iff$  for any  $x' \in [x], \exists y' \in [y]$  such that  $x' \leq y'$  and for any  $y'' \in [y], \exists x'' \in [x]$  such that  $x'' \leq y''.$

**Proposition 1.11** [2]. *If  $\phi : X \rightarrow Y$  ( $X, Y$  being two qmods over an unitary ring  $R$ ) be an order-morphism then  $\ker \phi$  is a congruence on  $X$ .*

We now give a quotient structure on  $X$  using the above congruence. For this let us construct the quotient set  $X/\ker \phi := \{[x] : x \in X\}$ , where  $[x]$  is the equivalence class containing  $x$  obtained by the congruence  $\ker \phi$ . We define addition, ring multiplication and partial order on  $X/\ker \phi$  as follows. For  $x, y \in X$  and  $r \in R$ ,

- (i)  $[x] + [y] := [x + y];$  (ii)  $r[x] := [rx];$  (iii)  $[x] \leq [y]$  if and only if  $\phi(x) \leq \phi(y).$

**Theorem 1.12** [2]. *If  $\phi : X \rightarrow Y$  ( $X, Y$  being two qmods over an unitary ring  $R$ ) be an order-morphism then  $X/\ker \phi$  is a quasi module over  $R$ .*

**Proposition 1.13** [2]. *Let  $\phi : X \rightarrow Y$  ( $X, Y$  being two qmods over an unitary ring  $R$ ) be an order-morphism. Then the canonical map  $\pi : X \rightarrow X/\ker \phi$  defined by  $\pi(x) := [x], \forall x \in X$  is an order-epimorphism.*

**Lemma 1.14** [2]. *Let  $X, Y, Z$  be three quasi modules over the unitary ring  $R$ ,  $\alpha : X \rightarrow Y$  be an order-epimorphism and  $\beta : X \rightarrow Z$  be an order-morphism such that  $\ker \alpha \subseteq \ker \beta$ . Then  $\exists$  a unique order-morphism  $\gamma : Y \rightarrow Z$  such that  $\gamma \circ \alpha = \beta$ .*

$$\begin{array}{ccc}
 X & \xrightarrow{\beta} & Z \\
 \alpha \downarrow & \nearrow \gamma & \\
 Y & & 
 \end{array}$$

## 2. SECOND ORDER-ISOMORPHISM THEOREM

In this section we shall use the concept of congruence from the above section to prove the Second order-isomorphism theorem. For this we need three quasi modules over a common unitary ring and two order-morphisms between them.

Let  $X, Y, Z$  be three quasi modules over an unitary ring  $R$  and  $\phi_1 : X \rightarrow Y$ ,  $\phi_2 : X \rightarrow Z$  be two order-morphisms such that  $\ker \phi_2 \subseteq \ker \phi_1$ . So if  $[x]_1 := \{y \in X : (x, y) \in \ker \phi_1\}$  and  $[x]_2 := \{y \in X : (x, y) \in \ker \phi_2\}$  denote the equivalence classes containing  $x$  with respect to the congruences  $\ker \phi_1$  and  $\ker \phi_2$  respectively then we must have

$$[x]_1 = \bigcup \{[y]_2 : (y, x) \in \ker \phi_1\}.$$

Also  $\ker \phi_2$  being a congruence on  $X$ ,  $X/\ker \phi_2$  is a quasi module over  $R$  (by Theorem 1.10). It is thus natural to define a relation on  $X/\ker \phi_2$  as follows:

$$\ker \phi_1 / \ker \phi_2 := \left\{ ([x]_2, [y]_2) \in X/\ker \phi_2 \times X/\ker \phi_2 : (x, y) \in \ker \phi_1 \right\}.$$

Now the question is whether  $\ker \phi_1 / \ker \phi_2$  is a congruence on  $X/\ker \phi_2$  and if so, whether it generates a quotient qmod from  $X/\ker \phi_2$  which is order-isomorphic to  $X/\ker \phi_1$ . We shall give answers to these in affirmative.

**Proposition 2.1.**  $\ker \phi_1 / \ker \phi_2$  is a congruence on  $X/\ker \phi_2$ .

**Proof.** For convenience let us denote  $\Gamma \equiv \ker \phi_1 / \ker \phi_2$ . Now  $\ker \phi_1$  being an equivalence relation it follows that  $\Gamma$  is also an equivalence relation. To show that  $\Gamma$  is a congruence let  $([x]_2, [y]_2) \in \Gamma$  and  $[z]_2 \in X/\ker \phi_2$ ,  $r \in R$ . Then  $(x, y) \in \ker \phi_1$ . So  $\ker \phi_1$  being a congruence we have

$$(i) (x+z, y+z) \in \ker \phi_1 \Rightarrow ([x+z]_2, [y+z]_2) \in \Gamma \Rightarrow ([x]_2+[z]_2, [y]_2+[z]_2) \in \Gamma.$$

$$(ii) (rx, ry) \in \ker \phi_1 \Rightarrow ([rx]_2, [ry]_2) \in \Gamma \Rightarrow (r[x]_2, r[y]_2) \in \Gamma.$$

(iii) Now let  $[x]_2 \preceq [y]_2 \preceq [z]_2$  and  $([x]_2, [z]_2) \in \Gamma$ . Then  $(x, z) \in \ker \phi_1$  and  $x \leq y' \leq z'$  for some  $y' \in [y]_2$ ,  $z' \in [z]_2$ . So  $(z', z) \in \ker \phi_2 \subseteq \ker \phi_1 \Rightarrow (x, z') \in \ker \phi_1 \Rightarrow (x, y') \in \ker \phi_1$  [ $\because \ker \phi_1$  is a congruence]  $\Rightarrow ([x]_2, [y]_2) = ([x]_2, [y']_2) \in \Gamma$ .

(iv) Next let  $[a]_2 \preceq [x]_2 \preceq [b]_2$  and  $([x]_2, [y]_2) \in \Gamma$ . Then  $(x, y) \in \ker \phi_1$  and  $a \leq x' \leq b'$  for some  $x' \in [x]_2$ ,  $b' \in [b]_2$ . Now  $(x', x) \in \ker \phi_2 \subseteq \ker \phi_1 \Rightarrow (x', y) \in \ker \phi_1 \Rightarrow \exists c, d \in X$  such that  $c \leq y \leq d$  and  $(a, c), (b', d) \in \ker \phi_1$  [ $\because \ker \phi_1$  is a congruence]  $\Rightarrow ([a]_2, [c]_2), ([b']_2, [d]_2) \in \Gamma$ . Again  $b' \in [b]_2 \Rightarrow [b']_2 = [b]_2$ . Thus  $([b]_2, [d]_2) \in \Gamma$ .

We now show that  $[c]_2 \preceq [y]_2 \preceq [d]_2$ . For this let  $c' \in [c]_2$  be arbitrary. Then  $\ker \phi_2$  being a congruence,  $c \leq y \Rightarrow \exists y' \in [y]_2$  such that  $c' \leq y'$  (by axiom (iv) in the definition of congruence 1.9). Again by the same argument, for any  $y'' \in [y]_2$ ,  $\exists c'' \in [c]_2$  such that  $c'' \leq y''$ . This implies  $[c]_2 \preceq [y]_2$ . Similarly, we can show that  $[y]_2 \preceq [d]_2$ , since  $y \leq d$ .

This justifies that  $\Gamma$  is a congruence on  $X/\ker\phi_2$ .  $\blacksquare$

**Theorem 2.2** (Second Order-isomorphism Theorem). *Let  $X, Y, Z$  be three quasi modules over an unitary ring  $R$  and  $\phi_1 : X \rightarrow Y$ ,  $\phi_2 : X \rightarrow Z$  be two order-morphisms such that  $\ker\phi_2 \subseteq \ker\phi_1$ . Then the quotient  $\text{qmod } \frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2}$  is order-isomorphic to  $X/\ker\phi_1$ .*

**Proof.**  $\ker\phi_1/\ker\phi_2$  being a congruence on  $X/\ker\phi_2$  by Proposition 2.1, we have by Theorem 1.10 that  $\frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2}$  is a quasi module over  $R$ . If

$$\left. \begin{array}{l} \pi_1 : X \rightarrow X/\ker\phi_1 \\ x \mapsto [x]_1 \end{array} \right\} \text{ and } \left. \begin{array}{l} \pi_2 : X \rightarrow X/\ker\phi_2 \\ x \mapsto [x]_2 \end{array} \right\}$$

are the canonical order-epimorphisms then  $\ker\pi_2 = \{(x, y) \in X \times X : [x]_2 = [y]_2\} = \ker\phi_2 \subseteq \ker\phi_1 = \ker\pi_1$ . So by Lemma 1.14,  $\exists$  a unique order-morphism  $\gamma : X/\ker\phi_2 \rightarrow X/\ker\phi_1$  such that  $\gamma \circ \pi_2 = \pi_1$ .

Now let  $\pi : X/\ker\phi_2 \rightarrow \frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2}$  be the canonical order-epimorphism.

Then  $\ker\pi = \ker\phi_1/\ker\phi_2$

$$\begin{aligned} &= \left\{ ([x]_2, [y]_2) \in X/\ker\phi_2 \times X/\ker\phi_2 : (x, y) \in \ker\phi_1 \right\} \\ &= \left\{ (\pi_2(x), \pi_2(y)) : \pi_1(x) = \pi_1(y) \right\} \\ &= \left\{ (\pi_2(x), \pi_2(y)) : \gamma \circ \pi_2(x) = \gamma \circ \pi_2(y) \right\} \\ &= \ker\gamma. \end{aligned}$$

So by Lemma 1.14,  $\exists$  a unique order-morphism  $\Psi : \frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2} \rightarrow X/\ker\phi_1$  such that  $\Psi \circ \pi = \gamma$ . So  $\Psi \circ \pi \circ \pi_2 = \gamma \circ \pi_2 = \pi_1$ . The following commutative diagram clarifies this.

$$\begin{array}{ccc} X & \xrightarrow{\pi_1} & X/\ker\phi_1 \\ \pi_2 \downarrow & \nearrow \gamma & \uparrow \Psi \\ X/\ker\phi_2 & \xrightarrow{\pi} & \frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2} \end{array}$$

$\pi_1$  being onto it follows from above diagram that  $\Psi$  is onto. To prove that  $\Psi$  is injective let  $\Psi \circ \pi \circ \pi_2(x) = \Psi \circ \pi \circ \pi_2(y)$  for some  $x, y \in X$ . Then  $\gamma \circ \pi_2(x) = \gamma \circ \pi_2(y) \Rightarrow (\pi_2(x), \pi_2(y)) \in \ker\gamma = \ker\pi \Rightarrow \pi \circ \pi_2(x) = \pi \circ \pi_2(y)$ . This justifies that  $\Psi$  is injective. Consequently,  $\Psi$  is an order-isomorphism.  $\blacksquare$



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