

ISOMORPHISM THEOREMS ON QUASI MODULE

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Abstract

A quasimodel is an algebraic axiomatisation of the hyperspace structure based on a module. We initiated this structure in our paper [2]. It is a generalisation of the module structure in the sense that every module can be embedded into a quasi module and every quasi module contains a module. The structure a quasimodel is a conglomeration of a commutative semigroup with an external ring multiplication and a compatible partial order. In the entire structure partial order has an intrinsic effect and plays a key role in any development of the theory of quasi module. In the present paper we have discussed order-morphism which is a morphism like concept. Also with the help of the quotient structure of a quasi module by means of a suitable compatible congruence, we have proved order-isomorphism theorem.

Keywords: module, quasi module, order-morphism, congruence, quotient.

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1. INTRODUCTION

Quasi module is an algebraic axiomatisation of the hyperspace structure based on a module. We proposed this structure in our paper [2], while we were studying

the family $\mathcal{C}(M)$ of all nonempty compact subsets of a Hausdorff topological module M over some topological unitary ring R . This family, commonly known as *hyperspace*, is closed under usual addition of two sets and the ring multiplication of a set defined by: $A + B := \{a + b : a \in A, b \in B\}$ and $rA := \{ra : a \in A\}$, for any $A, B \in \mathcal{C}(M)$ and $r \in R$. Moreover, in the semigroup $\mathcal{C}(M)$ singletons are the only invertible elements, $\{\theta\}$ acting as the identity (θ being the identity in M). Considering these singletons as the minimal elements of $\mathcal{C}(M)$ with respect to the usual set-inclusion as partial order, we can identify the collection $\{\{m\} : m \in M\}$ of all minimal elements of $\mathcal{C}(M)$ with the module M through the isomorphism $\{m\} \mapsto m$ ($m \in M$). Again for any two $r, s \in R$ and $A, B \in \mathcal{C}(M)$ we have $(r + s)A \subseteq rA + sA$ and $rA \subseteq rB$, whenever $A \subseteq B$. We have axiomatised these properties of the hyperspace $\mathcal{C}(M)$ and introduced the concept of *quasi module* whose definition is as follows:

Definition 1.1 [2]. Let (X, \leq) be a partially ordered set, ‘+’ be a binary operation on X [called *addition*] and ‘ \cdot ’: $R \times X \rightarrow X$ be another composition [called *ring multiplication*, R being a unitary ring]. If the operations and partial order satisfy the following axioms then $(X, +, \cdot, \leq)$ is called a *quasi module* (in short *qmod*) over R .

A_1 : $(X, +)$ is a commutative semigroup with identity θ .

A_2 : $x \leq y$ ($x, y \in X$) $\Rightarrow x + z \leq y + z$, $r \cdot x \leq r \cdot y$, $\forall z \in X, \forall r \in R$.

A_3 : (i) $r \cdot (x + y) = r \cdot x + r \cdot y$,

(ii) $r \cdot (s \cdot x) = (rs) \cdot x$,

(iii) $(r + s) \cdot x \leq r \cdot x + s \cdot x$,

(iv) $1 \cdot x = x$, ‘1’ being the multiplicative identity of R ,

(v) $0 \cdot x = \theta$ and $r \cdot \theta = \theta \forall x, y \in X, \forall r, s \in R$.

A_4 : $x + (-1) \cdot x = \theta$ if and only if $x \in X_0 := \{z \in X : y \not\leq z, \forall y \in X \setminus \{z\}\}$.

A_5 : For each $x \in X, \exists y \in X_0$ such that $y \leq x$.

The elements of the set X_0 are the minimal elements of X with respect to the defined partial order of X . These elements of X_0 are called ‘*one order*’ elements of X . In [2] we have shown that this X_0 becomes a module over the same unitary ring R . In the same paper [2] it has also been shown that every module can be embedded into a quasi module in the following sense: “*Given any module M over some unitary ring R , there exists a quasi module X over R such that M is isomorphic with X_0 as a module.*” For this reason we call ‘quasi module’ a generalisation of the module structure.

Example 1.2. Let \mathbb{Z} be the ring of integers and $\mathbb{Z}^+ := \{n \in \mathbb{Z} : n \geq 0\}$. Then under the usual addition, \mathbb{Z}^+ is a commutative semigroup with the identity 0.

Also it is a partially ordered set with respect to the usual order (\leq) of integers. If we define the ring multiplication ‘ \cdot ’ : $\mathbb{Z} \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ by $(m, n) \mapsto |m|n$, then it is a routine work to verify that $(\mathbb{Z}^+, +, \cdot, \leq)$ is a quasi module over \mathbb{Z} . Here the set of all one order elements is given by $[\mathbb{Z}^+]_0 = \{0\}$.

To prove some isomorphism theorem we need first some morphism-like concept between two quasi modules over a common unitary ring. So we start with the concept of ‘order-morphism’ which is capable enough to have some adequate theory on isomorphisms. We shall also discuss with the help of suitable examples some properties of order-morphisms.

Definition 1.3 [2]. A mapping $f : X \longrightarrow Y$ (X, Y being two quasi modules over a unitary ring R) is called an *order-morphism* if

- (i) $f(x + y) = f(x) + f(y), \forall x, y \in X$
- (ii) $f(rx) = rf(x), \forall r \in R, \forall x \in X$
- (iii) $x \leq y (x, y \in X) \Rightarrow f(x) \leq f(y)$
- (iv) $p \leq q (p, q \in f(X)) \Rightarrow f^{-1}(p) \subseteq \downarrow f^{-1}(q)$ and $f^{-1}(q) \subseteq \uparrow f^{-1}(p)$, where $\uparrow A := \{x \in X : x \geq a \text{ for some } a \in A\}$ and $\downarrow A := \{x \in X : x \leq a \text{ for some } a \in A\}$ for any $A \subseteq X$.

A surjective (injective, bijective) order-morphism is called an *order-epimorphism* (*order-monomorphism*, *order-isomorphism* respectively).

If $f : X \longrightarrow Y$ is an order-morphism and θ, θ' be the identity elements of X, Y respectively then $f(\theta) = f(0.\theta) = 0.f(\theta) = \theta'$.

Example 1.4. Let $f : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ be defined by $f(n) := 2n, \forall n \in \mathbb{Z}^+$. Then for any $n, m \in \mathbb{Z}^+$ we have $f(n + m) = 2(n + m) = 2n + 2m = f(n) + f(m)$. Also $f(r \cdot n) = f(|r|n) = 2|r|n = |r|f(n) = r \cdot f(n)$, for any $r \in \mathbb{Z}$ [Note that the ring multiplication ‘ \cdot ’ in \mathbb{Z}^+ is defined by $r \cdot n := |r|n, \forall r \in \mathbb{Z}, \forall n \in \mathbb{Z}^+$, see Example 1.2]. Now for $n, m \in \mathbb{Z}^+, n \leq m \Leftrightarrow 2n \leq 2m \Leftrightarrow f(n) \leq f(m)$. This justifies that f is an order-monomorphism, since $f^{-1}(2n) = \{n\}$, for all $n \in \mathbb{Z}^+$. This is not onto, since $f^{-1}(3) = \emptyset$.

Example 1.5. Let us consider the ring of integers \mathbb{Z} which can be thought of as a topological module over the ring \mathbb{Z} with respect to the discrete topology on \mathbb{Z} . Then the set $\mathcal{C}(\mathbb{Z})$ of all nonempty compact subsets of \mathbb{Z} form a quasi module over \mathbb{Z} with respect to the operations defined as:

- (i) $A + B := \{a + b : a \in A, b \in B\}$,
- (ii) $n.A := \{na : a \in A\}$,

where $A, B \in \mathcal{C}(\mathbb{Z}), n \in \mathbb{Z}$ and usual set-inclusion as the partial order of $\mathcal{C}(\mathbb{Z})$.

Let $f : \mathbb{Z}^+ \rightarrow \mathcal{C}(\mathbb{Z})$ be defined by $f(n) := [-n, n]$, $\forall n \in \mathbb{N}$, where

$$[-n, n] := \{-n, -n+1, \dots, -1, 0, 1, \dots, n-1, n\}, \quad \text{for } n \in \mathbb{N}.$$

and $f(0) := \{0\}$. Since $[-n, n]$ is a finite subset of \mathbb{Z} it follows that $[-n, n]$ is compact in \mathbb{Z} and hence $[-n, n] \in \mathcal{C}(\mathbb{Z})$, $\forall n \in \mathbb{N}$. This justifies that f is well-defined. We now show that f is an order-morphism.

Let $m, n \in \mathbb{Z}^+$. Then $[-m, m] + [-n, n] = [-m-n, m+n] \Rightarrow f(m) + f(n) = f(m+n)$. Here by the set $[-0, 0]$ we mean $\{0\}$. Again for any $r \in \mathbb{Z}$ we have $f(r \cdot n) = f(|r|n) = [-|r|n, |r|n] = r[-n, n] = rf(n)$. Now let $n \leq m$ in \mathbb{Z}^+ . Then $[-n, n] \subseteq [-m, m] \Rightarrow f(n) \subseteq f(m)$.

To complete our justification that f is an order-morphism let $A, B \in f(\mathbb{Z}^+)$ with $A \subseteq B$. Then $\exists n, m \in \mathbb{Z}^+$ such that $A = f(n) = [-n, n]$ and $B = f(m) = [-m, m]$. So we can say that $n \leq m$. Since f is injective we have $f^{-1}(A) = \{n\}$ and $f^{-1}(B) = \{m\}$. So $n \leq m$ implies $f^{-1}(A) \subseteq \downarrow f^{-1}(B)$ and $f^{-1}(B) \subseteq \uparrow f^{-1}(A)$. Thus f is an order-monomorphism which is not surjective. In fact, for $C = \{1\} \in \mathcal{C}(\mathbb{Z})$, $f^{-1}(C) = \emptyset$.

Example 1.6. Let $\mathcal{C}_s(\mathbb{Z}) := \{A \in \mathcal{C}(\mathbb{Z}) : 0 \in A, A \text{ is symmetric about } 0\}$. Then $\mathcal{C}_s(\mathbb{Z})$ is a quasi module over \mathbb{Z} with respect to the operations and partial order as defined in above Example 1.5.

Let $f : \mathcal{C}_s(\mathbb{Z}) \rightarrow \mathbb{Z}^+$ be defined by $f(A) := \max A$, $\forall A \in \mathcal{C}_s(\mathbb{Z})$. Since each $A \in \mathcal{C}_s(\mathbb{Z})$ is compact and hence finite so $\max A$ exists. Also A being symmetric about 0 it follows that $\max A \in \mathbb{Z}^+$. This justifies that f is well-defined. We now show that f is **not** an order-morphism although it satisfies *almost all* the axioms of an order-morphism.

We first show that f preserves the addition and ring multiplication on $\mathcal{C}_s(\mathbb{Z})$. For this let $A, B \in \mathcal{C}_s(\mathbb{Z})$. Then $\max(A+B) = \max A + \max B \Rightarrow f(A+B) = f(A) + f(B)$. Again for any $r \in \mathbb{Z}$ and $A \in \mathcal{C}_s(\mathbb{Z})$ we have

$$\begin{aligned} \max(rA) &= \begin{cases} r \cdot \max A, & \text{if } r \geq 0 \\ r \cdot \min A, & \text{if } r < 0 \end{cases} \\ &= \begin{cases} |r| \cdot \max A, & \text{if } r \geq 0 \\ -|r| \cdot \min A, & \text{if } r < 0 \end{cases} \\ &= \begin{cases} |r| \cdot \max A, & \text{if } r \geq 0 \\ |r| \cdot \max(-A), & \text{if } r < 0 \end{cases} \\ &= |r| \cdot \max A \quad [\because A = -A \text{ for, } A \text{ is symmetric}]. \end{aligned}$$

The above calculation implies that $f(rA) = |r|f(A) = r \cdot f(A)$.

If $A, B \in \mathcal{C}_s(\mathbb{Z})$ such that $A \subseteq B$ then $\max A \leq \max B \Rightarrow f(A) \leq f(B)$.

Before verifying the remaining axiom for f to be an order-morphism let us observe that f is surjective. In fact, for any $n \in \mathbb{Z}^+$ the set $[-n, n] \in \mathcal{C}_s(\mathbb{Z})$, where $[-n, n] := \{-n, -n+1, \dots, -1, 0, 1, \dots, n-1, n\}$, for $n \in \mathbb{N}$ and $[-0, 0] \equiv \{0\}$, as is explained in the above Example 1.5. Then $f([-n, n]) = \max[-n, n] = n$.

Now let $n \leq m$ in \mathbb{Z}^+ and $A \in f^{-1}(n)$. Then $f(A) = \max A = n$. Put $B := A \cup \{m, -m\}$. Then B is compact in \mathbb{Z} and symmetric about 0 so that $B \in \mathcal{C}_s(\mathbb{Z})$. Since $\max A = n \leq m$ it follows that $f(B) = \max B = m$. Clearly $A \subseteq B$. Thus we have $f^{-1}(n) \subseteq\downarrow f^{-1}(m)$.

Next let $D \in f^{-1}(m)$. Then $f(D) = \max D = m$.

Case I. If $n \in D$ then put $C := \{x \in D : |x| \leq n\}$. Then C being a finite subset of D is compact, symmetric (by construction) and hence $C \in \mathcal{C}_s(\mathbb{Z})$. Now $f(C) = \max C = n$ and $C \subseteq D \Rightarrow D \in\uparrow f^{-1}(n)$.

Case II. If $n \notin D$ then D cannot contain any symmetric proper subset E such that $\max E = n$, since $\max E \in E$. So such a $D \in f^{-1}(m)$ cannot lie in $\uparrow f^{-1}(n)$. Consequently, $f^{-1}(m) \not\subseteq\uparrow f^{-1}(n)$.

Thus although f satisfies *almost all* the axioms of an order-morphism, it fails to do so the last axiom.

Definition 1.7 [2]. Let $f : X \rightarrow Y$ (X, Y being two qmods over the same unitary ring R) be an order-morphism. We define $\ker f := \{(x, y) \in X \times X : f(x) = f(y)\}$ and call it the ‘kernel of f ’.

It is immediate from definition that $(x, x) \in \ker f$, $\forall x \in X$ and thus if we write $\Delta := \{(x, x) : x \in X\}$ then $\Delta \subseteq \ker f$, equality holds iff f is injective.

We now discuss some concepts which will be necessary for the further development of the theory in this paper.

Definition 1.8 [2]. Let $\{X_\mu : \mu \in \Lambda\}$ be an arbitrary family of quasi modules over the unitary ring R . Let $X := \prod_{\mu \in \Lambda} X_\mu$ be the Cartesian product of these quasi modules defined as: $x \in X$ if and only if $x : \Lambda \rightarrow \bigcup_{\mu \in \Lambda} X_\mu$ is a map such that $x(\mu) \in X_\mu$, $\forall \mu \in \Lambda$. Then by the axiom of choice we know that X is nonempty, since Λ is nonempty and each X_μ contains at least the additive identity θ_μ (say).

Let us denote $x_\mu := x(\mu)$, $\forall \mu \in \Lambda$. Also we write each $x \in X$ as $x = (x_\mu)$, where $x_\mu = p_\mu(x)$, $p_\mu : X \rightarrow X_\mu$ being the projection map, $\forall \mu \in \Lambda$. Now we define addition, ring multiplication and partial order as follows: for $x = (x_\mu)$, $y = (y_\mu) \in X$ and $r \in R$

- (i) $x + y = (x_\mu + y_\mu)$; (ii) $r.x = (rx_\mu)$; (iii) $x \leq y$ if $x_\mu \leq y_\mu$, $\forall \mu \in \Lambda$.

Definition 1.9 [4]. Let E be an equivalence relation on a qmod X over an unitary ring R . Then E is said to be a *congruence* on X if it satisfies the following:

- (i) $(a, b) \in E \implies (x + a, x + b) \in E$, $\forall x \in X$,

- (ii) $(a, b) \in E \implies (ra, rb) \in E, \forall r \in R,$
- (iii) $x \leq y \leq z \ \& \ (x, z) \in E \implies (x, y) \in E$ [and hence $(y, z) \in E$],
- (iv) $a \leq x \leq b \ \& \ (x, y) \in E \implies \exists c, d \in X$ such that $c \leq y \leq d$ and $(a, c) \in E, (b, d) \in E.$

Any congruence E on a qmod X (over a unitary ring R) produces the quotient set $X/E := \{[x] : x \in X\}$, where $[x]$ denotes the equivalence class containing x (with respect to E) i.e., $[x] := \{y \in X : (x, y) \in E\}$. We now make this quotient set a quasi module by defining operations and partial order suitably.

Theorem 1.10 [4]. *For any congruence E on a qmod X over a unitary ring R , X/E becomes a qmod over R with respect to the following operations and partial order.*

- (i) $[x] + [y] := [x + y], \forall [x], [y] \in X/E,$
- (ii) $r[x] := [rx], \forall [x] \in X/E, \forall r \in R,$
- (iii) $[x] \preceq [y] \iff$ for any $x' \in [x], \exists y' \in [y]$ such that $x' \leq y'$ and for any $y'' \in [y], \exists x'' \in [x]$ such that $x'' \leq y''.$

Proposition 1.11 [2]. *If $\phi : X \rightarrow Y$ (X, Y being two qmods over an unitary ring R) be an order-morphism then $\ker \phi$ is a congruence on X .*

We now give a quotient structure on X using the above congruence. For this let us construct the quotient set $X/\ker \phi := \{[x] : x \in X\}$, where $[x]$ is the equivalence class containing x obtained by the congruence $\ker \phi$. We define addition, ring multiplication and partial order on $X/\ker \phi$ as follows. For $x, y \in X$ and $r \in R$,

- (i) $[x] + [y] := [x + y];$ (ii) $r[x] := [rx];$ (iii) $[x] \leq [y]$ if and only if $\phi(x) \leq \phi(y).$

Theorem 1.12 [2]. *If $\phi : X \rightarrow Y$ (X, Y being two qmods over an unitary ring R) be an order-morphism then $X/\ker \phi$ is a quasi module over R .*

Proposition 1.13 [2]. *Let $\phi : X \rightarrow Y$ (X, Y being two qmods over an unitary ring R) be an order-morphism. Then the canonical map $\pi : X \rightarrow X/\ker \phi$ defined by $\pi(x) := [x], \forall x \in X$ is an order-epimorphism.*

Lemma 1.14 [2]. *Let X, Y, Z be three quasi modules over the unitary ring R , $\alpha : X \rightarrow Y$ be an order-epimorphism and $\beta : X \rightarrow Z$ be an order-morphism such that $\ker \alpha \subseteq \ker \beta$. Then \exists a unique order-morphism $\gamma : Y \rightarrow Z$ such that $\gamma \circ \alpha = \beta$.*

$$\begin{array}{ccc}
 X & \xrightarrow{\beta} & Z \\
 \alpha \downarrow & \nearrow \gamma & \\
 Y & &
 \end{array}$$

2. SECOND ORDER-ISOMORPHISM THEOREM

In this section we shall use the concept of congruence from the above section to prove the Second order-isomorphism theorem. For this we need three quasi modules over a common unitary ring and two order-morphisms between them.

Let X, Y, Z be three quasi modules over an unitary ring R and $\phi_1 : X \rightarrow Y$, $\phi_2 : X \rightarrow Z$ be two order-morphisms such that $\ker \phi_2 \subseteq \ker \phi_1$. So if $[x]_1 := \{y \in X : (x, y) \in \ker \phi_1\}$ and $[x]_2 := \{y \in X : (x, y) \in \ker \phi_2\}$ denote the equivalence classes containing x with respect to the congruences $\ker \phi_1$ and $\ker \phi_2$ respectively then we must have

$$[x]_1 = \bigcup \{[y]_2 : (y, x) \in \ker \phi_1\}.$$

Also $\ker \phi_2$ being a congruence on X , $X/\ker \phi_2$ is a quasi module over R (by Theorem 1.10). It is thus natural to define a relation on $X/\ker \phi_2$ as follows:

$$\ker \phi_1 / \ker \phi_2 := \left\{ ([x]_2, [y]_2) \in X/\ker \phi_2 \times X/\ker \phi_2 : (x, y) \in \ker \phi_1 \right\}.$$

Now the question is whether $\ker \phi_1 / \ker \phi_2$ is a congruence on $X/\ker \phi_2$ and if so, whether it generates a quotient qmod from $X/\ker \phi_2$ which is order-isomorphic to $X/\ker \phi_1$. We shall give answers to these in affirmative.

Proposition 2.1. $\ker \phi_1 / \ker \phi_2$ is a congruence on $X/\ker \phi_2$.

Proof. For convenience let us denote $\Gamma \equiv \ker \phi_1 / \ker \phi_2$. Now $\ker \phi_1$ being an equivalence relation it follows that Γ is also an equivalence relation. To show that Γ is a congruence let $([x]_2, [y]_2) \in \Gamma$ and $[z]_2 \in X/\ker \phi_2$, $r \in R$. Then $(x, y) \in \ker \phi_1$. So $\ker \phi_1$ being a congruence we have

$$(i) (x+z, y+z) \in \ker \phi_1 \Rightarrow ([x+z]_2, [y+z]_2) \in \Gamma \Rightarrow ([x]_2 + [z]_2, [y]_2 + [z]_2) \in \Gamma.$$

$$(ii) (rx, ry) \in \ker \phi_1 \Rightarrow ([rx]_2, [ry]_2) \in \Gamma \Rightarrow (r[x]_2, r[y]_2) \in \Gamma.$$

(iii) Now let $[x]_2 \preccurlyeq [y]_2 \preccurlyeq [z]_2$ and $([x]_2, [z]_2) \in \Gamma$. Then $(x, z) \in \ker \phi_1$ and $x \leq y' \leq z'$ for some $y' \in [y]_2$, $z' \in [z]_2$. So $(z', z) \in \ker \phi_2 \subseteq \ker \phi_1 \Rightarrow (x, z') \in \ker \phi_1 \Rightarrow (x, y') \in \ker \phi_1$ [$\because \ker \phi_1$ is a congruence] $\Rightarrow ([x]_2, [y]_2) = ([x]_2, [y']_2) \in \Gamma$.

(iv) Next let $[a]_2 \preccurlyeq [x]_2 \preccurlyeq [b]_2$ and $([x]_2, [y]_2) \in \Gamma$. Then $(x, y) \in \ker \phi_1$ and $a \leq x' \leq b'$ for some $x' \in [x]_2$, $b' \in [b]_2$. Now $(x', x) \in \ker \phi_2 \subseteq \ker \phi_1 \Rightarrow (x', y) \in \ker \phi_1 \Rightarrow \exists c, d \in X$ such that $c \leq y \leq d$ and $(a, c), (b', d) \in \ker \phi_1$ [$\because \ker \phi_1$ is a congruence] $\Rightarrow ([a]_2, [c]_2), ([b']_2, [d]_2) \in \Gamma$. Again $b' \in [b]_2 \Rightarrow [b']_2 = [b]_2$. Thus $([b]_2, [d]_2) \in \Gamma$.

We now show that $[c]_2 \preccurlyeq [y]_2 \preccurlyeq [d]_2$. For this let $c' \in [c]_2$ be arbitrary. Then $\ker \phi_2$ being a congruence, $c \leq y \Rightarrow \exists y' \in [y]_2$ such that $c' \leq y'$ (by axiom (iv) in the definition of congruence 1.9). Again by the same argument, for any $y'' \in [y]_2$, $\exists c'' \in [c]_2$ such that $c'' \leq y''$. This implies $[c]_2 \preccurlyeq [y]_2$. Similarly, we can show that $[y]_2 \preccurlyeq [d]_2$, since $y \leq d$.

This justifies that Γ is a congruence on $X/\ker\phi_2$. \blacksquare

Theorem 2.2 (Second Order-isomorphism Theorem). *Let X, Y, Z be three quasi modules over an unitary ring R and $\phi_1 : X \rightarrow Y$, $\phi_2 : X \rightarrow Z$ be two order-morphisms such that $\ker\phi_2 \subseteq \ker\phi_1$. Then the quotient $\text{qmod } \frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2}$ is order-isomorphic to $X/\ker\phi_1$.*

Proof. $\ker\phi_1/\ker\phi_2$ being a congruence on $X/\ker\phi_2$ by Proposition 2.1, we have by Theorem 1.10 that $\frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2}$ is a quasi module over R . If

$$\left. \begin{array}{l} \pi_1 : X \rightarrow X/\ker\phi_1 \\ x \mapsto [x]_1 \end{array} \right\} \text{ and } \left. \begin{array}{l} \pi_2 : X \rightarrow X/\ker\phi_2 \\ x \mapsto [x]_2 \end{array} \right\}$$

are the canonical order-epimorphisms then $\ker\pi_2 = \{(x, y) \in X \times X : [x]_2 = [y]_2\} = \ker\phi_2 \subseteq \ker\phi_1 = \ker\pi_1$. So by Lemma 1.14, \exists a unique order-morphism $\gamma : X/\ker\phi_2 \rightarrow X/\ker\phi_1$ such that $\gamma \circ \pi_2 = \pi_1$.

Now let $\pi : X/\ker\phi_2 \rightarrow \frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2}$ be the canonical order-epimorphism.

Then $\ker\pi = \ker\phi_1/\ker\phi_2$

$$\begin{aligned} &= \left\{ ([x]_2, [y]_2) \in X/\ker\phi_2 \times X/\ker\phi_2 : (x, y) \in \ker\phi_1 \right\} \\ &= \left\{ (\pi_2(x), \pi_2(y)) : \pi_1(x) = \pi_1(y) \right\} \\ &= \left\{ (\pi_2(x), \pi_2(y)) : \gamma \circ \pi_2(x) = \gamma \circ \pi_2(y) \right\} \\ &= \ker\gamma. \end{aligned}$$

So by Lemma 1.14, \exists a unique order-morphism $\Psi : \frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2} \rightarrow X/\ker\phi_1$ such that $\Psi \circ \pi = \gamma$. So $\Psi \circ \pi \circ \pi_2 = \gamma \circ \pi_2 = \pi_1$. The following commutative diagram clarifies this.

$$\begin{array}{ccc} X & \xrightarrow{\pi_1} & X/\ker\phi_1 \\ \pi_2 \downarrow & \nearrow \gamma & \uparrow \Psi \\ X/\ker\phi_2 & \xrightarrow{\pi} & \frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2} \end{array}$$

π_1 being onto it follows from above diagram that Ψ is onto. To prove that Ψ is injective let $\Psi \circ \pi \circ \pi_2(x) = \Psi \circ \pi \circ \pi_2(y)$ for some $x, y \in X$. Then $\gamma \circ \pi_2(x) = \gamma \circ \pi_2(y) \Rightarrow (\pi_2(x), \pi_2(y)) \in \ker\gamma = \ker\pi \Rightarrow \pi \circ \pi_2(x) = \pi \circ \pi_2(y)$. This justifies that Ψ is injective. Consequently, Ψ is an order-isomorphism. \blacksquare

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