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COMMUTATIVITY OF PRIME RINGS WITH SYMMETRIC BIDERIVATIONS

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Abstract

The present paper shows some results on the commutativity of R: Let R be a prime ring and for any nonzero ideal I of R, if R admits a biderivation B such that it satisfies any one of the following properties (i) $B([x,y],z) = [x,y]$, (ii) $B([x,y],m) + [x,y] = 0$, (iii) $B(xoy,z) = xoy$, (iv) $B(xoy,z) + xoy = 0$, (v) $B(x,y)oB(y,z) = 0$, (vi)$B(x,y)oB(y,z) = xoz$, (vii) $B(x,y)oB(y,z) + xoy = 0$, for all $x,y,z \in R$, then R is a commutative ring.

Keywords: prime ring, biderivation, commutativity and ideals.

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1. Introduction

In this paper, we study the relationship between the action of a prime ring R and the behaviour of some biadditive mappings defined on R. In all that follows unless specifically stated otherwise, R will be an associative ring with center $Z(R)$, for all $x, y \in R$. The symbols $[x,y]$ and xoy render the commutator $(xy - yx)$ and the anticommutator $(xy + yx)$ respectively. We use the following basic identities without any specific mention in the entire paper:

$$\begin{aligned}[xy, z] &= x[y, z] + [x, z]y, [x, yz] = y[x, z] + [x, y]z \\ xo(yz) &= (xoy)z - y[x, z] = y(xoz) + [x, y]z \\ (xy)oz &= x(yoz) - [x, z]y = (xoz)y + x[y, z].\end{aligned}$$

Recall that R is prime if $aRb = (0)$, implies either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$, implies $a = 0$. An additive mapping $d : R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$, holds for all pairs $x, y \in R$. Herstein [6] proved that a prime ring of characteristic not two with a nonzero derivation satisfying $d(x)d(y) = d(y)d(x)$, for all $x, y \in R$ must be commutative. Bell and Daif [4] showed that a prime ring of arbitrary characteristic with nonzero derivation d satisfying $d(xy) = d(yx)$, for all $x, y \in R$ must be commutative. Derivations with certain properties were investigated in various papers (see [2, 3, 10], and [7]). Further, Ashraf and Rehman [1] investigated the commutativity of R satisfying any one of the properties (i) $d([x, y]) = [x, y]$, (ii) $d(xoy) = xoy$, (iii) $d(x)od(y) = 0$, (iv) $d(x)od(y) = xoy$, for all $x, y \in R$. A mapping $B(.,.) : R \times R \to R$ is said to be symmetric if $B(x, y) = B(y, x)$ holds for all pairs $x, y \in R$. A symmetric biadditive mapping $B(.,.) : R\times R \to R$ is called a symmetric biderivation if either $B(xy, z) = B(x, z)y + xB(y, z)$ or $B(x, yz) = B(x, y)z + yB(x, z)$ is fulfilled for all $x, y, z \in R$. The concept of symmetric biderivation has been introduced by Maksa [8]. A mapping $f : R \to R$ is said to be commuting on R if $[f(x), x] = 0$ holds for all $x \in R$. Currently, many researchers are focussed to analyze the concept of commutings, related mappings with symmetric biderivations on a prime and semiprime rings (see [11, 12], and [13]). Motivated by these work of Ashraf [1] on derivations, we investigated the commutativity of biadditive mappings on R i.e., if R admits a biderivation B such that it satisfies any one of the following (i) $B([x, y], z) = [x, y]$, (ii) $B([x, y], m) + [x, y] = 0$, (iii) $B(xoy, z) = xoy$, (iv) $B(xoy, z) + xoy = 0$, (v) $B(x, y)oB(y, z) = 0$, (vi) $B(x, y)oB(y, z) = xoz$, (vii) $B(x, y)oB(y, z) + xoy = 0$, for all $x, y, z \in R$, then R is a commutative ring.

To prove the main Theorems we need the following lemmas.

Lemma 1 ([5], Theorem 4)**.** *Let R be a prime ring and I a nonzero left ideal of R. If R admits a nonzero biderivation B such that $[x, B(x, y)]$ is central for all $x, y \in I$, then R is commutative.*

Lemma 2 ([9], Lemma 3)**.** *If a prime ring R contains a nonzero commutative right ideal, then R is commutative.*

Lemma 3. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B^2(x, y) = 0$, for all $x, y \in I$, then $B(x, y) = 0$.*

Proof. We have $B^2(x,y) = 0$, for all $x, y \in I$. Put yz instead of y, for any $z \in I$ to get $0 = B(x,y)B(z,m) + B(y,m)B(x,z)$, for any $m \in I$. Put zr instead of z, for any $r \in I$ and then put $m = x$ to get $B(x,y)zB(x,y) = 0$, then $B(x,y)IRB(x,y) = 0$, by the primeness of R forces that either $B(x,y) = 0$ or $B(x,y)I = 0$. If $B(x,y)I = 0$, for all $x, y \in I$, then $B(x,y)RI = 0$. Because of $I \neq 0$, we find that $B(x,y) = 0$ for all $x, y \in I$. In both the cases $B(x,y) = 0$. ■

Theorem 1. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B([x,y],z) = [x,y]$, for all $x, y, z \in I$, then R is commutative.*

Proof. We have B is a biderivation such that, $B([x,y],z) = [x,y]$, for any $x, y, z \in I$. If $B = 0$, then $[x,y] = 0$, which implies that $[x,y]r = 0$, when the substitution of y by yr, for any $r \in I$. That is $[x,y]I = 0$, since I is non zero, obviously R is commutative using Lemma 2. Now consider a nonzero symmetric biderivation B such that $B([x,y],z) = [x,y]$ and use the commutator identity to get the equation

$$B(x,z)y + xB(y,z) - B(y,z)x - yB(x,z) = [x,y]$$

put ym instead of y, for any $m \in I$ in the above equation, we get

$$\begin{aligned} &B(y,z)xm + yB(x,z)m + xyB(m,z) - ymB(x,z) \\ &-B(y,z)mx - yB(m,z)x = y[x,m] \end{aligned}$$

again put x instead of m in the above equation, we find that $[x,y]B(x,z) = 0$. Again put ym instead of y in the above equation, we get $[x,y]IRB(x,z) = 0$. Thus the primeness of R forces either $[x,y]I = 0$ or $B(x,z) = 0$, since our assumption that B is a non zero biderivation, therefore $[x,y]I = 0$. So using Lemma 2, R is commutative. ■

Theorem 2. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B([x,y],m) + [x,y] = 0$, for all $x, y, z \in I$, then R is commutative.*

Proof. We have B is a biderivation such that, $B([x,y],m) + [x,y] = 0$, for all $x, y \in I$. If $B = 0$, then our result is obvious as in the proof of Theorem 4. So assume that a non zero biderivation and use the basic identity to get

$$B(x,m)y + xB(y,m) - B(y,m)x - yB(x,m) + [x,y] = 0.$$

Replace y by yz, for any $z \in I$ in the above equation, we get $B(y,m)[x,z] + [x,y]B(z,m) = 0$. Again replace z by x in the above equation to get $[x,y]B(x,m) = 0$, then replace y by yz for $z \in I$ in the above equation, we find that

$[x, y]IRB(x, m) = 0$. Thus the primeness of R forces either $[x, y]I = 0$ or $B(x, m) = 0$, since our assumption that B is a non zero biderivation, therefore $[x, y]I = 0$. So using Lemma 2, R is commutative. ■

Theorem 3. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B(xoy, z) = xoy$, for all $x, y, z \in I$, then R is commutative.*

Proof. For any $x, y, z \in I$, we have $B(xoy, z) = xoy$. If $B = 0$, then $xoy = 0$, for all $x, y \in I$. Put yz instead of y in above equation and using the identity to get $(xoy)z - y[x, z] = 0$ which implies $y[x, z] = 0$, for any $y \in I$ then $IR[x, z] = 0$. Since I is nonzero and by the primeness of R, $[x, z] = 0$. Hence by Lemma 2, R is commutative. Hence onwards we assume that $B \neq 0$ for any $x, y, z \in I$, $B(xoy, z) = xoy$. Using the identity it can be rewritten as $B(x, z)oy + xoB(y, z) = xoy$. Put yz instead of y in above equation, we get $(B(x, z)oy + xoB(y, z))x + (xoy)B(x, z) = (xoy)x$, using the above equation, to get $(xoy)B(x, z) = 0$, again put my instead of y, for any $m \in I$, in above equation to find $(m(xoy) + [x, m]y)B(x, z) = 0$, which implies $[x, m]IRB(x, z) = 0$, by primeness of R, B is nonzero and by Lemma 2, we get R is commutative. ■

Utilizing the above procedure with necessary variations we can show the following.

Theorem 4. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B(xoy, z) + xoy = 0$, for all $x, y, z \in I$, then R is commutative.*

Proof. A careful investigation of the proof of above theorem shows that a prime ring R is commutative if it satisfies the property $xoy = 0$. Thus it is natural to explore the behaviour of rings satisfying the property $B(xoy, z) + xoy = 0$, with a non zero biderivation is commutative using the same techniques with necessary variations. ■

In the present section we shall study the behaviour of the ring satisfying any one of the properties $B(x, y)oB(y, z) = 0$, $B(x, y)oB(y, z) = xoz$ and $B(x, y)oB(y, z) + xoy = 0$, for all $x, y, z \in I$.

Theorem 5. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B(x, y)oB(y, z) = 0$, for all $x, y, z \in I$, then R is commutative.*

Proof. For any $x, y, z \in I$, we have $B(x, y)oB(y, z) = 0$. If $B = 0$, then the result is trival. So assume that B is nonzero biderivation. Replacing z by zm, for any $m \in I$ in the hypothesis equation, we get this result that $[B(x, y), z]B(y, m)-$

$B(y,z)[B(x,y),m] = 0$ and again replace m by $mB(x,y)$ in above equation, we find

$$[B(x,y),z]B(y,m)B(x,y)+[B(x,y),z]mB^2(x,y)-B(y,z)[B(x,y),m]B(x,y) = 0.$$

On simplifying above equation , we have $[B(x,y),z]mB^2(x,y) = 0$. Hence $[B(x,y),z]IRB^2(x,y) = 0$. By the primeness of R forces that either $B^2(x,y) = 0$ or $[B(x,y),z]I = 0$. If $B^2(x,y) = 0$, by Lemma 3, $B(x,y) = 0$ which is a contradiction. So $[B(x,y),z]I = 0$. Since I is a nonzero and by Lemma 2, R is commutative. ∎

Theorem 6. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B(x,y)oB(y,z) = xoz$, for all $x,y,z \in I$, then R is commutative.*

Proof. For any $x,y,z \in I$, we have $B(x,y)oB(y,z) = xoz$. If $B = 0$ then $xoz = 0$. Remember that the procedure given in the beginning of the proof of Theorem 3 are still valid in the present situation and hence we get the required result. Therefore we assume that $B \neq 0$, we have $B(x,y)oB(y,z) = xoz$. Replacing z by zm, for any $m \in I$ in the above equation, we get

$$(B(x,y)oB(y,z))m - B(y,z)[B(x,y),m] + (B(x,y)oz)B(y,m)$$
$$-z[B(x,y),B(y,m)] = (xoz)m - z[x,m]$$

now by our hypothesis the above relation yields that

$$-B(y,z)[B(x,y),m] + (B(x,y)oz)B(y,m) - z[B(x,y),B(y,m)] + z[x,m] = 0$$

for any $r \in R$, replace z by rz in above equation, we get $[B(x,y),r]zB(y,m) - B(y,r)z[B(x,y),m] = 0$. Now substituting $B(x,y)$ for r in the above relation , we find that $B^2(x,y)RI[B(x,y),m] = (0)$. By the primeness of R, either $B^2(x,y) = 0$ or $I[B(x,y),m] = 0$. If $B^2(x,y) = 0$ then by Lemma 3, $B(x,y) = 0$, which is a contradiction. On the other hand $I[B(x,y),m] = (0)$, since I is a nonzero ideal of R and R is prime, the above relation yields that $[B(x,y),m] = 0$, for all $x,y,m \in I$. Hence by Lemma 1, R is commutative. ∎

Using the similar arguments we can prove the following.

Theorem 7. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B(x,y)oB(y,z) + xoz = 0$, for all $x,y,z \in I$, then R is commutative.*

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