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# FOLDING THEORY OF IMPLICATIVE AND OBSTINATE IDEALS IN BL-ALGEBRAS

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**Abstract**

In this paper, the concepts of $n$-fold implicative ideals and $n$-fold obstinate ideals in $BL$-algebras are introduced. With respect to this concepts, some related results are given. In particular, it is proved that an ideal is an $n$-fold implicative ideal if and only if is an $n$-fold Boolean ideal. Also, it is shown that a $BL$-algebra is an $n$-fold integral $BL$-algebra if and only if trivial ideal $\{0\}$ is an $n$-fold obstinate ideal. Moreover, the relation between $n$-fold obstinate ideals and $n$-fold (integral) obstinate filters in $BL$-algebras are studied by using the set of complement elements. Finally, it is proved that ideal $I$ of $BL$-algebra $L$ is an $n$-fold obstinate ideal if and only if $\frac{L}{I}$ is an $n$-fold obstinate $BL$-algebra.

**Keywords:** $BL$-algebra, ideal, $n$-fold implicative ideal, $n$-fold obstinate ideal.

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## 1. Introduction

$BL$-algebras are the algebraic structure for Hájek basic logic [7] in order to investigate many valued logic by algebraic means. His motivations for introducing $BL$-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic ($BL$ for short) is proposed as "the most general"many-valued logic with truth values in [0, 1] and $BL$-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0, 1]. In 1958, Chang [1] introduced the concept of an $MV$-algebra which is one of the most classes of $BL$-algebras. Turunen [12] introduced the notion of an implicative filter and a Boolean filter in $BL$-algebras. Boolean filters are an important class of filters, because the quotient $BL$-algebra induced by these filters are Boolean algebras. The notion of (fuzzy) ideal has been introduced in many algebraic structures such as lattices, rings, $MV$-algebras. Ideal theory is very effective tool for studying various algebraic and logical systems. In the theory of $MV$-algebras, as various algebr aic structures, the notion of ideal is at the center, while in $BL$-algebras, the focus has been on deductive systems also filters. The study of $BL$-algebras has experienced a tremendous growth over resent years and the main focus has been on filters. In 2013, Lele [6], introduced the notions of (Boolean, prime) ideals and analyzed the relationship between ideals and filters by using the set of complement elements. In 2017, Yang and Xin [11], introduced implicative ideals in BL-algebras and studied some characterizations of them by the pseudo implication operation and proved the implicative ideals coincide with Boolean ideals in $BL$-algebras.

This motivates us to introduce the notions of $n$-fold implicative and $n$-fold obstinate ideals in $BL$-algebras and investigate the relations among $n$-fold implicative ideals, $n$-fold obstinate ideals and the other ideals in $BL$-algebras. In particular, we prove that an ideal is an $n$-fold implicative ideal if and only if is an $n$-fold Boolean. Also, we prove that a $BL$-algebra is an $n$-fold integral $BL$-algebra if and only if trivial ideal $\{0\}$ is an $n$-fold obstinate ideal. Moreover, we study rela tion between $n$-fold obstinate ideals and $n$-fold (integral) obstinate filters in $BL$-algebras by using the set of complement elements. Finally, we prove that ideal $I$ of $BL$-algebra $L$ is an $n$-fold obstinate ideal if and only if $\frac{L}{I}$ is an $n$-fold obstinate $BL$-algebra.

## 2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, refer to the references.

**Definition** [7]**.** A $BL$-algebra is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

($BL1$) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

($BL2$) $(L, \odot, 1)$ is a commutative monoid,

($BL3$) $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,

($BL4$) $x \wedge y = x \odot (x \rightarrow y)$,

($BL5$) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

We denote $x^n = \overbrace{x \odot \cdots \odot x}^{n-times}$, if $n > 0$ and $x^0 = 1$, for all $x, y \in L$.

A $BL$-algebra $L$ is called a Gödel algebra (1-fold implicative $BL$-algebra) if $x^2 = x \odot x = x$, for all $x \in L$ and $L$ is called an $MV$-algebra if $(x^-)^- = x$, for all $x \in L$, where $x^- = x \to 0$. A $BL$-algebra $L$ is called a Boolean algebra if $x \vee x^- = 1$, for all $x \in L$.

**Proposition 1** [2, 3]**.** *In any BL-algebra the following hold:*

($BL6$) $x \leq y$ *if and only if* $x \to y = 1$,

($BL7$) $y \leq x \to y$, *and* $x \odot y \leq x, y$,

($BL8$) $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$,

($BL9$) $(x \to y)^{--} = x^{--} \to y^{--}$,

($BL10$) $(x \odot y)^{--} = x^{--} \odot y^{--}$,

($BL11$) $(x \odot y)^- = x \to y^-$,

($BL12$) $x^{---} = x^-$, $x \leq x^{--}$ *and* $x \odot x^- = 0$,

($BL13$) $x \to (y \to z) = (x \odot y) \to z$,

($BL14$) $x \leq y$ *implies* $y^- \leq x^-$,

($BL15$) $x \leq y$ *implies* $z \odot x \leq z \odot y$,

($BL16$) $(x \wedge y)^{--} = x^{--} \wedge y^{--}$, *for all* $x, y, z \in L$.

Note that by ($BL13$) $(\overbrace{x \to (\cdots (x \to (x}^{n-times} \to y))) \cdots ) = x^n \to y$, for all $x, y \in L$. The following theorems and definitions are from [4, 5, 8, 10] and we refer the reader to them, for more details.

**Definition.** Let $L$ be a $BL$-algebra, $n$ be a natural number and $F$ be a nonempty subset of $L$. Then

(i) $F$ is called a *filter* of $L$ if $x \odot y \in F$, for any $x, y \in F$ and if $x \in F$ and $x \leq y$ then $y \in F$, for all $x, y \in L$. A proper filter $F$ is called a *maximal filter* of $L$ if it is not properly contained in any other proper filter of $L$.

(ii) $F$ is called an *n-fold implicative filter* of $L$ if $1 \in F$ and for all $x, y, z \in L$,

$$x^n \to (y \to z) \in F \text{ and } x^n \to y \in F \text{ imply } x^n \to z \in F.$$

(iii) A proper filter $F$ is called an *n-fold obstinate filter* if for all $x, y \in L$,

$$x, y \notin F \text{ imply } x^n \to y \in F \text{ and } y^n \to x \in F.$$

(iv) A proper filter $F$ is called an *n-fold integral filter* if for all $x, y \in L$,

$$(x^n \odot y^n)^- \in F \text{ implies } (x^n)^- \in F \text{ or } (y^n)^- \in F.$$

**Definition** [10]**.** Let $L$ be a $BL$-algebra and $n$ be a natural number. Then

(i) $L$ is called an *$n$-fold integral $BL$-algebra* if for all $x, y \in L$

$$x^n \odot y^n = 0 \text{ then } x^n = 0 \text{ or } y^n = 0.$$

(ii) $L$ is called an $n$-fold obstinate $BL$-algebra if $L$ is an $MV$-algebra and $x^n = 0$, for all $x \in L \backslash \{1\}$.

**Definition** [6, 8, 9]**.** Let $L$ be a $BL$-algebra and $I$ be a nonempty subset of $L$. Then

(i) $I$ is called an *ideal* of $L$, if $x \oslash y := x^- \to y \in I$, for any $x, y \in I$ and if $y \in I$ and $x \leq y$ then $x \in I$, for all $x, y \in L$. The operation $\oslash$ is associative. Moreover, a set $I$ containing 0 of $L$ is an ideal if and only if for all $x, y \in L$, $x^- \odot y \in I$ and $x \in I$ imply $y \in I$.

(ii) A proper ideal $I$ of $L$ is called a *prime ideal* of $L$ if $x \wedge y \in I$ implies $x \in I$ or $y \in I$, for all $x, y \in L$.

(iii) A proper ideal $I$ is called a *maximal ideal* of $L$ if it is not properly contained in any other proper ideal of $L$.

(iv) An ideal $I$ of $L$ is called a *$n$-fold Boolean ideal* if $x^n \wedge (x^n)^- \in I$, for all $x \in L$ and an ideal $I$ of $L$ is called a *Boolean ideal* if $x \wedge x^- \in I$, for all $x \in L$.

(v) An ideal $I$ of $L$ is called an *$n$-fold integral ideal*, if for all $x, y \in L$,

$$(x \odot y)^n \in I \text{ implies } x^n \in I \text{ or } y^n \in I.$$

Let $L$ be a $BL$-algebra, we define the pseudo implication operation $\rightharpoonup$ by $x \rightharpoonup y := x \odot y^-$, for any $x, y \in L$. It is easy to see that $z \leq x \oslash y$ if and only if $z \rightharpoonup x \leq y$.

Moreover, we denote $x^n_{\oslash} = \overbrace{x \oslash \cdots \oslash x}^{n-times}$, when $n$ is a natural number.

**Lemma 2** [11]**.** *Let $L$ be a $BL$-algebra, for any $x, y, z \in L$, we have:*

(i) $x \leq y$ *implies* $z \rightharpoonup y \leq z \rightharpoonup x$ *and* $x \rightharpoonup z \leq y \rightharpoonup z$,

(ii) $(x \rightharpoonup y) \rightharpoonup z = (x \rightharpoonup z) \rightharpoonup y = x \rightharpoonup (y \oslash z)$,

(iii) $x \rightharpoonup 0 = x$, $0 \rightharpoonup x = 0$, $x \rightharpoonup x = 0$,

(iv) $(x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \leq x \rightharpoonup y$,

(v) $(x \rightharpoonup z) \leq (y \rightharpoonup z) \oslash (x \rightharpoonup y)$,

(vi) $x \leq x \oslash x$.

**Lemma 3** [11]**.** *Let $I$ be a nonempty subset of a $BL$-algebra $L$. Then $I$ is an ideal of $L$ if and only if it satisfies:*

(i) $0 \in I$,

(ii) *for any* $x, y \in L$, *if* $x \rightharpoonup y \in I$ *and* $y \in I$, *then* $x \in I$.

**Lemma 4** [11]**.** *Let* $I$ *be an ideal of* $BL$*-algebra* $L$*. Then the following hold: for any* $x, y, z \in L$

(i) $x \rightharpoonup y \in I$ *if and only if* $y^- \rightharpoonup x^- \in I$.

(ii) $x \rightharpoonup y \in I$ *if and only if* $x^{--} \rightharpoonup y \in I$.

(iii) $(y \rightharpoonup x^-) \rightharpoonup z \in I$ *if and only if* $(z^- \rightharpoonup y^-) \rightharpoonup x^- \in I$.

(iv) $x \in I$ *if and only if* $x^{--} \in I$.

**Theorem 5** [11]**.** *Let* $P$ *be a proper ideal of* $BL$*-algebra* $L$*. Then* $P$ *is a prime ideal if and only if* $x \rightharpoonup y \in P$ *or* $y \rightharpoonup x \in P$*, for all* $x, y \in L$.

**Definition** [6]**.** Let $L$ be a $BL$-algebra and $X$ any subset of $L$. Then the set of complement elements (with respect to $X$) is denoted by $N(X)$ and is defined by

$$N(X) = \{x \in L \mid x^- \in X\}.$$

**Theorem 6** [6]**.** *Let* $I$ *be an ideal of* $BL$*-algebra* $L$*. Then the binary relation* $\equiv_I$ *on* $L$ *which is defined by*

$$x \equiv_I y \quad \text{if and only if } x^- \odot y \in I \quad \text{and } y^- \odot x \in I$$

*is a congruence relation on* $L$*. Define* $\cdot$, $\rightharpoonup$, $\sqcup$, $\sqcap$ *on* $\frac{L}{I}$ *, the set of all congruence classes of* $L$*, as follows:*

$$[x] \cdot [y] = [x \odot y], \;\; [x] \rightharpoonup [y] = [x \to y]$$

$$[x] \sqcup [y] = [x \vee y], \;\; [x] \sqcap [y] = [x \wedge y].$$

*Then* $\left(\frac{L}{I}, \cdot, \rightharpoonup, \sqcup, \sqcap, [0], [1]\right)$ *is a* $BL$*-algebra which is called quotient* $BL$*-algebra with respect to* $I$*. In addition, it is clear* $[x]^{--} = [x]$*, for all* $x \in L$*. Consequently, the quotient* $BL$*-algebra via any ideal is always an* $MV$*-algebra.*

**Theorem 7** [9]**.** *Let* $I$ *be an ideal of* $L$*. Then the following conditions are equivalent:*

(i) $I$ *is an* $n$*-fold integral ideal of* $L$,

(ii) $I$ *is a maximal and* $n$*-fold Boolean ideal of* $L$,

(iii) $I$ *is a prime and* $n$*-fold Boolean ideal of* $L$,

(iv) $I$ *is a proper ideal and for all* $x \in L$, $x^n \in I$ *or* $(x^n)^- \in I$.

**Theorem 8** [9]**.** *Let* $I$ *be an ideal of* $L$*. Then* $I$ *is an* $n$*-fold integral ideal if and only if* $N(I)$ *is an* $n$*-fold obstinate filter of* $L$.

**Theorem 9** [9]**.** *Let $F$ be a proper filter of $L$. Then $F$ is an $n$-fold integral filter if and only if $N(F)$ is an $n$-fold integral ideal of $L$.*

**Theorem 10** [9]**.** *In any $BL$-algebra $L$, the following conditions are equivalent:*

(i) $\{0\}$ *is an $n$-fold integral ideal of $L$,*

(ii) *any ideal of $L$ is an $n$-fold integral ideal,*

(iii) *$L$ is an $n$-fold integral $BL$-algebra.*

**Theorem 11** [9]**.** *Let $I$ be an ideal of $L$. Then $I$ is an $n$-fold integral ideal of $L$ if and only if $\frac{L}{I}$ is an $n$-fold obstinate $BL$-algebra.*

**Theorem 12** [9]**.** *Let $L$ be a Boolean algebra or a Gödel algebra. Then every ideal of $L$ is implicative.*

From now on, in this paper $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ (or simply) $L$ is a $BL$-algebra, unless otherwise stated.

## 3. n-fold implicative ideals in BL-algebras

In this section we introduce two new class of ideals in $BL$-algebras that called $n$-fold implicative ideals and we give some related results.

**Definition.** A nonempty subset $I$ of $L$ is called an $n$-fold implicative ideal if it satisfies:

(i) $0 \in I$,

(ii) $(x \rightharpoonup y) \rightharpoonup z^n_{\oslash} \in I$ and $y \rightharpoonup z^n_{\oslash} \in I$ imply $x \rightharpoonup z^n_{\oslash} \in I$, for all $x, y, z \in L$.

An 1-fold implicative ideal is called an implicative ideal of $L$.

**Example 13** [6]**.** Let $L = \{0, a, b, c, d, e, f, 1\}$ be such that $0 < a < b < c < 1$, $0 < d < e < f < 1$, $a < e$ and $b < f$. Define $\odot$ and $\rightarrow$ as follows:

Table 1

| $\odot$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | $a$ | $a$ | $a$ | 0 | $a$ | $a$ | $a$ |
| $b$ | 0 | $a$ | $a$ | $b$ | 0 | $a$ | $a$ | $b$ |
| $c$ | 0 | $a$ | $b$ | $c$ | 0 | $a$ | $b$ | $c$ |
| $d$ | 0 | 0 | 0 | 0 | $d$ | $d$ | $d$ | $d$ |
| $e$ | 0 | $a$ | $a$ | $a$ | $d$ | $e$ | $e$ | $e$ |
| $f$ | 0 | $a$ | $a$ | $b$ | $d$ | $e$ | $e$ | $f$ |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |

Table 2

| $\to$ | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $a$ | $d$ | 1 | 1 | 1 | $d$ | 1 | 1 | 1 |
| $b$ | $d$ | $f$ | 1 | 1 | $d$ | $f$ | 1 | 1 |
| $c$ | $d$ | $e$ | $f$ | 1 | $d$ | $e$ | $f$ | 1 |
| $d$ | $c$ | $c$ | $c$ | $c$ | 1 | 1 | 1 | 1 |
| $e$ | 0 | $c$ | $c$ | $c$ | $d$ | 1 | 1 | 1 |
| $f$ | 0 | $b$ | $c$ | $c$ | $d$ | $f$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | 1 |

Then $(L, \wedge, \vee, \odot, \to, 0, 1)$ is a $BL$-algebra. Let $I = \{0, d\}$. Then $I$ is a 2-fold implicative ideal of $L$.

**Proposition 14.** *Let $I$ be an $n$-fold implicative ideal of $L$. Then $I$ is an ideal of $L$.*

***Proof.*** Suppose that $I$ is an $n$-fold implicative ideal of $L$ and $x, y \in L$. If $x \rightharpoonup y \in I$ and $y \in I$, then $(x \rightharpoonup y) \rightharpoonup 0^n_\oslash = x \rightharpoonup y \in I$ and $y \rightharpoonup 0^n_\oslash = y \in I$. By hypothesis $x = x \rightharpoonup 0^n_\oslash \in I$, hence $I$ is an ideal of $L$. ■

The following example shows that the converse of Proposition 14, does not hold in general.

**Example 15** [6]**.** Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\to$ are defined as the following tables:

Table 3

| $\odot$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | $a$ |
| $b$ | 0 | 0 | $a$ | $b$ |
| 1 | 0 | $a$ | $b$ | 1 |

Table 4

| $\to$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | 1 | 1 | 1 |
| $b$ | $a$ | $b$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | 1 |

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a $BL$-algebra. Now, let $I = \{0\}$. Then $I$ is an ideal of $L$ and since $(1 \rightharpoonup b) \rightharpoonup b = b^- \odot b^- = a \odot a = 0 \in I$, $b \rightharpoonup b = b \odot b^- = b \odot a = 0 \in I$ and $1 \rightharpoonup b = 1 \odot b^- = a \notin I$, then $I$ is not a 1-fold implicative ideal of $L$.

**Theorem 16.** *Let $I$ be an ideal of $L$. Then the following conditions are equivalent:*

(i) *$I$ is an $n$-fold implicative ideal of $L$,*

(ii) *for any $a \in L$, the set $I_{a^n_\oslash} := \{x \in L \mid x \rightharpoonup a^n_\oslash \in I\}$ is an ideal of $L$.*

***Proof.*** (i) $\Rightarrow$ (ii) Suppose that $I$ is an $n$-fold implicative ideal of $L$ and $a \in L$. For any $x, y \in L$, if $x \rightharpoonup y \in I_{a^n_\oslash}$ and $y \in I_{a^n_\oslash}$, then $(x \rightharpoonup y) \rightharpoonup a^n_\oslash \in I$ and $y \rightharpoonup a^n_\oslash \in I$, hence $x \rightharpoonup a^n_\oslash \in I$, and so $x \in I_{a^n_\oslash}$. Moreover, since $0 \rightharpoonup a^n_\oslash = 0 \odot (a^n_\oslash)^- = 0 \in I$, we obtain $0 \in I_{a^n_\oslash}$. Therefore, $I_{a^n_\oslash}$ is an ideal of $L$.

(ii) $\Rightarrow$ (i) Suppose that $I_{a^n_\oslash}$ is an ideal of $L$, for any $a \in L$. For any $x, y, z \in L$, if $(x \rightharpoonup y) \rightharpoonup z^n_\oslash \in I$ and $y \rightharpoonup, z^n_\oslash \in I$, then $x \rightharpoonup y \in I_{z^n_\oslash}$ and $y \in I_{z^n_\oslash}$. Now, sinc e $I_{z^n_\oslash}$ is an ideal of $L$, we have $x \in I_{z^n_\oslash}$, and so $x \rightharpoonup z^n_\oslash \in I$. Therefore, $I$ is an $n$-fold implicative ideal of $L$. ■

**Theorem 17.** *Let $I$ be an $n$-fold implicative ideal of $L$. Then for any $a \in L$, $I_{a^n_\oslash}$ is the least ideal of $L$ containing $I$ and $a$.*

***Proof.*** Let $I$ be an $n$-fold implicative ideal of $L$ and $a \in L$. Then by Theorem 16, $I_{a^n_\oslash}$ is an ideal of $L$ and by $(BL7)$, for any $x \in I$, $x \rightharpoonup a^n_\oslash = x \odot (a^n_\oslash)^- \leq x$, we get $x \rightharpoonup a^n_\oslash \in I$, and so $x \in I_{a^n_\oslash}$. Hence $I \subseteq I_{a^n_\oslash}$. Moreover, by $(BL7)$, $(BL12)$, $(BL14)$ and $(BL15)$,

$$\begin{aligned} a \rightharpoonup a^n_\oslash &= a \rightharpoonup \left(a^{n-1}_\oslash \oslash a\right) = a \odot \left(a^{n-1}_\oslash \oslash a\right)^- \\ &= a \odot \left(\left(a^{n-1}_\oslash\right)^- \to a\right)^- \leq a \odot a^- = 0. \end{aligned}$$

Hence, $a \rightharpoonup a^n_\oslash = 0 \in I$, and so $a \in I_{a^n_\oslash}$. Now, if $J$ is an ideal of $L$ containing $I$ and $a$, then for any $x \in I_{a^n_\oslash}$, we get that $x \rightharpoonup a^n_\oslash \in I \subseteq J$. Since $J$ is an ideal of $L$ and $a \in J$, we have $a^n_\oslash = \overbrace{a \oslash \cdots \oslash a}^{n-times} \in J$ and so $x \in J$. Therefore, $I_{a^n_\oslash} \subseteq J$ and so $I_{a^n_\oslash}$ is the least ideal of $L$ containing $I$ and $a$. ■

**Theorem 18.** *Let $I$ be a nonempty subset of $L$. Then the following conditions are equivalent:*

(i) *$I$ is an $n$-fold implicative ideal of $L$,*

(ii) *$I$ is an ideal of $L$ and for any $x, y \in L$, $x \rightharpoonup y^{n+1}_\oslash \in I$ implies $x \rightharpoonup y^n_\oslash \in I$,*

(iii) *$I$ is an ideal of $L$ and for any $x, y, z \in L$, $(x \rightharpoonup y) \rightharpoonup z^n_\oslash \in I$ implies $(x \rightharpoonup z^n_\oslash) \rightharpoonup (y \rightharpoonup z^n_\oslash) \in I$,*

(iv) *$0 \in I$, and if $(x \rightharpoonup y^{n+n}_\oslash) \rightharpoonup z \in I$ and $z \in I$, then $x \rightharpoonup y^n_\oslash \in I$, for any $x, y, z \in L$.*

(v) *$0 \in I$, and if $(x \rightharpoonup y^{n+1}_\oslash) \rightharpoonup z \in I$ and $z \in I$, then $x \rightharpoonup y^n_\oslash \in I$, for any $x, y, z \in L$.*

***Proof.*** (i) $\Rightarrow$ (ii) Let $I$ be an $n$-fold implicative ideal of $L$. Then by Proposition 14, $I$ is an ideal of $L$. Now, if $x \rightharpoonup y^{n+1}_\oslash \in I$, for $x, y \in L$, then by Lemma

2(ii), $(x \rightharpoonup y) \rightharpoonup y_{\oslash}^{n} = x \rightharpoonup y_{\oslash}^{n+1} \in I$ and since by Lemma 2(ii) and (iii), $y \rightharpoonup y_{\oslash}^{n} = y \rightharpoonup y \oslash y_{\oslash}^{n-1} = (y \rightharpoonup y) \rightharpoonup y_{\oslash}^{n-1} = 0 \rightharpoonup y_{\oslash}^{n-1} = 0 \in I$, we get $x \rightharpoonup y_{\oslash}^{n} \in I$.

(ii) $\Rightarrow$ (iii) Assume that (ii) holds. Let $x, y, z \in L$ and $(x \rightharpoonup y) \rightharpoonup z_{\oslash}^{n} \in I$. By Lemma 2(i), (ii) and (iv),

$$((x \rightharpoonup (y \rightharpoonup z_{\oslash}^{n})) \rightharpoonup z_{\oslash}^{n}) \rightharpoonup z_{\oslash}^{n} = ((x \rightharpoonup z_{\oslash}^{n}) \rightharpoonup (y \rightharpoonup z_{\oslash}^{n})) \rightharpoonup z_{\oslash}^{n} \leq (x \rightharpoonup y) \rightharpoonup z_{\oslash}^{n}.$$

Then $((x \rightharpoonup (y \rightharpoonup z_{\oslash}^{n})) \rightharpoonup z_{\oslash}^{n}) \rightharpoonup z_{\oslash}^{n} \in I$, and so $(x \rightharpoonup (y \rightharpoonup z_{\oslash}^{n})) \rightharpoonup z_{\oslash}^{(n+n-1)+1} \in I$ and by hypothesis $(x \rightharpoonup (y \rightharpoonup z_{\oslash}^{n})) \rightharpoonup z_{\oslash}^{(n+n-1)} \in I$. By continuing this process we get that $(x \rightharpoonup (y \rightharpoonup z_{\oslash}^{n})) \rightharpoonup z_{\oslash}^{(n+1)} \in I$. Hence, $(x \rightharpoonup (y \rightharpoonup z_{\oslash}^{n})) \rightharpoonup z_{\oslash}^{n} \in I$. Therefore, $(x \rightharpoonup z_{\oslash}^{n}) \rightharpoonup (y \rightharpoonup z_{\oslash}^{n}) \in I$.

(iii) $\Rightarrow$ (iv) Assume that (iii) holds. Obviously, $0 \in I$. Let $(x \rightharpoonup y_{\oslash}^{n+n}) \rightharpoonup z \in I$ and $z \in I$, for $x, y, z \in L$. Since $I$ is an ideal of $L$, we have $x \rightharpoonup y_{\oslash}^{n+n} \in I$. Now, since by Lemma 2(ii), $(x \rightharpoonup y_{\oslash}^{n}) \rightharpoonup y_{\oslash}^{n} = x \rightharpoonup y_{\oslash}^{n+n} \in I$, then by (iii), $(x \rightharpoonup y_{\oslash}^{n}) \rightharpoonup (y_{\oslash}^{n} \rightharpoonup y_{\oslash}^{n}) \in I$ and since $y_{\oslash}^{n} \rightharpoonup y_{\oslash}^{n} = 0$, then $x \rightharpoonup y_{\oslash}^{n} \in I$.

(iv) $\Rightarrow$ (i) Suppose that (iv) is valid. Firstly, we show that $I$ is an ideal of $L$. For any $x, y \in L$, if $x \rightharpoonup y \in L$ and $y \in I$, then

$$\begin{aligned}
(x \rightharpoonup 0_{\oslash}^{n+n})) \rightharpoonup y &= (\cdots(x \rightharpoonup \overbrace{0) \rightharpoonup 0) \cdots \rightharpoonup 0}^{(n+n)-times})\cdots) \rightharpoonup y \\
&= (\cdots(x \rightharpoonup \overbrace{0) \rightharpoonup 0) \cdots \rightharpoonup 0}^{(2n-1)-times})\cdots) \rightharpoonup y \\
&\vdots \\
&= x \rightharpoonup y \in I.
\end{aligned}$$

And since $y \in I$, it follows that by (iv), $x = x \rightharpoonup 0_{\oslash}^{n} \in I$. Hence, $I$ is an ideal of $L$. Now, let $(x \rightharpoonup y) \rightharpoonup z_{\oslash}^{n} \in I$ and $y \rightharpoonup z_{\oslash}^{n} \in I$, for $x, y, z \in L$. Then by Lemma 2(ii) and (iv),

$$((x \rightharpoonup z_{\oslash}^{n}) \rightharpoonup z_{\oslash}^{n}) \rightharpoonup (y \rightharpoonup z_{\oslash}^{n}) \leq ((x \rightharpoonup z_{\oslash}^{n}) \rightharpoonup y = (x \rightharpoonup y) \rightharpoonup z_{\oslash}^{n}.$$

And since $(x \rightharpoonup y) \rightharpoonup z_{\oslash}^{n} \in I$, we obtain $((x \rightharpoonup z_{\oslash}^{n}) \rightharpoonup z_{\oslash}^{n}) \rightharpoonup (y \rightharpoonup z_{\oslash}^{n}) \in I$, hence $(x \rightharpoonup z_{\oslash}^{n+n}) \rightharpoonup (y \rightharpoonup z_{\oslash}^{n}) \in I$. Now, since $y \rightharpoonup z_{\oslash}^{n} \in I$, so by (iv), $x \rightharpoonup z_{\oslash}^{n} \in I$. Therefore, $I$ is an $n$-fold implicative ideal of $L$.

(iv) $\Rightarrow$ (v) Let $(x \rightharpoonup y_{\oslash}^{n+1}) \rightharpoonup z \in I$ and $z \in I$, for $x, y, z \in L$. Then by the similarly proof ((iv) $\Rightarrow$ (i)), $I$ is an ideal of $L$. Moreover, since $y_{\oslash}^{n+1} \leq y_{\oslash}^{n+n}$, we conclude that by Lemma 2(i), $x \rightharpoonup y_{\oslash}^{n+n} \leq x \rightharpoonup y_{\oslash}^{n+1}$. Hence, $(x \rightharpoonup y_{\oslash}^{n+n}) \rightharpoonup z \leq x \rightharpoonup (y_{\oslash}^{n+1}) \rightharpoonup z$ and since $(x \rightharpoonup y_{\oslash}^{n+1}) \rightharpoonup z \in I$, we get $(x \rightharpoonup y_{\oslash}^{n+n}) \rightharpoonup z \in I$. Now, since $z \in I$, we have by (iv), $x \rightharpoonup y_{\oslash}^{n} \in I$.

(v) $\Rightarrow$ (ii) By the similarly proof ((iv) $\Rightarrow$ (i)), $I$ is an ideal of $L$. Now, if $x \rightharpoonup y_{\oslash}^{n+1} \in I$, then $(x \rightharpoonup y_{\oslash}^{n+1}) \rightharpoonup 0 \in I$ and so by (v), $x \rightharpoonup y_{\oslash}^{n} \in I$. ∎

**Theorem 19.** *Let $I \subseteq J$, where $I$ and $J$ be two ideals of $L$ and $I$ be an $n$-fold implicative ideal of $L$. Then $J$ is an $n$-fold implicative ideal, too.*

***Proof.*** Let $I$ be an $n$-fold implicative ideal of $L$, $I \subseteq J$ and $(x \rightharpoonup y) \rightharpoonup z^n_{\oslash} \in J$, for $x, y, z \in L$. Denote $u = (x \rightharpoonup y) \rightharpoonup z^n_{\oslash}$. Then by Lemma 2(i) and (iii), $((x \rightharpoonup u) \rightharpoonup y) \rightharpoonup z^n_{\oslash}) = ((x \rightharpoonup y) \rightharpoonup z^n_{\oslash}) \rightharpoonup u = u \rightharpoonup u = 0 \in I$. Since $I$ is an $n$-fold implicative ideal of $L$, it follows by by Theorem 18,

$$((x \rightharpoonup u) \rightharpoonup z^n_{\oslash}) \rightharpoonup (y \rightharpoonup z^n_{\oslash}) \in I \subseteq J.$$

Hence, by Lemma 2(ii), $((x \rightharpoonup z^n_{\oslash}) \rightharpoonup (y \rightharpoonup z^n_{\oslash})) \rightharpoonup u \in J$ and since $J$ is an ideal of $L$ and $u \in J$, we have $(x \rightharpoonup z^n_{\oslash}) \rightharpoonup (y \rightharpoonup z^n_{\oslash}) \in J$. Therefore, by Theorem 18, $J$ is an $n$-fold implicative ideal of $L$. ■

**Lemma 20.** *For any $BL$-algebra $L$ and $x, y \in L$,*

(i) $(x^n_{\oslash})^- = (x^-)^n$.

(ii) $(x^n)^- = (x^-)^n_{\oslash}$.

(iii) $(x \oslash y)^{--} = x^{--} \oslash y^{--}$.

(iv) $(x \oslash y)^- = x^- \rightharpoonup y^{--}$.

***Proof.*** (i) For any $x \in L$, by $(BL9)$, $(BL11)$ and $(BL12)$,

$$(x^- \to x)^{--} = x^{---} \to x^{--} = x^- \to x^{--} = (x^- \odot x^-)^-.$$

Then $(x^- \to x)^- = (x^- \to x)^{---} = (x^- \odot x^-)^{--} = x^{---} \odot x^{---} = x^- \odot x^-$. Hence,

$$(x \oslash x)^- = (x^- \to x)^- = x^- \odot x^-.$$

Now, since the operation $\oslash$ is associative, we get

$$\begin{aligned}
(x^n_{\oslash})^- &= (\overbrace{x \oslash \cdots \oslash x}^{n-times})^- \\
&= ((\overbrace{x \oslash \cdots \oslash x}^{(n-1)-times}) \oslash x)^- \\
&= (\overbrace{x \oslash \cdots \oslash x}^{(n-1)-times})^- \odot x^- \\
&\vdots \\
&= (x \oslash x)^- \odot (\overbrace{x^- \odot \cdots \odot x^-}^{(n-2)-times}) \\
&= \overbrace{x^- \odot \cdots \odot x^-}^{n-times} \\
&= (x^-)^n.
\end{aligned}$$

(ii) For any $x \in L$, by $(BL9)$, $(BL11)$ and $(BL12)$,

$$\begin{aligned}(x \odot x)^- &= ((x \odot x)^-)^{--} \\ &= (x \to x^-)^{--} \\ &= x^{--} \to x^{---} \\ &= x^{--} \to x^- \\ &= x^- \oslash x^-.\end{aligned}$$

Now,

$$\begin{aligned}(x^n)^- &= (\overbrace{x \odot \cdots \odot x}^{n-times})^- \\ &= ((\overbrace{x \odot \cdots \odot x}^{(n-1)-times}) \odot x)^- \\ &= (\overbrace{x \odot \cdots \odot x}^{(n-1)-times})^- \oslash x^- \\ &\vdots \\ &= (x \odot x)^- \oslash (\overbrace{x^- \oslash \cdots \oslash x^-}^{(n-2)-times}) \\ &= \overbrace{x^- \oslash \cdots \oslash x^-}^{n-times} \\ &= (x^-)^n_{\oslash}.\end{aligned}$$

(iii) Let $x, y \in L$. Then by the definition $\oslash$ and $(BL9)$, $(x \oslash y)^{--} = (x^- \to y)^{--}$ $= x^{---} \to y^{--} = x^{--} \oslash y^{--}$.

(iv) Let $x, y \in L$. Then by the definition $\oslash$, $(x \oslash y)^- = (x^- \to y)^-$. Now, by $(BL9)$, $(BL11)$ and $(BL12)$,

$$\begin{aligned}((x^- \to y)^-)^- &= (x^- \to y)^{--} \\ &= x^{---} \to y^{--} \\ &= x^- \to y^{--} \\ &= (x^- \odot y^-)^-.\end{aligned}$$

And by $(BL10)$ and $(BL12)$,

$$\begin{aligned}(x^- \longrightarrow y)^- &= ((x^- \longrightarrow y)^-)^{--} \\ &= ((x^- \odot y^-)^-)^- \\ &= x^- \odot y^- \\ &= x^- \odot y^{---} \\ &= x^- \rightharpoonup y^{--}.\end{aligned}$$

Therefore, $(x \oslash y)^- = x^- \rightharpoonup y^{--}$. ∎

**Theorem 21.** *Let $I$ be an ideal of $L$. Then $I$ is an $n$-fold implicative ideal of $L$ if and only if it satisfies the condition*

**(n-PI)**: $(y \rightharpoonup (x^n)^-) \rightharpoonup z \in I$ *and* $x^n \rightharpoonup y \in I$ *imply* $x^n \rightharpoonup z \in I$, *for any* $x, y, z \in L$.

***Proof.*** Let $I$ be an $n$-fold implicative ideal of $L$. For any $x, y, z \in L$, let $(y \rightharpoonup (x^n)^-) \rightharpoonup z \in I$ and $x^n \rightharpoonup y \in I$. Then by Lemma 4(i) and (iii), $(z^- \rightharpoonup y^-) \rightharpoonup (x^n)^- \in I$ and $y^- \rightharpoonup (x^n)^- \in I$. Now, by Lemma 20(ii), $(z^- \rightharpoonup y^-) \rightharpoonup (x^-)^n_\oslash \in I$ and $y^- \rightharpoonup (x^-)^n_\oslash \in I$ and since $I$ is an $n$-fold implicative ideal of $L$, we have $z^- \rightharpoonup (x^-)^n_\oslash \in I$. Now, by Lemma 4(i), $((x^-)^n_\oslash)^- \rightharpoonup z^{--} \in I$ and so by Lemma 20(i), $(x^{--})^n \rightharpoonup z^{--} \in I$. Moreover, since by $(BL12)$ and $(BL15)$, $x^n \leq (x^{--})^n$, it follows that by Lemma 2(i), $x^n \rightharpoonup z^{--} \leq (x^{--})^n \rightharpoonup z^{--}$ and so $x^n \rightharpoonup z^{--} \in I$. Hence, $x^n \odot (z^{--})^- \in I$ and so $x^n \odot z^- \in I$. Therefore, $x^n \rightharpoonup z \in I$.

Conversely, let $I$ satisfy the condition **(n-PI)** and $(x \rightharpoonup y) \rightharpoonup z^n_\oslash \in I$, $y \rightharpoonup z^n_\oslash \in I$, for $x, y, z \in L$. Then by Lemma 2(ii), $x \rightharpoonup y \oslash z^n_\oslash \in I$ and so by Lemma 4(i), $(y \oslash z^n_\oslash)^- \rightharpoonup x^- \in I$ and $(z^n_\oslash)^- \rightharpoonup y^- \in I$. Now, by Lemma 20(iv), $(y^- \rightharpoonup ((z^n_\oslash)^-)^-) \rightharpoonup x^- \in I$ and so Lemma 20(i), $(y^- \rightharpoonup ((z^-)^n)^-) \rightharpoonup x^- \in I$ and since $(z^-)^n \rightharpoonup y^- \in I$, we get by condition **(n-PI)**, $(z^-)^n \rightharpoonup x^- \in I$. Hence, by Lemma 20(ii), $(z^n_\oslash)^- \rightharpoonup x^-$, and so by Lemma 4(i), $x \rightharpoonup z^n_\oslash \in I$. Therefore, $I$ is an $n$-fold implicative ideal of $L$. ■

**Theorem 22.** *Let $I$ be an $n$-fold implicative ideal of $L$. Then $I$ is an $(n+1)$-fold implicative ideal of $L$.*

***Proof.*** Let $I$ be an $n$-fold implicative ideal of $L$ and $x \rightharpoonup y^{n+2}_\oslash$, for $x, y \in L$. Then by Lemma 2(ii),

$$(x \rightharpoonup y) \rightharpoonup y^{n+1}_\oslash = x \rightharpoonup y \oslash y^{n+1}_\oslash = x \rightharpoonup y^{n+2}_\oslash \in I.$$

Now, by Theorem 18, $(x \rightharpoonup y) \rightharpoonup y^n_\oslash \in I$ and so $x \rightharpoonup y^{n+1}_\oslash = (x \rightharpoonup y) \rightharpoonup y^n_\oslash \in I$. Hence, by Theorem 18, $I$ is an $(n+1)$-fold implicative ideal of $L$. ■

**Theorem 23.** *Let $I$ be an ideal of $L$. Then $I$ is an $n$-fold implicative ideal of $L$ if and only if $x^{2n}_\oslash \rightharpoonup x^n_\oslash \in I$, for any $x \in L$.*

***Proof.*** Let $I$ be an $n$-fold implicative ideal of $L$ and $x \in L$. Since by Lemma 2(ii), $(x^{2n}_\oslash \rightharpoonup x^n_\oslash) \rightharpoonup x^n_\oslash = x^{2n}_\oslash \rightharpoonup x^n_\oslash \oslash x^n_\oslash = x^{2n}_\oslash \rightharpoonup x^{2n}_\oslash = 0 \in I$, and $x^n_\oslash \rightharpoonup x^n_\oslash = 0 \in I$, we get $x^{2n}_\oslash \rightharpoonup x^n_\oslash \in I$.

Conversely, suppose that for any $x \in L$, $x^{2n}_\oslash \rightharpoonup x^n_\oslash \in I$, and $(x \rightharpoonup y) \rightharpoonup z^n_\oslash \in I$, $y \rightharpoonup z^n_\oslash \in I$, for $x, y, z \in L$. Then by Lemma 2(ii) and (iv), $((x \rightharpoonup z^n_\oslash) \rightharpoonup z^n_\oslash) \rightharpoonup (y \rightharpoonup z^n_\oslash) \leq (x \rightharpoonup z^n_\oslash) \rightharpoonup y = (x \rightharpoonup y) \rightharpoonup z^n_\oslash$. Since $(x \rightharpoonup y) \rightharpoonup z^n_\oslash \in I$ and $I$ is an ideal of $L$, we have

$$((x \rightharpoonup z^n_\oslash) \rightharpoonup z^n_\oslash) \rightharpoonup (y \rightharpoonup z^n_\oslash) \in I.$$

And since $y \rightharpoonup z_\oslash^n \in I$, by Lemma 3,

$$(x \rightharpoonup z_\oslash^n) \rightharpoonup z_\oslash^n \in I.$$

Moreover, by Lemma 2(v),

$$x \rightharpoonup z_\oslash^n \leq (z_\oslash^n \oslash z_\oslash^n \rightharpoonup z_\oslash^n) \oslash (x \rightharpoonup z_\oslash^n \oslash z_\oslash^n).$$

And since $x \rightharpoonup z_\oslash^n \oslash z_\oslash^n = (x \rightharpoonup z_\oslash^n) \rightharpoonup z_\oslash^n \in I$ and by hypothesis $z_\oslash^n \oslash z_\oslash^n \rightharpoonup z_\oslash^n = z_\oslash^{2n} \rightharpoonup z_\oslash^n \in I$, we have $(z_\oslash^n \oslash z_\oslash^n \rightharpoonup z_\oslash^n) \oslash (x \rightharpoonup z_\oslash^n \oslash z_\oslash^n) \in I$. Hence $x \rightharpoonup z_\oslash^n \in I$. Therefore, $I$ is an $n$-fold implicative ideal of $L$. ■

**Theorem 24.** *Let $I$ be an ideal of $L$. Then $I$ is an $n$-fold implicative ideal of $L$ if and only if $I$ is an $n$-fold Boolean ideal of $L$.*

***Proof.*** Let $I$ be an $n$-fold implicative ideal of $L$. Then by Theorem 22, $x_\oslash^{2n} \rightharpoonup x_\oslash^n \in I$, for any $x \in L$. By Lemma 2(ii),

$$\begin{aligned} x_\oslash^{2n} \rightharpoonup x_\oslash^n &= x_\oslash^n \oslash x_\oslash^n \rightharpoonup x_\oslash^n \\ &= ((x_\oslash^n)^- \to x_\oslash^n) \rightharpoonup x_\oslash^n \\ &= ((x_\oslash^n)^- \to x_\oslash^n) \odot (x_\oslash^n)^- \\ &= (x_\oslash^n)^- \odot ((x_\oslash^n)^- \to x_\oslash^n) \\ &= (x_\oslash^n)^- \wedge x_\oslash^n \\ &= x_\oslash^n \wedge (x_\oslash^n)^-. \end{aligned}$$

Hence, for any $x \in L$, $x_\oslash^n \wedge (x_\oslash^n)^- \in I$ and since $(x^-)_\oslash^{2n} \rightharpoonup (x^-)_\oslash^n \in I$, by similar way $(x^-)_\oslash^n \wedge ((x^-)_\oslash^n)^- \in I$. Now, since by Lemma 20(i), $((x^-)_\oslash^n)^- = (x^{--})^n$, then $(x^-)_\oslash^n \wedge (x^{--})^n \in I$ and since by $(BL12)$, $(x^-)_\oslash^n \wedge x^n \leq (x^-)_\oslash^n \wedge (x^{--})^n$, we get $(x^-)_\oslash^n \wedge x^n \in I$. Moreover, by Lemma 4(iv), $((x^-)_\oslash^n \wedge x^n)^{--} \in I$. Hence, applying $(BL16)$, we have $((x^-)_\oslash^n)^{--} \wedge (x^n)^{--} \in I$. Now, by Lemma 20(i), $((x^-)_\oslash^n)^{--} = (((x^-)_\oslash^n)^-)^- = ((x^{--})^n)^- = ((x^n)^{--})^- = (x^n)^-$. Hence,

$$(x^n)^- \wedge (x^n)^{--} = ((x^-)_\oslash^n)^{--} \wedge (x^n)^{--} \in I.$$

By $(BL12)$, $x^n \leq (x^n)^{--}$ and so $(x^n)^- \wedge x^n \leq (x^n)^- \wedge (x^n)^{--}$ and since $I$ is an ideal of $L$, we have $(x^n)^- \wedge x^n \in I$, for any $x \in L$. Therefore, $I$ is an $n$-fold Boolean ideal of $L$.

Conversely, Let $I$ be an $n$-fold Boolean ideal of $L$. Then for any $x \in L$, $((x^-)^n)^- \wedge (x^-)^n \in I$. By Lemma 20(i),

$$((x_\oslash^n)^-)^- \wedge (x_\oslash^n)^- = ((x^-)^n)^- \wedge (x^-)^n \in I.$$

Since $I$ is an ideal of $L$ and by $(BL12)$,

$$x^n_\oslash \wedge (x^n_\oslash)^- \leq (x^n_\oslash)^{--} \wedge (x^n_\oslash)^- = ((x^n_\oslash)^-)^- \wedge (x^n_\oslash)^-,$$

we obtain $x^n_\oslash \wedge (x^n_\oslash)^- \in I$, and so $x^{2n}_\oslash \rightharpoonup x^n_\oslash \in I$. Therefore, by Theorem 23, $I$ is an $n$-fold implicative ideal of $L$. ■

**Theorem 25.** *In a $BL$-algebra $L$, the following conditions are equivalent:*

(i) *any ideal $I$ of $L$ is an $n$-fold implicative,*

(ii) *$\{0\}$ is an $n$-fold implicative ideal of $L$,*

(iii) *for any $a \in L$, the set $L(a) = \{x \in L \mid x \rightharpoonup a^n_\oslash = 0\}$ is an ideal of $L$.*

***Proof.*** (i) ⇔ (ii) It follows from Theorem 19.

(ii) ⇔ (iii) For any $a, x, y \in L$, if $x \rightharpoonup y \in L(a)$ and $y \in L(a)$, then $(x \rightharpoonup y) \rightharpoonup a^n_\oslash = 0 \in \{0\}$, $y \rightharpoonup a^n_\oslash = 0 \in \{0\}$ and since $\{0\}$ is an $n$-fold implicative ideal of $L$, we have $x \rightharpoonup a^n_\oslash \in \{0\}$. Hence, $x \rightharpoonup a^n_\oslash = 0$ and so $x \in L(a)$. Therefore, $L(a)$ is an ideal of $L$.

(iii) ⇔ (ii) Let $(x \rightharpoonup y) \rightharpoonup z^n_\oslash \in \{0\}$ and $y \rightharpoonup z^n_\oslash \in \{0\}$, for $x, y, z \in L$. Then $(x \rightharpoonup y) \in L(z^n_\oslash)$ and $y \in L(z^n_\oslash)$ and since $L(z^n_\oslash)$ is an ideal of $L$, we get $x \in L(z^n_\oslash)$, and so $x \rightharpoonup z^n_\oslash = 0$. Hence, $\{0\}$ is an $n$-fold implicative ideal of $L$. ■

**Proposition 26.** *Let $L$ be Boolean algebra or Gödel algebra. Then any ideal of $L$ is an $n$-fold implicative ideal of $L$ for any natural number $n$.*

***Proof.*** It follows from Theorems 12 and 22. ■

**Theorem 27.** *Let $I$ be a proper ideal of a $L$. Then the following conditions are equivalent:*

(i) *$I$ is a maximal and $n$-fold implicative ideal of $L$,*

(ii) *$x, y \notin I$ imply $x \rightharpoonup y^n_\oslash \in I$ and $y \rightharpoonup x^n_\oslash \in I$, for all $x, y \in L$,*

(iii) *if $x \notin I$, then there exists natural number $m$ such that $((x^n_\oslash)^-)^m_\oslash \in I$,*

(iv) *$(x^-)^n_\oslash \in I$ or $((x^-)^n_\oslash)^- \in I$, for all $x \in L$,*

(v) *$I$ is a prime and $n$-fold implicative ideal of $L$,*

(vi) *$I$ is a prime and $n$-fold Boolean ideal of $L$.*

***Proof.*** (i) ⇔ (ii) Let $I$ be a maximal and $n$-fold implicative ideal of $L$ and $x, y \notin I$. Then by Theorem 17, $I_{y^n_\oslash} = \{z \in L \mid z \rightharpoonup y^n_\oslash \in I\}$ is the least ideal of $L$ containing $I$ and $y$ and since $I$ is maximal ideal of $L$ and $y \notin I$, we have $I_{y^n_\oslash} = L$, and so $x \in I_{y^n_\oslash}$. Therefore, $x \rightharpoonup y^n_\oslash \in I$. By similar way $y \rightharpoonup x^n_\oslash \in I$.

(ii) ⇔ (iii) Suppose that $x \notin I$. Since $I$ is a proper ideal, we have $1 \notin I$ and so by hypothesis $1 \rightharpoonup x^n_\oslash = (x^n_\oslash)^- \in I$. Hence, for some natural number $m$, $((x^n_\oslash)^-)^m_\oslash \in I$.

(iii) ⇔ (iv) For any $x \in L$, if $x^- \in I$, then $(x^-)^n_\oslash \in I$. Assume that $x^- \notin I$, then there exists natural number $m$ such that $(((x^-)^n_\oslash)^-)^m_\oslash \in I$ and since by Lemma 2(vi), $((x^-)^n_\oslash)^- \leq (((x^-)^n_\oslash)^-)^m_\oslash$ and $I$ is an ideal of $L$, we get that $((x^-)^n_\oslash)^- \in I$. Thus, (iv) is valid.

(iv) ⇔ (v) Let $(x^-)^n_\oslash \in I$ or $((x^-)^n_\oslash)^- \in I$, for all $x \in L$. Then by Lemma 20(ii), $(x^n)^- \in I$ or $(x^n)^{--} \in I$, for all $x \in L$, and since $I$ is an ideal of $L$, we obtain $(x^n)^- \in I$ or $x^n implicationalgebranI$, for all $x \in L$. Now, by Theorem 7, $I$ is a prime and $n$-fold Boolean ideal of $L$ and so by Theorem 24, $I$ is a prime and $n$-fold implicative ideal of $L$.

(v) ⇔ (vi) It follows from Theorem 24.

(vi) ⇔ (i) Let $I$ be a prime and $n$-fold Boolean ideal of $L$. Then by Theorem 7, $I$ is a maximal and $n$-fold Boolean ideal of $L$. Hence, by Theorem 24, $I$ is a maximal and $n$-fold implicative ideal of $L$. ■

## 4. N-FOLD OBSTINATE IDEALS IN BL-ALGEBRAS

In this section we introduce a new class of ideals in $BL$-algebras that called $n$-fold obstinate ideals and we give some results.

**Definition.** Let $I$ be an ideal of $L$. $I$ is called an *n-fold obstinate ideal* if it satisfies:

$x, y \notin I$ imply $x \rightharpoonup y^n_\oslash \in I$ and $y \rightharpoonup x^n_\oslash \in I$, for all $x, y \in L$

**Example 28.** [6] Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\rightarrow$ are defined as the following tables:

Table 3

| $\odot$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | $a$ | $a$ |
| $b$ | 0 | $a$ | $b$ | $b$ |
| 1 | 0 | $a$ | $b$ | 1 |

Table 4

| $\rightarrow$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $a$ | $a$ | 1 | 1 | 1 |
| $b$ | 0 | $a$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | 1 |

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a $BL$-algebra. Now, let $I = \{0\}$. Then $I$ is a 2-fold obstinate ideal of $L$, but it is not a 1-fold obstinate ideal. Indeed, $a, b \notin \{0\}$ and $b \rightharpoonup a = b \odot a^- = b \odot a = a \notin \{0\}$.

**Theorem 29.** *Let $I$ be an ideal of $L$. Then $I$ is an $n$-fold obstinate ideal of $L$ if and only if $I$ is an $n$-fold integral ideal of $L$.*

***Proof.*** It follows from Theorems 7 and 27. ■

**Theorem 30.** *Let $I$ be a proper ideal and $F$ be a proper filter of $L$. Then*

(i) *$I$ is an $n$-fold obstinate ideal if and only if $N(I)$ is an $n$-fold obstinate filter of $L$.*

(ii) *$F$ is an $n$-fold integral filter if and only if $N(F)$ is an $n$-fold obstinate ideal of $L$.*

***Proof.*** It follows from Theorems 8, 9 and 29. ■

The following theorem describes the relationship between $n$-fold obstinate ideals and $n$-fold integral $BL$-algebras.

**Theorem 31.** *In any $BL$-algebra $L$, the following conditions are equivalent:*

(i) *$\{0\}$ is an $n$-fold obstinate ideal of $L$,*

(ii) *any ideal of $L$ is an $n$-fold obstinate ideal,*

(iii) *$L$ is an $n$-fold integral $BL$-algebra.*

***Proof.*** It follows from Theorems 10 and 29. ■

**Theorem 32.** *Let $I$ be an ideal of $L$. Then $I$ is an $n$-fold obstinate ideal of $L$ if and only if $\frac{L}{I}$ is an $n$-fold obstinate $BL$-algebra.*

***Proof.*** It follows from Theorems 11 and 29. ■

**Example 33.** Let $L$ be $BL$-algebra given in Example 28 and $I = \{0\}$, which is a 2-fold obstinate ideal of $L$. We have $\frac{L}{I} = \{[0], [a], [1]\}$, where $[0] = \{0\}$, $[a] = \{a\}$ and $[1] = \{b, 1\}$. Note that $\frac{L}{I}$ is an $MV$-algebra and $[a]^2 = [a^2] = [0]$. Hence, $\frac{L}{I}$ is a 2-fold obstinate $BL$-algebra.

## 5. Conclusion

The results of this paper are devoted to study two new classes of ideals that is called $n$-fold implicative ideals and $n$-fold obstinate ideals. We presented a characterization and several important properties of $n$-fold implicative ideals and $n$-fol d obstinate ideals. In particular, we proved that an ideal is $n$-fold implicative ideal if and only if is an $n$-fold Boolean ideal. Also, we proved that a $BL$-algebra is an $n$-fold integral $BL$-algebra if and only if trivial ideal $\{0\}$ is an $n$-fold obstinate ideal. Moreover, we studied the relation between n-fold obstinate ideals and $n$-fold (integral) obstinate filters in $BL$-algebras by using the set of complement elements.

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