

## CONRAD'S PARTIAL ORDER ON P.Q.-BAER \*-RINGS

ANIL KHAIRNAR

*Department of Mathematics*  
*Abasaheb Garware College, Pune-411004, India*

**e-mail:** anil.khairnar@mesagc.org  
anil\_maths2004@yahoo.com

AND

B.N. WAPHARE

*Center for Advanced Studies in Mathematics*  
*Department of Mathematics*  
*Savitribai Phule Pune University, Pune-411007, India*

**e-mail:** bnwaph@math.unipune.ac.in  
waphare@yahoo.com

### Abstract

We prove that a p.q.-Baer \*-ring forms a pseudo lattice with Conrad's partial order and also characterize p.q.-Baer \*-rings which are lattices. The initial segments of a p.q.-Baer \*-ring with the Conrad's partial order are shown to be an orthomodular posets.

**Keywords:** Conrad's partial order, p.q.-Baer \*-ring, central cover, orthomodular set.

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### 1. INTRODUCTION

A \*-ring  $R$  is a ring equipped with an involution  $x \rightarrow x^*$ , that is an additive anti-automorphism of a period at most two. An element  $e$  of a \*-ring  $R$  is a *projection* if  $e = e^2$  (idempotent) and  $e = e^*$  (self-adjoint). For a nonempty subset  $B$  of  $R$ , we write  $r_R(B) = \{x \in R \mid bx = 0, \text{ for every } b \in B\}$ , and call the *right annihilator* of  $B$  in  $R$ . Similarly, we define the *left annihilator* of  $B$  in  $R$  (denoted by  $l_R(B)$ ). A ring is said to be *abelian* if its every idempotent is central. A ring

without nonzero nilpotent elements is called a *reduced* ring. Let  $P$  be a poset and  $a, b \in P$ , then the *join* of  $a$  and  $b$ , denoted by  $a \vee b$  is defined as  $a \vee b = \sup\{a, b\}$  and the *meet* of  $a$  and  $b$ , denoted by  $a \wedge b$  is defined as  $a \wedge b = \inf\{a, b\}$ . A poset  $P$  is said to be a pseudo lattice, if for  $a, b \in P$ , whenever  $a, b$  have a common upper bound, then  $a \wedge b$  and  $a \vee b$  both exist.

Kaplansky [16] introduced Baer rings and Baer  $*$ -rings to abstract various properties of  $AW^*$  algebras, von Neumann algebras and complete  $*$ -regular rings. The subject of Baer  $*$ -rings is essentially pure algebra, with historic roots in operator algebras and lattice theory.

The set of projections in a Rickart  $*$ -ring forms an orthomodular lattice under the partial order ' $e \leq_p f$  if and only if  $e = fe = ef$ '. This lattice is extensively studied in [3, 16, 24]. In [2, 9, 10, 12, 25] the authors studied partial orders on complex matrices or  $\mathcal{B}(H)$  (the algebra of all bounded linear operators on an infinite-dimensional Hilbert space  $H$ ). In [11, 15, 22] the authors studied partial orders on Rickart  $*$ -rings. In [26], authors introduced multiplicatively finite elements in a ring. By restricting multiplicatively finite elements, Khairnar and Waphare [18] introduced generalized projections, a partial order on them and studied this poset in a Rickart  $*$ -ring. In [19], authors studied Generalized Projections in  $\mathbb{Z}_n$ . Hartwig [12] defined the plus partial order on the set of regular elements in a semigroup. For  $m \times n$  matrices over a division ring  $D$  (that is  $D_{m \times n}$ ) Hartwig [12] use the concept of rank  $\rho(\cdot)$  and obtained the following result, which characterize the plus order for the ring  $D_{m \times n}$ .

**Theorem 1.1** (Theorem 2, [12]). *Let  $A, B \in D_{m \times n}$ . Then  $A \leq B$  if and only if  $\rho(B - A) = \rho(B) - \rho(A)$ . In particular, rank-subtractivity is a partial-ordering relation on  $D_{m \times n}$ .*

Also in the same paper [12], Hartwig posed the following open problems for regular rings.

**Problem 1.** Can one induce a partial ordering on a ring  $R$ , by a subtractive rank-like function  $\rho : R \rightarrow G$ , where  $G$  is a well-ordered abelian group and  $\rho(b - a) = \rho(b) - \rho(a)$ ?

**Problem 2.** Does  $a \leq c, b \leq c, aR \cap bR = \{0\} = Ra \cap Rb \Rightarrow a + b \leq c$ ? (here  $\leq$  denote the plus partial order on regular elements of a ring  $R$ ).

Conrad [8] extended the work of Abian [1] by showing that a ring  $R$  is partially ordered by the relation  $a \leq_c b$  if and only if  $arb = ara$  for all  $r \in R$  (this is called Conrad's relation) precisely when it is semiprime. Burgess and Raphael [6] proved that this relation, when defined on a semigroup  $S$ , is a partial order whenever  $S$  is weakly separative.

Birkenmeier *et al.* [5] introduced principally quasi-Baer (p.q.-Baer)  $*$ -rings. A  $*$ -ring  $R$  is said to be a *p.q.-Baer  $*$ -ring* if, for every principal right ideal  $aR$

of  $R$ ,  $r_R(aR) = eR$ , where  $e$  is a projection in  $R$ . From the above definition, it follows that  $l_R(aR) = Rf$  for a suitable projection  $f$ . In [20], authors studied a sheaf representation of p.q.-Baer \*-Rings. There is an abelian p.q.-Baer \*-ring which is not a Rickart \*-ring. Also, reduced Rickart \*-rings are p.q.-Baer \*-rings. In [5], Birkenmeier *et al.* have given examples of p.q.-Baer \*-rings those are neither Rickart \*-rings nor quasi-Baer \*-rings.

**Example 1.2** [5, Exercise 10.2.24.4]. Let  $A$  be a domain,  $A_n = A$  for all  $n = 1, 2, \dots$ , and  $B$  be the ring of  $(a_n)_{n=1}^\infty \in \prod_{n=1}^\infty A_n$  such that  $a_n$  is eventually constant, which is a subring of  $\prod_{n=1}^\infty A_n$ . Take  $R = M_n(B)$ , where  $n$  is an integer such that  $n > 1$ . Let  $*$  be the transpose involution of  $R$ . Then  $R$  is a p.q.-Baer \*-ring which is not quasi-Baer (hence not a quasi-Baer \*-ring). Also, if  $A$  is commutative which is not Prüfer, then  $R$  is not a Rickart \*-ring.

**Example 1.3** [5, Exercise 10.2.24.5]. Let  $R$  be a \*-ring. If  $R$  is a right (or left) p.q.-Baer ring and  $*$  is semiproper, then  $R$  is a p.q.-Baer \*-ring. Hence, if  $R$  is biregular and  $*$  is semiproper, then  $R$  is a p.q.-Baer \*-ring.

**Example 1.4** [20, Example 2.3]. Let  $T$  be a commutative regular ring with unity such that  $|T| > 1$ , and  $S = \prod_{\lambda \in \Lambda} T_\lambda$ , where  $T_\lambda = T$  and  $\Lambda$  is an infinite indexing set. If  $R$  is a subring of  $S$  generated by  $\bigoplus_{\lambda \in \Lambda} T_\lambda$  and either  $1 \in S$  or  $\{f : \Lambda \rightarrow T \mid f \text{ is a constant function}\}$ , then by [4, Example 1.5],  $R$  is a p.q.-Baer ring that is not quasi-Baer. Since  $R$  is commutative,  $R$  is a \*-ring with an identity involution. Therefore  $R$  is a p.q.-Baer \*-ring but not a quasi-Baer \*-ring.

**Example 1.5** [20, Example 2.6]. Let

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) \mid a \equiv d, b \equiv 0, \text{ and } c \equiv 0 \pmod{2} \right\}.$$

Consider involution  $*$  on  $R$  as the transpose of the matrix. In [14, Example 2(1)], it is shown that  $R$  is neither right p.p. nor left p.p. (hence not a Rickart \*-ring) but  $r_R(uR) = \{0\} = 0R$  for any nonzero element  $u \in R$ . Therefore  $R$  is a p.q.-Baer \*-ring.

Recall the following remark from [20].

**Remark 1.6** [20, Remark 2.2]. Let  $R$  be a p.q.-Baer \*-ring. Then,

- (1)  $R$  is semiprime.
- (2)  $R$  is reflexive (see [21, Proposition 4]).
- (3) Involution  $*$  is semiproper.
- (4) For any central projection  $e \in R$ ,  $C(e) = e$ . Moreover, for any  $x \in R$  and any central projection  $e \in R$ ,  $C(ex) = eC(x)$ .

- (5) Let  $a \in R$ , then  $C(a^*) = C(a)$ .  
 (6) For any central projection  $e \in R$ ,  $eR$  is a p.q.-Baer  $*$ -ring.

As p.q.-Baer  $*$ -rings are semiprime, therefore Conrad's relation is a partial order on a p.q.-Baer  $*$ -ring. Analogous to Problem 1 and Problem 2, we raise the following problems for a p.q.-Baer  $*$ -ring.

**Problem 3.** Can one induce a partial ordering on a ring  $R$ , by a subtractive rank-like function  $\rho : R \rightarrow G$ , where  $G$  is a partially-ordered abelian group and  $\rho(b - a) = \rho(b) - \rho(a)$ ?

**Problem 4.** Does  $a \leq_c d, b \leq_c d, aR \cap bR = \{0\} = Ra \cap Rb \Rightarrow a + b \leq_c d$ ?

Let  $R$  be a  $*$ -ring and  $x \in R$ , we say that  $x$  possesses a *central cover* if there exists a smallest central projection  $h$  such that  $hx = x$ . If such a projection  $h$  exists, then it is unique and is called the central cover of  $x$ , denoted by  $h = C(x)$  (see [3]). In [17] the authors proved the existence of central cover of every element of a p.q.-Baer  $*$ -ring. In the second section of this paper, we characterize Conrad's partial order on p.q.-Baer  $*$ -rings in terms of central covers. This essentially gives a solution of Problem 3. In the third section, we answer Problem 4 positively.

Janowitz [15] proved that the initial segments of an arbitrary Rickart  $*$ -ring with the  $*$ -order are orthomodular posets. The same result is proved by Kr emere [22] for the left-star order. In the last section, we prove that the initial segments of a p.q.-Baer  $*$ -ring with Conrad's partial order are orthomodular posets.

## 2. CONRAD'S RELATION ON P.Q.-BAER $*$ -RINGS

Hence fourth,  $\leq$  denotes Conrad's partial order relation. In the following remark we list some basic observations.

**Remark 2.1.** Let  $R$  be a  $*$ -ring and  $P(Z(R))$  denotes the set of central projections of  $R$ .

- (1) For  $e, f \in P(Z(R))$ ,  $e \leq f$  if and only if  $e = ef$ .  
 (2) For any  $e \in P(Z(R))$  the central cover of  $e$ ,  $C(e)$  exists and  $C(e) = e$ . Moreover, whenever  $C(x)$  exists for some  $x \in R$ , then for any  $e \in P(Z(R))$ ,  $C(ex)$  exists and  $C(ex) = eC(x)$ .  
 (3) Let  $a \in R$ . If  $C(a)$  exists in  $R$ , then  $C(a^*)$  exists in  $R$  and  $C(a^*) = C(a)$  (see [17]).

**Lemma 2.2.** Let  $R$  be a  $*$ -ring and  $x \in R$ . Let  $e \in R$  be a central projection in  $R$  such that (1)  $xe = x$  and (2)  $xRy = 0$  implies  $ey = 0$ . Then  $e = C(x)$ .

**Proof.** To prove that  $e = C(x)$ , it is sufficient to prove that  $e$  is the smallest central projection with  $xe = x$ . Let  $e' \in R$  be a central projection such that

$xe' = x$ . Then  $x(1 - e') = 0$ . Since  $1 - e'$  is central,  $xR(1 - e') = 0$ . By condition (2), we have  $e(1 - e') = 0$  and hence  $e = ee'$ . Therefore  $e \leq e'$ . Thus  $e = C(x)$ . ■

The existence of a central cover of every element in a p.q.-Baer \*-ring is guaranteed by the following theorem.

**Theorem 2.3** (Theorem 2.3, [17]). *Let  $R$  be a p.q.-Baer \*-ring and  $x \in R$ . Then  $x$  has a central cover  $e \in R$ . Further,  $xRy = 0$  if and only if  $yRx = 0$  if and only if  $ey = 0$ . That is  $r_R(xR) = r_R(eR) = l_R(Rx) = l_R(Re) = (1 - e)R = R(1 - e)$ .*

In the following lemma, we characterize Conrad's relation in terms of central cover.

**Theorem 2.4.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$ . Then the following statements are equivalent.*

- (1)  $a^*rb = a^*ra$  for all  $r \in R$ .
- (2)  $a = C(a)b$ .
- (3)  $arb = ara$  for all  $r \in R$  (that is  $a \leq b$ ).

**Proof.** (1)  $\Rightarrow$  (2): Suppose  $a^*rb = a^*ra$  for all  $r \in R$ . Hence  $a^*r(b - a) = 0$  for all  $r \in R$ . This gives  $a^*R(b - a) = 0$ . By Theorem 2.3, we get  $C(a^*)(b - a) = 0$ . By Remark 2.1, we have  $C(a)(b - a) = 0$ . Thus  $a = C(a)b$ .

(2)  $\Rightarrow$  (3): Suppose  $a = C(a)b$ . For  $r \in R$ , we have  $ara = arC(a)b = C(a)arb = arb$ . Therefore  $arb = ara$  for all  $r \in R$ .

(3)  $\Rightarrow$  (1): By the similar arguments as in the proof of (1)  $\Rightarrow$  (2), we get  $a = C(a)b$ . Further, for  $r \in R$ ,  $a^*ra = a^*rC(a)b = C(a)a^*rb = C(a^*)a^*rb = a^*rb$ . Thus  $a^*rb = a^*ra$  for all  $r \in R$ . ■

The above theorem essentially says that, in a p.q.-Baer \*-ring  $R$ , for  $a, b \in R$ ,  $a \leq b$  if and only if  $a = C(a)b$ . Therefore, we use the relation  $a = C(a)b$  as Conrad's relation (partial order) on a p.q.-Baer \*-ring. The following lemma leads to the result which constructs a subtractive function on a p.q.-Baer \*-ring.

**Lemma 2.5.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$  be such that  $a \leq b$ . Then,*

- (1)  $C(a) \leq C(b)$  and  $a = aC(b) = bC(a)$
- (2)  $C(b - a) = C(b) - C(a)$ .

**Proof.** (1) Since  $a \leq b$ , we have  $a = C(a)b$ . By Remark 2.1,  $C(a) = C(C(a)b) = C(a)C(b)$ . This yields  $C(a) \leq C(b)$ . Also,  $aC(a) = aC(a)C(b)$  implies that  $a = aC(b)$ . Therefore  $a = aC(b) = bC(a)$ .

(2) Since  $C(a) \leq C(b)$ ,  $C(b) - C(a)$  is a central projection. Also by part (1), we have  $(b - a)(C(b) - C(a)) = bC(b) - bC(a) - aC(b) + aC(a) = b - a - a + a = b - a$ . Further, for  $y \in R$ ,  $(b - a)Ry = 0$  if and only if  $br_y = ar_y$  for all  $r \in R$  if and only

if  $bC(b)ry = bC(a)ry$  for all  $r \in R$  if and only if  $bR(C(b) - C(a))y = 0$  if and only if  $C(b)(C(b) - C(a))y = 0$  (by Theorem 2.3) if and only if  $(C(b) - C(a))y = 0$ . Thus, by Lemma 2.2, we get  $C(b - a) = C(b) - C(a)$ , as required. ■

In the above lemma we have proved that in a p.q.-Baer  $*$ -ring  $R$ , for  $a, b \in R$ , if  $a \leq b$  then  $C(b - a) = C(b) - C(a)$ . The following lemma gives a sufficient condition so that the converse of this statement is true.

**Lemma 2.6.** *Let  $R$  be a p.q.-Baer  $*$ -ring in which 2 is invertible. Let  $a, b \in R$  be such that  $C(b - a) = C(b) - C(a)$ . Then  $a \leq b$ .*

**Proof.** Let  $a, b \in R$  be such that  $C(b - a) = C(b) - C(a)$ . Then  $(C(b) - C(a))^2 = C(b) - C(a)$ , which yields  $2C(b)C(a) = 2C(a)$ . Since 2 is invertible element in  $R$ , we have  $C(b)C(a) = C(a)$ . Further,  $C(b - a)C(a) = (C(b) - C(a))C(a) = 0$ . By Theorem 2.3,  $(b - a)RC(a) = 0$ . Consequently,  $(b - a)C(a) = 0$  and hence  $bC(a) = a$ . Therefore  $a \leq b$ . ■

The following theorem characterizes Conrad's partial order in terms of central covers, which gives a result similar to Theorem 1.1.

**Theorem 2.7.** *Let  $R$  be a p.q.-Baer  $*$ -ring in which 2 is invertible and let  $a, b \in R$ . Then  $a \leq b$  if and only if  $C(b - a) = C(b) - C(a)$ .*

**Proof.** The proof follows from Lemmas 2.5 and 2.6. ■

In the following corollary, we give a solution of Problem 3. Let  $B(R)$  denote the algebra of central projections in a  $*$ -ring  $R$ . Note that  $B(R)$  is a partial ordered abelian group.

**Corollary 2.8.** *Let  $R$  be a p.q.-Baer  $*$ -ring in which 2 is invertible. Then there exists a function  $\rho : R \rightarrow B(R)$  such that  $\rho(b - a) = \rho(b) - \rho(a)$  and  $\rho$  induces the Conrad's partial order on  $R$ .*

**Proof.** Let  $\rho : R \rightarrow B(R)$  defined as  $\rho(x) = C(x)$ . Then the proof follows from Theorem 2.7. ■

A  $*$ -regular ring is a regular ring with proper involution (i.e., for any element  $a$ ,  $a^*a = 0$  implies that  $a = 0$ ). Note that the  $*$ -regular rings whose lattice of principal right ideals is complete are Baer  $*$ -rings and hence are p.q.-Baer  $*$ -rings (see [3]). In connection to Problem 1 we have the following corollary.

**Corollary 2.9.** *Let  $R$  be a  $*$ -regular and p.q.-Baer  $*$ -ring in which 2 is invertible. Then there exists a subtractive rank like function  $\rho : R \rightarrow B(R)$  such that  $\rho(b - a) = \rho(b) - \rho(a)$  and  $\rho$  induces Conrad's partial order on  $R$ .*

An abelian group admits an order if and only if it is torsion free (see [23]). Since  $B(R)$  is a Boolean algebra, it is well-ordered with respect to Conrad's partial order if and only if the cardinality of  $B(R)$  is two.

3. WHEN DOES A P.Q.-BAER \*-RING BECOME A LATTICE?

Hartwig [13] showed that a \*-regular ring  $R$  forms a pseudo upper semilattice under the \*-orthogonal partial ordering. That is,  $a, b \in R$  have a common upper bound if and only if  $a \vee b$  exists in  $R$ . In this section, we prove that a p.q.-Baer \*-ring  $R$  forms a pseudo lattice under Conrad's partial order. Also, we characterize p.q.-Baer \*-rings those form lattices. As a consequence, we answer Problem 4 positively.

In [8], a concept of orthogonality is introduced as follows.

**Definition 3.1.** Let  $R$  be a semiprime ring and  $a, b \in R$ . Then  $a$  is said be *orthogonal* to  $b$  if  $aRb = 0$ . In a p.q.-Baer \*-ring this condition is equivalent to  $C(a)C(b) = 0$  (see [17]). We write  $a \perp b$ , whenever  $a$  is orthogonal to  $b$ .

Recall the following definition and theorem from [6].

**Definition 3.2.** Let  $R$  be a semiprime ring. For an ideal  $I$  of  $R$ ,  $Ann I = \{r \in R \mid rI = 0\}$ . If for each ideal  $I$ ,  $Ann I$  contains a nonzero central idempotent then  $R$  is called *weakly  $i$ -dense*.  $R$  is *orthogonally complete* if every orthogonal set has a supremum.

**Theorem 3.3** (Theorem 9, [6]). *An orthogonally complete semiprime ring which is weakly  $i$ -dense is complete.*

We give an example of a commutative, reduced, weakly  $i$ -dense p.q.-Baer \*-ring which is not orthogonally complete.

**Example 3.4.** Let  $R = \{x \in \prod_{i=1}^{\infty} \mathbb{Q} \mid \text{for almost all } i, x_i \in \mathbb{Z}\}$ . Then  $R$  is a commutative \*-ring with an identity involution. For  $a = (a_1, a_2, \dots) \in R$ ,  $r_R(a) = bR$  where  $b = (b_1, b_2, \dots)$  with  $b_i = 1$  if  $a_i = 0$ ; and  $b_i = 0$  if  $a_i \neq 0$ . Note that  $b^2 = b = b^*$ . Therefore  $R$  is a Rickart \*-ring. Since an abelian Rickart \*-ring is a reduced p.q.-Baer \*-ring,  $R$  becomes a commutative reduced p.q.-Baer \*-ring. Since every ideal of  $R$  is a principal ideal and  $R$  is a p.q.-Baer \*-ring, therefore by Theorem 2.3,  $R$  is weakly  $i$ -dense. Let  $c_1 = (\frac{1}{2}, 0, 0, \dots)$ ,  $c_2 = (0, \frac{1}{2}, 0, 0, \dots), \dots$ , and  $S = \{c_n \mid n \in \mathbb{N}\}$ . Then  $S$  is an orthogonal subset of  $R$  which does not have the supremum in  $R$ . Thus  $R$  is not orthogonally complete.

In the following theorem, we prove that a p.q.-Baer \*-ring forms a pseudo lattice with respect to Conrad's partial order.

**Theorem 3.5.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$  have a common upper bound. Then,*

- (1)  $aC(b) = bC(a)$ ;
- (2)  $a^*rb = C(a)b^*rb = C(b)a^*ra$  for all  $r \in R$ . Hence,  $a^*b$  is self adjoint;

- (3)  $arb^* = C(a)brb^* = C(b)ara^*$  for all  $r \in R$ . Hence,  $ab^*$  is self adjoint;  
 (4)  $a \wedge b = aC(b) = bC(a)$ ; and  
 (5)  $a \vee b = a + b - a \wedge b$ .

**Proof.** Let  $a, b, c \in R$  and  $c$  be a common upper bound of  $a$  and  $b$ . Then  $a = C(a)c$  and  $b = C(b)c$ . By Theorem 2.4,  $a^*ra = a^*rc$ ,  $b^*rb = b^*rc$  for all  $r \in R$ . Also,  $b^*rb = c^*rb$  for all  $r \in R$ .

(1) Since  $a = C(a)c$  and  $b = C(b)c$ , we have  $aC(b) = C(a)cC(b) = bC(a)$ .

(2) Let  $r \in R$ . Then  $a^*rb = a^*rC(b)c = C(b)a^*rc = C(b)a^*ra$ . Also,  $a^*rb = (C(a)c)^*rb = C(a)c^*rb = C(a)b^*rb$ . Consequently,  $a^*rb = C(a)b^*rb = C(b)a^*ra$  for all  $r \in R$ . In particular for  $r = 1$ , we have  $a^*b = C(b)a^*a$ . Therefore  $(a^*b)^* = C(b)a^*a = a^*b$ . Thus  $a^*b$  is self adjoint.

(3) The proof is similar to the proof of part (2).

(4) To prove  $a \wedge b = aC(b)$ , first we prove that  $aC(b)$  is a common lower bound of  $a$  and  $b$ . By Remark 2.1,  $C(aC(b))a = C(a)C(b)a = aC(b)$ . This implies that  $aC(b) \leq a$ . Similarly,  $bC(a) \leq b$ . By part (1), we get  $aC(b) \leq b$ . Let  $d \in R$  be such that  $d \leq a$  and  $d \leq b$ . Then  $d = C(d)a = C(d)b$  and hence  $dC(b) = C(d)b$ . Further,  $C(d)aC(b) = dC(b) = C(d)b = d$ . Therefore  $d \leq aC(b)$ . Thus  $a \wedge b = aC(b) = bC(a)$ .

(5) By parts (1) and (4),  $C(a)(a + b - a \wedge b) = C(a)(a + b - aC(b)) = aC(a) + bC(a) - aC(a)C(b) = a + bC(a) - aC(b) = a$ . This yields  $a \leq (a + b - a \wedge b)$ . Similarly,  $b \leq (a + b - a \wedge b)$ . Let  $d \in R$  be such that  $a \leq d$  and  $b \leq d$ . Then  $a = C(a)d$  and  $b = C(b)d$ . Let  $r \in R$ . By part (2), we have  $(a + b - a \wedge b)^*r(a + b - a \wedge b) = (a^* + b^* - a^*C(b))r(a + b - aC(b)) = a^*ra + a^*rb - a^*raC(b) + b^*ra + b^*rb - b^*raC(b) - a^*raC(b) - a^*rbC(b) + a^*raC(b) = a^*ra + a^*rb - a^*rb + b^*ra + b^*rb - C(b)b^*ra - a^*rb - a^*raC(b) + a^*raC(b) = a^*ra + b^*ra + b^*rb - b^*ra - a^*rb = a^*rdC(a) + b^*rdC(b) - a^*rdC(b) = a^*rd + b^*rd - a^*rdC(b) = (a^* + b^* - a^*C(b))rd = (a + b - aC(b))^*rd = (a + b - a \wedge b)^*rd$ . By Theorem 2.4, we get  $(a + b - a \wedge b) \leq d$ . Therefore  $a \vee b = a + b - a \wedge b$ . ■

As an immediate consequence of above theorem we have the following corollaries.

**Corollary 3.6.** *Let  $R$  be a p.q.-Baer  $*$ -ring. Then  $R$  is a pseudo lattice with respect to Conrad's partial order.*

**Corollary 3.7.** *Let  $R$  be a p.q.-Baer  $*$ -ring and  $a, b \in R$ . If  $a \vee b$  exists in  $R$  then  $a \vee b = a + b(1 - C(a)) = b + a(1 - C(b))$ .*

By Theorem 3.5(1), in a p.q.-Baer  $*$ -ring  $R$ , if  $a, b \in R$  have a common upper bound then  $aC(b) = bC(a)$ . In the following lemma, we prove that the converse of this statement is also true.



**Lemma 3.8.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$ . If  $aC(b) = bC(a)$  then  $a, b$  have a common upper bound.*

**Proof.** Let  $a, b \in R$  be such that  $aC(b) = bC(a)$ . We prove that  $a + b - aC(b)$  is a common upper bound of  $a$  and  $b$ . Clearly  $C(a)(a + b - aC(b)) = a + C(a)b - aC(b) = a$ . Also,  $C(b)(a + b - aC(b)) = aC(b) + b - aC(b) = b$ . Therefore  $a \leq a + b - aC(b)$  and  $b \leq a + b - aC(b)$ , as required. ■

The following theorem, characterizes p.q.-Baer \*-rings which form lattices with Conrad's partial order.

**Theorem 3.9.** *Let  $R$  be a p.q.-Baer \*-ring. Then  $R$  is a lattice with respect to Conrad's partial order if and only if  $aC(b) = bC(a)$  for all  $a, b \in R$ .*

**Proof.** The proof follows from Theorem 3.5 and Lemma 3.8. ■

We conclude this section with a positive answer to Problem 4.

**Theorem 3.10.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b, c \in R$ . If  $a \leq c$ ,  $b \leq c$ ,  $aR \cap bR = \{0\}$  then  $a + b \leq c$ .*

**Proof.** Let  $a, b, c \in R, a \leq c, b \leq c$  and  $aR \cap bR = \{0\}$ . Then, by Theorem 3.5,  $aC(b) = bC(a)$ . This implies that  $aC(b) \in aR \cap bR$  and hence  $aC(b) = 0$ . Again, by using Theorem 3.5, we have  $a \vee b = a + b$ . Thus  $a + b \leq c$ . ■

#### 4. ORTHOGONALITY RELATION ON P.Q.-BAER \*-RINGS

In this section, we prove that the initial segments of an arbitrary p.q.-Baer \*-ring with Conrad's partial order are orthomodular posets.

We recall the following definitions from [7].

A binary relation  $\perp$  on a poset  $(P, \leq, 0)$ , where  $0$  is the least element of the poset, is called an *orthogonality relation* (for the order  $\leq$ ) if for all  $x, y, z \in P$ ,

- (1) if  $x \perp y$ , then  $y \perp x$ ;
- (2) if  $x \leq y$  and  $y \perp z$ , then  $x \perp z$ ; and
- (3)  $0 \perp x$ .

A poset with orthogonality  $(P, \leq, \perp, 0)$  is called *quasi-orthomodular* if for all  $x, y \in P$ ,

- (4) if  $x \perp y$ , then  $x \vee y$  exists;
- (5) if  $x \leq y$ , then  $y = x \vee z$  for some  $z \in P$  with  $x \perp z$ ;
- (6) if  $x \perp y, x \perp z$  and  $y \leq x \vee z$ , then  $y \leq z$ .

A poset  $(P, \leq, 0, 1)$  (where 0 is the least and 1 is the greatest element) is called an *orthocomplemented poset* if there is an operation  $^\perp : P \rightarrow P$  such that for all  $a, b \in P$ ,

- (1)  $a \wedge a^\perp$  and  $a \vee a^\perp$  exist, and  $a \wedge a^\perp = 0$  and  $a \vee a^\perp = 1$ ;
- (2)  $(a^\perp)^\perp = a$ ;
- (3) if  $a \leq b$ , then  $b^\perp \leq a^\perp$ .

The operation  $^\perp$  is called an *orthocomplementation*. In an orthocomplemented poset, we define the relation  $\perp$  by  $a \perp b$  if and only if  $a \leq b^\perp$ . This is an orthogonality relation. An orthocomplemented poset  $(P, \leq, ^\perp, 0, 1)$  is called *orthomodular* if for all  $a, b \in P$ ,

- (1) if  $a \perp b$ , then  $a \vee b$  exist;
- (2) if  $a \leq b$ , then there exists an element  $c \in P$  such that  $c \leq a^\perp$  and  $b = a \vee c$ .

Between orthomodularity and quasi-orthomodularity, the following connection holds.

**Theorem 4.1** [7]. *In a quasi-orthomodular poset  $(P, \leq, ^\perp)$ , all initial segments  $[0, p] = \{a \in P \mid a \leq p\}$  are orthomodular for some orthogonality  $\perp_p$  on  $([0, p], \leq)$ . Furthermore, if  $\perp_p$  is the orthogonality of the initial segment  $[0, p]$ , then for all  $a, b \in [0, p]$ ,  $a \perp_p b$  if and only if  $a \perp b$ . Moreover, if  $x \perp_p y$  and  $x, y \leq q$ , then  $x \perp_q y$ .*

By using above theorem, we prove that the initial segments of p.q.-Baer \*-rings with Conrad's partial order are orthomodular posets, for that we prove the following sequence of theorems and lemmas.

**Lemma 4.2.** *The relation  $\perp$  is an orthogonality relation on a p.q.-Baer \*-ring.*

**Proof.** Let  $R$  be a p.q.-Baer \*-ring. By definition of orthogonal elements, it is clear that for any  $x, y \in R$ , if  $x \perp y$  then  $y \perp x$ . Suppose  $a \leq b$  and  $b \perp c$ . Then  $a = C(a)b$  and  $C(b)C(c) = 0$ . By Lemma 2.5,  $C(a)C(c) = C(a)C(b)C(c) = 0$  and hence  $a \perp c$ . Further,  $C(0) = 0$ , therefore  $C(0)C(x) = 0$  for any  $x \in R$ . Consequently,  $0 \perp x$  for any  $x \in R$ . Thus the relation  $\perp$  is an orthogonality relation. ■

**Lemma 4.3.** *Let  $R$  be a p.q.-Baer \*-ring and  $a, b \in R$  be orthogonal elements. Then  $a \wedge b = 0$  and  $a \vee b = a + b$ .*

**Proof.** Let  $a, b \in R$  be such that  $a \perp b$ . Then  $C(a)C(b) = 0$ . This implies  $aC(b) = C(a)b = 0$ . Therefore by Lemma 3.8,  $a$  and  $b$  have a common upper bound. By Theorem 3.5, we have  $a \wedge b = 0$  and  $a \vee b = a + b$ . ■

The following lemma leads to the orthomodularity condition in a poset.

**Theorem 4.4.** *A p.q.-Baer \*-ring  $R$  with the order  $\leq$  and the orthogonality  $\perp$  is a quasi-orthomodular poset.*

**Proof.** By Lemma 4.2, the relation  $\perp$  is an orthogonality relation on  $R$ . Let  $a, b \in R$  and  $a \leq b$ . Then  $a = C(a)b$  and hence  $C(a) = C(a)C(b)$ . Let  $c = b - a$ . By Lemma 2.5,  $C(a)C(c) = C(a)C(b - a) = C(a)(C(b) - C(a)) = C(a)C(b) - C(a) = 0$ . Therefore  $a \perp c$ . Let  $e, f, d \in R$  be such that  $e \perp f$ ,  $e \perp d$  and  $f \leq e \vee d$ . Then  $C(e)C(f) = C(e)C(d) = 0$  and  $f = C(b)(e \vee d)$ . By Lemma 4.3,  $f = C(f)(e + d) = C(f)e + C(f)d = C(f)d$ . Hence  $f \leq d$ . Thus  $R$  is a quasi-orthomodular poset. ■

**Theorem 4.5.** *In a p.q.-Baer \*-ring  $R$ , the initial segments  $[0, m] = \{a \in R \mid a \leq m\}$  are orthomodular posets. Furthermore, if  $\perp_m$  is the local orthogonality of the initial segment  $[0, m]$ , then for all  $a, b \in [0, m]$ ,  $a \perp_m b$  if and only if  $a \perp b$ . Moreover, if  $a \perp_m b$  and  $a, b \leq n$ , then  $a \perp_n b$ .*

**Proof.** The proof follows from Theorems 4.1 and 4.4. ■

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