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IDEALS IN ORDERED Γ-SEMIRINGS

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Abstract

In this paper, we introduce the notion of k-ideal, $m-k$ ideal, prime ideal, maximal ideal, filter, irreducible ideal, strongly irreducible ideal in ordered Γ-semirings, study the properties of ideals in ordered Γ-semirings and the relations between them. We characterize $m-k$ ideals using derivation of ordered Γ-semirings and prove that every ideal in a mono regular ordered Γ-semiring is a prime ideal and field ordered Γ-semiring is simple.

Keywords: ordered Γ-semiring, integral ordered Γ-semiring, regular ordered Γ-semiring, mono ordered Γ-semiring, prime ideal, maximal ideal, k-ideal, $m-k$ ideal, filter, irreducible ideal, strongly irreducible ideal.

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1. Introduction

In 1995, Rao [9, 10, 11, 12] introduced the notion of Γ-semiring as a generalization of Γ-ring, ternary semiring and semiring. The notion of semiring was introduced by an American mathematician Vandiver [23] in 1934. The non trivial example of semiring first appeared in the work of German mathematician Richard Dedikind in 1894 in connection with the study of algebra of ideals of a commutative ring. A semiring is an algebraic structure with two associative binary operations where one of them distributes over the other. In particular, if I is the unit interval on the real line then $(I, \max, \min)$ is a semiring in which 0 is the additive identity and 1 is the multiplicative identity. Though semiring is a generalization of a ring, ideals of semiring do not coincide with ring ideals. Henriksen [5] defined k-ideals in semirings to obtain analogous of ring results for semiring. In structure, semirings lie between semigroups and rings. The results which hold in rings but not in semigroups hold in semirings, since semiring is a generalization of a ring. The study of rings shows that multiplicative structure of ring is an independent of additive structure whereas in semiring multiplicative structure is not independent of additive structure. The additive structure and the multiplicative structure of a semiring play an important role in determining the structure of a semiring. The theory of rings and theory of semigroups have considerable impact on the development of theory of semirings. Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in graph theory, optimization theory, in particular for studying automata,coding theory and formal languages. Semiring theory has many applications in other branches. Ahn *et al.* [2, 3] studied ideals, r-ideals in incline algebras. Rao *et al.* [19, 20] studied derivations of ordered Γ-semirings and Γ-incline.

The notion of Γ-ring was introduced by Nobusawa [21] as a generalization of ring in 1964. Sen [22] introduced the notion of Γ-semigroup in 1981. The notion of ternary algebraic system was introduced by Lehmer [7] in 1932, Lister [8] introduced the notion of ternary ring. Dutta and Kar [4] introduced the notion of ternary semiring which is a generalization of ternary ring and semiring. After the paper [9] was published, many mathematicians obtained interesting results on Γ-semirings. Rao and Venkateswarlu [17] introduced the notion of regular Γ-incline and field Γ-semiring. Jagatap and Pawar [6] studied quasi-ideals and minimal quasi-ideals in semirings. Rao [13, 15, 16] introduced bi-quasi-ideals in semirings, bi-interior ideals of semigroups, bi-quasi-ideals and fuzzy bi-quasi-ideals in Γ-semigroups. The notion of ideal was introduced by Dedekind for the theory of algebraic numbers and it was generalized by Noether for associative rings. The one and two sided ideals were introduced by her, are still central concepts in ring theory. In this paper, we introduce the notion of prime ideal, maximal ideal, filter, irreducible ideal, stronngly irreducible ideal in ordered Γ-semirings. We study the properties of ideals in ordered Γ-semirings. We prove that every ideal in a mono regular ordered Γ-semiring is a prime ideal and if an ideal is a $m-k$ ideal then ideal is a maximal ideal.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. Let $(M,+)$ and $(\Gamma,+)$ be commutative semigroups. Then we call M as a Γ-semiring, if there exists a mapping $M\times\Gamma\times M\to M$ is written (x,α,y) as $x\alpha y$ such that it satisfies the following axioms for all $x,y,z\in M$ and $\alpha,\beta\in\Gamma$

(i) $x\alpha(y+z) = x\alpha y + x\alpha z$

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z$

(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring R is a Γ-semiring with $\Gamma = R$ and ternary operation $x\gamma y$ as the usual semiring multiplication.

Example 2.2. Let S be a semiring and $M_{p,q}(S)$ denotes the additive abelian semigroup of all $p \times q$ matrices with identity element whose entries are from S. Then $M_{p,q}(S)$ is a Γ-semiring with $\Gamma = M_{p,q}(S)$ ternary operation is defined by $x\alpha z = x\ (\alpha^t)z$ as the usual matrix multiplication, where α^t denotes the transpose of the matrix α; for all x, y and $\alpha \in M_{p,q}(S)$.

Definition 2.3. Let M be a Γ-semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.4. In a Γ-semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M$, $\alpha \in \Gamma$ such that $b\alpha a = 1$ $(a\alpha b = 1)$.

Definition 2.5. In a Γ-semiring M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M$, $\alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

Definition 2.6. In a Γ-semiring M, an element $u \in M$ is said to be unit if there exist $a \in M$ and $\alpha \in \Gamma$ such that $a\alpha u = 1 = u\alpha a$.

Definition 2.7. A Γ-semiring M is said to be simple Γ-semiring if it has no proper ideals of M.

Definition 2.8. A non zero element a in a Γ-semiring M is said to be zero divisor if there exist non zero element $b \in M$, $\alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 0$.

Definition 2.9. A Γ-semiring M is said to be field Γ-semiring if M is a commutative Γ-semiring with unity 1 and every non zero element of M is invertible.

Definition 2.10. A Γ-semiring M with unity 1 and zero element 0 is called an integral Γ-semiring if it has no zero divisors.

Definition 2.11. A Γ-semiring M is said to be hold cancellation laws if $a\alpha b = a\alpha c$, $(b\alpha a = c\alpha a)$ where $a, b, c \in M$, $\alpha \in \Gamma$, then $b = c$.

Definition 2.12. Let M be a Γ-semiring is called a pre -integral Γ-semiring if M holds cancellation laws.

Example 2.13. Let M be a set of all rational numbers and Γ be a set of all natural numbers are commutative semigroups with respedt to usual addition. Define the mapping $M \times \Gamma \times M \Rightarrow M$ by $a\alpha b$ as usual multiplication for all $a, b \in M$, $\alpha \in \Gamma$. Then M is a field Γ-semiring.

Definition 2.14. A Γ-semiring M is called an ordered Γ-semiring if it admits a compatible relation $\leq$, i.e., $\leq$ is a partial ordering on M satisfies the following conditions. If $a \leq b$ and $c \leq d$ then

(i) $a + c \leq b + d$

(ii) $a\alpha c \leq b\alpha d$

(iii) $c\alpha a \leq d\alpha b$, for all $a, b, c, d \in M$, $\alpha \in \Gamma$.

Definition 2.15. An ordered Γ-semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M$, $\alpha \in \Gamma$.

Definition 2.16. Let M be an ordered Γ-semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.17. An ordered Γ-semiring M is said to be commutative Γ-semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.18. A non-empty subset A of an ordered Γ-semiring M is called a Γ-subsemiring M if $(A, +)$ is a subsemigroup of $(M, +)$ and $a\alpha b \in A$ for all $a, b \in A$ and $\alpha \in \Gamma$.

Definition 2.19. A non-empty subset A of an ordered Γ-semiring. M is called a left (right) ideal of an ordered Γ-semiring M if A is closed under addition, $M\Gamma A \subseteq A$ $(A\Gamma M \subseteq A)$ and if for any $a \in M$, $b \in A$, $a \leq b$ then $a \in A$.

Definition 2.20. A non-empty subset A of an ordered Γ-semiring M is called a k-ideal if A is an ideal and $x \in M$, $x + y \in A$, $y \in A$ then $x \in A$.

Definition 2.21. An ordered Γ-semiring M is said to be simple if it has no proper ideals.

Definition 2.22. Let M be an ordered Γ-semiring. An element $a \in M$ is said to be idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α idempotent.

Definition 2.23. Let M be an ordered Γ-semiring. Every element of M is an idempotent of M then M is said to be idempotent ordered Γ-semiring.

Definition 2.24. A non-zero element a in an ordered Γ-semiring M is said to be zero divisor if there exists non- zero element $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 0$.

Definition 2.25. An ordered Γ-semiring M with unity 1 and zero element 0 is called an integral ordered Γ-semiring if it has no zero divisors.

Definition 2.26. An ordered Γ-semiring M is said to be totally ordered Γ-semiring M if any two elements of M are comparable.

Definition 2.27. Let M and N be ordered Γ-semirings. A mapping $f : M \to N$ is called a homomorphism if

(i) $f(a + b) = f(a) + f(b)$,

(ii) $f(a\alpha b) = f(a)\alpha f(b)$, for all $a, b \in M$, $\alpha \in \Gamma$.

Definition 2.28. An ordered Γ-semiring M is said to be zero sum free ordered Γ-semiring if $x + y = 0 \Rightarrow x = 0$ and $y = 0$, for all $x, y \in M$.

Definition 2.29. In an ordered Γ-semiring M

(i) semigroup $(M, +)$ is said to be positively ordered if $a \leq a + b$ and $b \leq a + b$ for all $a, b \in M$,

(ii) semigroup $(M, +)$ is said to be negatively ordered if $a + b \leq a$ and $a + b \leq b$ for all $a, b \in M$,

(iii) Γ-semigroup M is said to be positively ordered if $a \leq a\alpha b$ and $b \leq a\alpha b$ for all $\alpha \in \Gamma, a, b \in M$,

(iv) Γ-semigroup M is said to be negatively ordered if $a\alpha b \leq a$ and $a\alpha b \leq b$ for all $\alpha \in \Gamma, a, b \in M$.

Definition 2.30. Let M be an ordered Γ-semiring. If a mapping $d : M \to M$ satisfies the following conditions

(i) $d(x + y) = d(x) + d(y)$,

(ii) $d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$,

then d is called a derivation of M.

3. IDEALS IN ORDERED Γ-SEMIRINGS

In this section, we introduce the notion of k-ideal, $m - k$ ideal, prime ideal, maximal ideal, filter, irreducible ideal, strongly irreducible ideal and homomorphism in ordered Γ-semiring. Throughout this paper,if $a \leq b$ then $a + b = b$ for all $a, b \in M$.

Definition 3.1. A Γ-subsemiring I of an ordered Γ-semiring M is called an ideal (filter) if it is a lower (upper) set, i.e., for any $x \in I$, $y \in M$ and $y \leq x \Rightarrow y \in I$. ($x \in F$, $y \in M$ and $x \leq y \Rightarrow y \in F$.)

Definition 3.2. A proper ideal P of an ordered Γ-semiring M is said to be prime ideal if for all $x, y \in M$, $\alpha \in \Gamma$, $x\alpha y \in P \Rightarrow x \in P$ or $y \in P$.

Definition 3.3. An ideal K of an ordered Γ-semiring M is said to be maximal ideal if $K \neq M$ and for every ideal I of M with $K \subseteq I \subseteq M$, then either $I = K$ or $I = M$.

Definition 3.4. A proper ideal I of an ordered Γ-semiring M is said to be irreducible ideal if $I = A \cap B$ then $I = A$ or $I = B$.

Example 3.5. Let $M = [0, 1]$ and $\Gamma = N$. A binary operation $+$ is defined as $a + b = \max\{a, b\}$, for all $a, b \in M$, $x + y = \max\{x, y\}$, for all $x, y \in N$ and ternary operation is defined as $x\gamma y = x\gamma y$ (usual product), for all $x, y \in M$ and $\gamma \in N$. Then M is an ordered Γ-semiring M with ususal ordering. All ideals of M are closed intervals, $[0, a]$ for some $a \in M$. Let $I = [0, 0.2]$. Then I is an irreducible ideal but not a prime ideal.

Definition 3.6. An ideal I of an ordered Γ-semiring M is strongly irreducible ideal if for ideals J and K of M, $J \cap K \subseteq I$ then $J \subseteq I$ or $K \subseteq I$.

Definition 3.7. An ideal I of an ordered Γ-semiring M is said to be k-ideal if $x + y \in I, x \in M$ and $y \in I$ then $x \in I$.

Definition 3.8. An ideal I of an ordered Γ-semiring M is said to be $m - k$ ideal if $x\alpha y \in I$, $x \in I$, $1 \neq y \in M$ and $\alpha \in \Gamma$ then $y \in I$.

Theorem 3.9. *Every $m - k$ ideal of an ordered Γ-semiring M is a k-ideal of M.*

Proof. Let I be a $m - k$ ideal of an ordered Γ-semiring M. Suppose $x + y \in I$, $x \in I$, $y \in M$ and $\alpha \in \Gamma$, then $(x + y)\alpha y \in I$. Therefore $y \in I$, since I is a $m - k$ ideal. Hence I is a k-ideal of M. ∎

Converse of the theorem need not be true.

Example 3.10. Let M be a the set of all non-negative integers and $\Gamma = \mathcal{N}$ be additive abelian semigroups. Tennary operation is defined as $(x, \alpha, y) \to x\alpha y$, usual multiplication of integers. Then M is an ordered Γ-semiring. A subset $I = 3M \setminus \{3\}$ of M is an ideal of M but not a k-ideal of M.

Example 3.11. Let M be the set of all natural numbers. Then $(M, \max, \min)$ with usual ordering is an ordered semiring. If $\Gamma = M$, then M is an ordered Γ-semiring. If $I_n = \{1, 2, \cdot, \cdot, \cdot, n\}$ then I_n forms a k-ideal but not $m - k$ ideal of ordered Γ-semiring.

Example 3.12. Let $\mathcal{N}$ be set of all non-negative integers and

$$M = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \mid a, b, c \in \mathcal{N} \right\}, \ \Gamma = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \mid a, c \in \mathcal{N} \right\}$$

be additive abeliean semigroups. Ternary operation is defined as $(x, \alpha, y) \to x\alpha y$, usual matrix multiplication for all $x, y \in M$ and $\alpha \in \Gamma$. Let $A = (a_{ij})$ and $B = (b_{ij}) \in M$. We define $A \subseteq B$ if and only if $a_{ij} \leq b_{ij}$, for all i, j. Then M is an ordered Γ-semiring. Define a derivation $d : M \longrightarrow M$ by

$$d \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}, \text{ for all } \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in M.$$

And define $Ker\ d = \left\{ A \mid A \in M \text{ and } d(A) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$. Then $Ker\ d$ is a $m - k$ ideal of an ordered Γ-semiring M.

Theorem 3.13. *Let I be a Γ-subsemiring of an ordered Γ-semiring M in which semigroup $(M, +)$ is band. Then I is an ideal of M if and only if I is a k-ideal of M.*

Proof. Let I be an ideal of an ordered Γ-semiring M and $x + y \in I$, $y \in I$.

$$\begin{aligned} x + y &= (x + x) + y \\ &= x + (x + y) \\ \Rightarrow x &\leq x + y. \end{aligned}$$

Therefore, by definition of ideal, $x \in I$. Hence I is a k-ideal.

Conversely suppose that I is a k-ideal of an ordered Γ-semiring M. Let $y \in M$, $x \in I$ and $y \leq x$.

$$\begin{aligned} &\Rightarrow y + x = x \\ &\Rightarrow y + x \in I \\ &\Rightarrow y \in I, \text{ since } I \text{ is a } k\text{-ideal of an ordered } \Gamma\text{-semiring } M. \end{aligned}$$

Hence I is an ideal of an ordered Γ-semiring M. ■

Theorem 3.14. *In an ordered Γ-semiring M, every maximal ideal of M is irreducible ideal of M.*

Proof. Let S be a maximal ideal of an ordered Γ-semiring M. Suppose S is not irreducible and $S = U \cap V$.

$$\begin{aligned} &\Rightarrow \ S \neq U \text{ and } S \neq V \\ &\Rightarrow \ S \subset U \subset M \text{ and } S \subset V \subset M. \end{aligned}$$

Which is a contradiction. Hence S is irreducible ideal of an ordered Γ-semiring M. ■

Theorem 3.15. *Let I be an ideal of an ordered Γ-semiring M.*

(i) *If I is a prime ideal then I is a strongly irreducible ideal.*

(ii) *If I is a strongly irreducible ideal then I is an irreducible ideal.*

Proof. Let I be an ideal of an ordered Γ-semiring M.

(i) Suppose I is a prime ideal, J and K are ideals of an ordered Γ-semiring M such that $J \cap K \subseteq I$. Then $J\Gamma K \subseteq I \Rightarrow J \subseteq I$ or $K \subseteq I$, since I is a prime ideal. Hence I is a strongly irreducible ideal.

(ii) Suppose I is a strongly irreducible ideal, J and K are ideals of an ordered Γ-semiring M such that $J \cap K = I$. Then

$$\begin{aligned} J \cap K &\subseteq I \\ \Rightarrow J \subseteq I &\text{ or } K \subseteq I \\ \text{Hence } J = I &\text{ or } K = I. \end{aligned}$$

Therefore I is an irreducible ideal of M. ■

Corollary 3.16. *Let M be an ordered Γ-semiring. If I is a prime ideal of M then I is an irreducible ideal of M.*

The following theorem proof is a straightforward verification.

Theorem 3.17. *If F is a non-empty subset of an ordered Γ-semiring M, then the following are equivalent*

(i) *F is a filter*

(ii) *$a + b \in F$, for all $a \in F$ and $b \in M$.*

Theorem 3.18. *Let M be an ordered Γ-semiring in which semigroup $(M, +)$ is positively ordered.Then F is a filter of an ordered Γ-semiring M if and only if F^c is a M if and only if F^c is a prime ideal of an ordered Γ-semiring M.*

Proof. Let F be a filter of an ordered Γ-semiring M and $a, b \in F^c$. Then $a, b \notin F$

$$\begin{aligned} &\Rightarrow a\alpha b \notin F \text{ and } a + b \notin F, \text{ for all } \alpha \in \Gamma \\ &\Rightarrow a\alpha b \in F^c \text{ and } a + b \in F^c, \text{ for all } \alpha \in \Gamma. \\ \text{Let } &a, b \in M \text{ and } a\alpha b \in F^c, \ \alpha \in \Gamma. \\ \text{Suppose } &a, b \notin F^c \\ &\Rightarrow a, b \in F \\ &\Rightarrow a\alpha b \in F, \alpha \in \Gamma. \end{aligned}$$

Which is a contradicts to our assumption. Therefore F^c is a prime ideal of an ordered Γ-semiring M.

Conversely suppose that F^c is a prime ideal of an ordered Γ-semiring M. Let $a, b \in F$, $\alpha \in \Gamma$. Then $a \leq a + b$. If $a + b \in F^c$, then $a \in F^c$. Hence $a + b \notin F^c$. Therefore $a + b \in F$. If $a\alpha b \notin F$.

$$\begin{aligned}
&\Rightarrow a\alpha b \in F^c \\
&\Rightarrow a \text{ or } b \in F^c, \text{ which is a contradiction.} \\
\text{Hence } & a\alpha b \in F. \\
\text{Let } a \in F,\ & a \leq b,\ b \in M. \text{ Suppose } b \notin F. \\
&\Rightarrow b \in F^c \\
&\Rightarrow a \in F^c,
\end{aligned}$$

which is a contradiction. Therefore $b \in F$. Hence F is a filter of an ordered Γ-semiring M. ■

Theorem 3.19. *Let $f : K \to L$ be a homomorphism of ordered Γ-semirings. If J is an ideal of L then $f^{-1}(J)$ is an ideal of an ordered Γ-semiring K.*

Proof. Suppose J is an ideal of L, $f : K \to L$ be a homomorphism of ordered Γ-semirings and $x, y \in f^{-1}(J)$, $\alpha \in \Gamma$.

$$\begin{aligned}
&\Rightarrow f(x),\ f(y) \in J \\
&\Rightarrow f(x) + f(y) = f(x + y) \in J \\
&\Rightarrow x + y \in f^{-1}(J) \\
x, y \in f^{-1}(J) &\Rightarrow f(x), f(y) \in J \\
&\Rightarrow f(x)\alpha f(y) \in J \\
&\Rightarrow f(x\alpha y) \in J \\
&\Rightarrow x\alpha y \in f^{-1}(J).
\end{aligned}$$

Hence $f^{-1}(J)$ is an ordered Γ-subsemiring of K. Let $x \in K$, $y \in f^{-1}(J)$ such that $x \leq y$.

$$\begin{aligned}
&\Rightarrow x + y = y \\
&\Rightarrow f(x + y) = f(y) \\
&\Rightarrow f(x) + f(y) = f(y) \in J \\
&\Rightarrow f(x) \leq f(y) \\
&\Rightarrow f(x) \in J \\
&\Rightarrow x \in f^{-1}(J).
\end{aligned}$$

Hence $f^{-1}(J)$ is an ideal of an ordered Γ-semiring K. ■

Theorem 3.20. *Let M be an ordered Γ-semiring with unity 1 and zero element 0. If I is an ideal containing a unit element then $I = M$.*

Proof. Let I be an ideal of an ordered Γ-semiring M containing a unit element u and $x \in M$. Then there exists $\alpha \in \Gamma$, such that $x\alpha 1 = x$. Since I is an ideal, $x\alpha u \in I$. Since u is a unit, there exist $\delta \in \Gamma$, $t \in M$ such that $u\delta t = 1$.

$$\Rightarrow x\alpha u\delta t = x\alpha 1 = x$$
$$\Rightarrow x \in I.$$

Hence $I = M$. ■

Theorem 3.21. *A field ordered Γ-semiring M is simple.*

Proof. Let I be a proper ideal of field ordered Γ-semiring M. Every nonzero element of I is a unit. By Theorem 3.20, we have $I = M$. Hence field ordered Γ-semiring M is simple. ■

Theorem 3.22. *Let M be an ordered Γ-semiring. If I is a $m-k$ ideal of M, then I is a maximal ideal of M.*

Proof. Let I be a $m-k$ ideal of an ordered Γ-semiring M. Suppose J is an ideal of M such that $I \subseteq J$, $x \in J$, $y \in I$ and $\alpha \in \Gamma$. Therefore $x\alpha y \in I$. Then $x \in I$, since I is a $m-k$ ideal of M. Therefore $I = J$. Hence $m-k$ ideal I of M is maximal ideal. ■

The following theorems are characterizations of $m-k$ ideal of an ordered Γ-semiring M.

Theorem 3.23. *Let d be a derivation of an ordered Γ-semiring M, in which semigroup $(M,+)$ is positively ordered, right cancellative, band and Γ-semigroup is left cancellative.*

Define a set $Fix_d(M) = \{x \in M / d(x) = x\}$. Then $Fix_d(M)$ is a k ideal and a $m-k$ ideal of M.

Proof. Let d be a derivation of M. Suppose $x, y \in Fix_d(M)$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} d(x) &= x, d(y) = y \\ d(x+y) &= d(x) + d(y) = x + y. \\ \text{Therefore } x + y &\in Fix_d(M) \\ d(x\alpha y) &= d(x)\alpha y + x\alpha d(y) \\ &= x\alpha y + x\alpha y \\ &= x\alpha y. \end{aligned}$$

Therefore $Fix_d(M)$ is a Γ-subsemiring of M.
Suppose $x \leq y$ and $y \in Fix_d(M)$.

$$\begin{aligned} & x \leq y \\ \text{Then } & x + y = y \\ & \Rightarrow d(x+y) = x + y \\ & \Rightarrow d(x) + d(y) = x + y \\ & \Rightarrow d(x) + y = x + y. \\ \text{Therefore } & d(x) = x. \end{aligned}$$

Therefore $d(x) = x$. Hence $Fix_d(M)$ is a k-ideal of M. Suppose $x\alpha y \in Fix_d(M)$, $x \in Fix_d(M)$ and $\alpha \in \Gamma$. Then $d(x\alpha y) = x\alpha y$

$$\begin{aligned} & \Rightarrow d(x)\alpha y + x\alpha d(y) = x\alpha y \\ & \Rightarrow x\alpha y + x\alpha d(y) = x\alpha y \\ & \Rightarrow x\alpha[y + d(y)] = x\alpha y \\ & \Rightarrow y + d(y) = y \\ & \Rightarrow d(y) \leq y + d(y) = y \end{aligned}$$

we have $y \leq d(y)$. Hence $d(y) = y$, $y \in Fix_d(M)$. Hence $Fix_d(M)$ is a $m - k$ ideal of M. ■

Definition 3.24. Let d be a derivation of an ordered Γ-semiring M. If $x \leq y$ $\Rightarrow d(x) \leq d(y)$, for all $x, y \in M$, then d is called an isotone derivation.

Theorem 3.25. *Let d be an isotone derivatipn of an ordered Γ-semiring M. Define $ker d = \{x \in M/d(x) = 0\}$. Then $ker d$ is a k-ideal of an Γ-semiring M.*

Proof. Let $x, y \in ker\ d$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} d(x) = 0, d(y) = 0 \\ d(x+y) = d(x) + d(y) = 0. \end{aligned}$$

Therefore $x + y \in ker\ d$.

$$\begin{aligned} d(x\alpha y) &= d(x)\alpha y + x\alpha d(y) \\ &= 0\alpha y + x\alpha 0 = 0. \end{aligned}$$

Therefore $x\alpha y \in ker\ d$. Suppose $y \in ker\ d, x \in M$ and $x \leq y$. Then

$$\Rightarrow d(x) \leq d(y) = 0$$

$$\Rightarrow d(x) = 0.$$

Hence $ker\ d$ is an ideal. Suppose $x+y \in ker\ d$ and $y \in ker\ d$. Then $d(x+y)=0$

$$\begin{aligned} &\Rightarrow d(x)+d(y)=0 \\ &\Rightarrow d(x)=0 \\ &\Rightarrow x \in ker\ d. \end{aligned}$$

Hence $ker\ d$ is a k-ideal of an ordered Γ-semiring M. ∎

Theorem 3.26. *Let d be an isotone derivation of an integral ordered Γ-semiring M in which semigroup $(M,+)$ is positively ordered. Define $ker\ d = \{x \in M / d(x)=0\}$. Then $ker\ d$ is a $m-k$ ideal of M.*

Proof. By Theorem [3.25], $ker\ d$ is an ideal. Let $0 \neq y \in ker\ d$, $x \in M$ $\alpha \in \Gamma$ and $x\alpha y \in ker\ d$, then $d(x\alpha y)=0$

$$\begin{aligned} &\Rightarrow d(x)\alpha y + x\alpha d(y) = 0 \\ &\Rightarrow d(x)\alpha y = 0 \\ &\Rightarrow d(x) = 0, \end{aligned}$$

since M is an integral ordered Γ-semiring. Therefore $ker\ d$ is a $m-k$ ideal of M. ∎

Let M be an ordered Γ-semiring. $E[+]$ denotes the set $\{x \in M \mid x+x=x\}$.

Theorem 3.27. *Let M be a pre integral ordered Γ-semiring in which $(M,+)$ is cancellative semigroup. If $E[+] \neq \emptyset$, then $E[+]$ is a k ideal and a $m-k$ ideal of an ordered Γ-semiring M.*

Proof. Let $x \in E[+]$, $y \in M$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} x &= x+x \\ \Rightarrow x\alpha y &= (x+x)\alpha y \\ &= x\alpha y + x\alpha y. \end{aligned}$$

Therefore $x\alpha y \in E[+]$. Similarly $y\alpha x \in E[+]$. Suppose $x, y \in E[+]$. Then

$$\begin{aligned} &x+x=x, y+y=y \\ &\Rightarrow (x+y)+(x+y) = (x+x)+(y+y) = x+y \\ &\Rightarrow x+y \in E[+]. \end{aligned}$$

Suppose $x \leq y$, $y \in E[+]$. Then $x+y=y$

$$\begin{aligned} &\Rightarrow x+x+y = x+y \\ &\Rightarrow x+x = x. \end{aligned}$$

Therefore $x \in E[+]$. Hence $E[+]$ is an ideal of an ordered Γ-semiring. Suppose $x, x+y \in E[+]$. Then

$$\begin{aligned} & x+x=x, x+y+x+y=x+y \\ \Rightarrow\ & (x+y)+(x+y)=x+y \\ \Rightarrow\ & (x+x)+(y+y)=x+y \\ \Rightarrow\ & x+(y+y)=x+y \\ \Rightarrow\ & y+y=y \\ \Rightarrow\ & y \in E[+]. \end{aligned}$$

Hence $E[+]$ is a k-ideal of an ordered Γ-semiring M. Suppose $x\alpha y \in E[+]$, $x \in E[+]$ and $\alpha \in \Gamma$. Then

$$\begin{aligned} & x\alpha y + x\alpha y = x\alpha y \\ \Rightarrow & x\alpha(y+y) = x\alpha y \\ \Rightarrow & y+y=y \\ \Rightarrow & y \in E[+]. \end{aligned}$$

Hence $E[+]$ is a $m-k$ ideal of M. ∎

4. IDEALS IN QUOTIENT ORDERED Γ-SEMIRING

In this section, we introduce the notion of a quotient ordered Γ-semiring and study the properties of ideals of quotient ordered Γ-semiring.

Suppose that I is an ideal of an ordered Γ-semiring M with zero element 0. We define a relation' $\sim$' on an ordered Γ-semiring M by '$x \sim y$' if and only if $x+i_1 = y+i_2$ for some $i_1, i_2 \in I$, $x, y \in M$. Obviously '$\sim$' is an equivalence relation.

Let M be an ordered Γ-semiring. The equivalence class of $x \in M$ is determined by an ideal I is denoted by $x+I$. The set of all equivalence classes $\{x+I \mid x \in M\}$ is denoted by M/I. We define two operations on M/I by

(i) $(x+I)+(y+I) = x+y+I$

(ii) $(x+I)\alpha(y+I) = x\alpha y + I$, for all $x, y \in M$, $\alpha \in \Gamma$.

Then M/I is an ordered Γ-semiring.The ordered Γ-semiring M/I is called a quotient ordered Γ-semiring. If M is a commutative ordered Γ-semiring M then M/I is a commutative ordered Γ-semiring. Define $\phi : M \to M/I$ by $\phi(x) = x+I$, for all $x \in M$. Clearly ϕ is a homomorphism. We define the order relation on an ordered Γ-semiring M/I by $a+I \leq b+I$ if and only if $a \leq b$, i.e., $a+b=b$. Obviously $\leq$ is a partial order relation of M/I.

Theorem 4.1. *Let M be an ordered Γ-semiring. If $a, b \in M$, $a \sim b$, then $a\alpha x \sim b\alpha x$ and $a + x \sim b + x$, for all $x \in M$, $\alpha \in \Gamma$. Then the relation '$\sim$' is a congruence relation.*

Proof. Let M be an ordered Γ-semiring. If $a \sim b$, $\alpha \in \Gamma$, $a, b, x \in M$ there exist $i_1, i_2 \in I$ such that $a + i_1 = b + i_2$

$$\begin{aligned} &\Rightarrow (a + i_1)\alpha x = (b + i_2)\alpha x \\ &\Rightarrow a\alpha x + i_1\alpha x = b\alpha x + i_2\alpha x \\ &\Rightarrow a\alpha x \sim b\alpha x \\ \text{since } &\; i_1\alpha x,\ i_2\alpha x \in I. \\ \text{Now } &\; a + i_1 = b + i_2 \\ &\Rightarrow a + i_1 + x = b + i_2 + x \\ &\Rightarrow (a + x) + i_1 = (b + x) + i_2 \\ &\Rightarrow a + x \sim b + x. \end{aligned}$$

Hence relation $\sim$ on ordered Γ-semiring M is congruence relation. ■

Theorem 4.2. *Let I be an ideal of an ordered Γ-semiring M in which semigroup $(M, +)$ is positively ordered and $a \in I$. Then $a + I = b + I$ for every $b \in M$ if and only if $b \in I$. In particular $c + I = I$ if and only if $c \in I$.*

Proof. Let $a + I = b + I$, then $a + u = b + v$ for some $u, v \in I$.

$$\begin{aligned} &\Rightarrow b + v \in I \\ &\Rightarrow b \leq b + v \in I \\ &\Rightarrow b \in I. \end{aligned}$$

Converse is obvious. ■

The proofs of the following theorems are similar to incline Theorems 2.4 and 2.7 in [2].

Theorem 4.3. *If I and J are any ideals of an ordered Γ-semiring M and $I \subseteq J$, then*

(i) *I is also an ideal of the Γ-subsemiring J.*

(ii) *J/I is an ideal of the quotient ordered Γ-semiring M/I.*

Theorem 4.4. *Let I be an ideal of an ordered Γ-semiring M. If A is an ideal of a quotient ordered Γ-semiring M/I then $\phi^{-1}(A)$ is an ideal of an ordered Γ-semiring M containing I where ϕ is a natural homomorphism from M onto M/I.*

Theorem 4.5. *Let M be an ordered Γ-semiring M with unity 1 and zero element 0. If $a \in M$ then $0 \leq a \leq 1$.*

Theorem 4.6. *Let M/I be a quotient ordered Γ-semiring. Then*

(i) *M/I is a zero sum free quotient ordered Γ-semiring.*

(ii) *I is the least element of M/I.*

(iii) *$1+I$ is the greatest element of M/I.*

(iv) *$I \leq a+I \leq 1+I$, for all $a \in M$.*

Proof. Let M/I be a quotient ordered Γ-semiring.

(i) Suppose $(a+I),(b+I) \in M/I$ such that $(a+I)+(b+I)=I$

$$\Rightarrow a+b+I=I$$
$$\Rightarrow a+b \in I.$$

We have $a \leq a+b,\ b \leq a+b$

$$\Rightarrow a,\ b \in I.$$

Therefore $a+I+b+I=I$

$$\Rightarrow a+I=I \text{ and } b+I=I.$$

Hence M/I is a zero sum free quotient ordered Γ-semiring.

(ii) Let $a,b \in M$ and $a \leq b$.

$$\begin{aligned} &\Rightarrow a+b=b \\ &\Rightarrow a+b+I=b+I \\ &\Rightarrow (a+I)+(b+I)=b+I \\ &\Rightarrow a+I \leq b+I. \\ \text{We have }\ &0 \leq a,\ \text{for all } a \in M. \\ &\Rightarrow 0+I \leq a+I \\ &\Rightarrow I \leq a+I,\ \text{for all } a+I \in M. \end{aligned}$$

Hence I is the least element of M/I.

(iii) We have $a \leq 1 \Rightarrow a+I \leq 1+I$, for all $a \in M$. Hence $1+I$ is the greatest element of M/I.

(iv) Obvious. ■

Theorem 4.7. *If $a+I$ is a regular element of an ordered Γ-semiring M/I in which Γ-semigroup M/I is negatively ordered, then there exist $x+I \in M/I$, $\alpha,\beta \in \Gamma$ such that $a+I=(a+I)\alpha(x+I)=(x+I)\beta(a+I)$.*

Proof. Suppose $a+I$ is a regular element of an ordered Γ-semiring M/I Then there exist $x+I \in M/I, \alpha, \beta \in \Gamma$ such that $a+I=(a+I)\alpha(x+I)\beta(a+I) \leq (a+I)\alpha(x+I) \leq a+I$. Therefore $(a+I)\alpha(x+I)=a+I$. Now $a+I = (a+I)\alpha(x+I)\beta(a+I) \leq (x+I)\beta(a+I) \leq (a+I)$. Therefore $a+I=(x+I)\beta(a+I)$. Hence $a+I=(a+I)\alpha(x+I)=(x+I)\beta(a+I)$. ∎

Theorem 4.8. *Let M/I be an ordered Γ-semiring in which Γ-semigroup M/I is negatively ordered.Then M/I is a regular ordered Γ-semiring if and only if M/I is an idempotent ordered Γ-semiring.*

Proof. Suppose M/I is a regular ordered Γ-semiring and $a+I \in M/I$. Since $a+I$ is a regular element, there exist $x+I \in M/I$, $\alpha, \beta \in \Gamma$ such that $a+I = (a+I)\alpha(x+I)\beta(a+I)$. By Theorem 4.7, $a+I=(a+I)\alpha(x+I)=(x+I)\beta(a+I)$. Now $a+I=(a+I)\alpha(x+I)\beta(a+I)=(a+I)\alpha(a+I)$. Therefore $a+I$ is an idempotent. Hence M/I is an idempotent ordered Γ-semiring.

Conversely suppose that $a+I$ is an $\alpha-$idempotent of M/I, $\alpha \in \Gamma$. $a+I = (a+I)\alpha(a+I) = (a+I)\alpha(a+I)\alpha(a+I)$. Hence M/I is a regular ordered Γ-semiring. ∎

Theorem 4.9. *Let M/I be a commutative ordered Γ-semiring in which Γ-semigroup M/I is negatively ordered and if $b+I$, $c+I \in M/I$ are α, β idempotents respectively, $\alpha, \beta \in \Gamma$, then $(b+I)\alpha(c+I)=(b+I)\beta(c+I)$.*

Proof. Let M/I be a commutative ordered Γ-semiring and $b+I$, $c+I \in M/I$ be α, β idempotents respectively, $\alpha, \beta \in \Gamma$. Then we have

$$\begin{aligned}
(b+I)\alpha(b+I) &= b+I, (c+I)\beta(c+I)=c+I. \\
\text{Now } (b+I)\alpha(c+I) &= \big((b+I)\alpha(b+I)\big)\alpha\big((c+I)\beta(c+I)\big) \\
&= (b+I)\alpha\big((b+I)\alpha(c+I)\beta(c+I)\big) \\
&= (b+I)\alpha\big((c+I)\alpha(b+I)\big)\beta(c+I) \\
&= \big((b+I)\alpha(c+I)\alpha(b+I)\beta(c+I)\big) \\
&\leq (b+I)\beta(c+I).
\end{aligned}$$

Similarly we can prove $(b+I)\beta(c+I) \leq (b+I)\alpha(c+I)$.

Hence $(b+I)\alpha(c+I)=(b+I)\beta(c+I)$. ∎

Theorem 4.10. *Let M/I be a commutative ordered Γ-semiring in which Γ-semigroup M/I is negatively ordered and $(b+I)\alpha(a+I)+(b+I)=(b+I)$ for all $b+I$, $a+I \in M/I$, $\alpha \in \Gamma$. If $b+I$, $c+I \in M/I$ are α, β idempotents respectively, $\alpha, \beta \in \Gamma$, $a+I \in M/I$, $(a+I)+(b+I)=(a+I)+(c+I)$ and $(b+I)\alpha(a+I)=(c+I)\beta(a+I)$, then $b+I=c+I$.*

Proof. Suppose M/I is a commutative ordered Γ-semiring and $b+I$, $c+I \in M/I$ are α, β idempotents respectively, $\alpha, \beta \in \Gamma$,

$$\begin{aligned} a+I \in M/I,\ (a+I)+(b+I) &= (a+I)+(c+I) \\ \text{and } (b+I)\alpha(a+I) &= (c+I)\beta(a+I). \\ \text{We have } (b+I)\alpha(b+I) &= (b+I), (c+I)\beta(c+I) = c+I. \\ \text{By Theorem 4.9, } (b+I)\alpha(c+I) &= (b+I)\beta(c+I). \end{aligned}$$

$$\begin{aligned} & (a+I)+(b+I) = (a+I)+(c+I) \\ \Rightarrow\ & (b+I)\alpha\big((a+I)+(b+I)\big) = (b+I)\alpha\big((a+I)+(c+I)\big) \\ \Rightarrow\ & (b+I)\alpha(a+I)+(b+I) = (b+I)\alpha(a+I)+(b+I)\alpha(c+I) \\ \Rightarrow\ & (b+I) = (c+I)\beta(a+I)+(b+I)\beta(c+I) \\ \Rightarrow\ & (b+I) = (a+I)\beta(c+I)+(b+I)\beta(c+I) \\ \Rightarrow\ & (b+I) = \big((a+I)+(b+I)\big)\beta(c+I) \\ \Rightarrow\ & (b+I) = \big((a+I)+(c+I)\big)\beta(c+I) \\ \Rightarrow\ & (b+I) = (a+I)\beta(c+I)+(c+I)\beta(c+I) \\ \Rightarrow\ & (b+I) = (a+I)\beta(c+I)+(c+I) \\ \Rightarrow\ & (b+I) = (c+I). \end{aligned}$$

Hence the theorem. ■

Theorem 4.11. *Let I and J be ideals of an ordered Γ-semiring M with $I \subseteq J$. Then following*

(i) *If $1+I \in J/I$ then $M/I = J/I$.*

(ii) *If $a+I$ is an invertible element of M/I with $a+I \in J/I$, then $M/I = J/I$.*

Proof. Let I and J be ideals of an ordered Γ-semiring M with $I \subseteq J$.

(i) Suppose $x+I \in M/I$ and $1+I \in J/I$. Then $(x+I)\alpha(1+I) \in J/I$, for all $\alpha \in \Gamma$.

$$\begin{aligned} &\Rightarrow x\alpha 1 + I \in J/I, \text{ for all } \alpha \in \Gamma \\ &\Rightarrow x+I \in J/I. \end{aligned}$$

Hence $M/I = J/I$.

(ii) Since $a+I$ is an invertible, there exist $\alpha \in \Gamma, b+I$ such that $(a+I)\alpha(b+I) = 1+I \Rightarrow 1+I \in J/I$. By (i) $M/I = J/I$. ■

The following proof of the theorem is similar to proof of Theorem 3.20 in [2].

Theorem 4.12. *Let M be an ordered Γ-semiring with unity 1 and zero element 0. Then ideal I is a maximal if and only if quotient ordered Γ-semiring M/I is simple.*

Theorem 4.13. *Let P be a proper ideal of a commutative ordered Γ-semiring M with unity. Then P is a maximal ideal if and only if M/P is an ordered field Γ-semiring.*

Proof. Let P be a maximal ideal of a commutative ordered Γ-semiring M with unity and $P \neq a + P \in M/P$.

$$\begin{aligned} &\Rightarrow a \notin P \\ &\Rightarrow P + M\Gamma a = M, \quad \text{by maximality of } P. \\ \text{There exist } & r \in M, \alpha \in \Gamma, \quad p \in P \text{ such that } p + r\alpha a = 1 \\ &\Rightarrow (r+P)\alpha(a+P) = 1 + P \\ &\Rightarrow a + P \text{ is invertible.} \end{aligned}$$

Hence M/P is an ordered field Γ-semiring.

Conversely suppose that M/P is an ordered field Γ-semiring and $P \subsetneq J$. Then there exists $b \in J \setminus P$ such that $P \neq b + P \in M/P \Rightarrow b + P$ is an invertible $\Rightarrow$ $\alpha \in \Gamma, c + P \in M/P$ and

$$\begin{aligned} &(b+P)\alpha(c+P) = 1 + P \\ \Rightarrow\ & b\alpha c + P = 1 + P \in J/P \\ \Rightarrow\ & J/P = M/P, \text{ by Theorem 4.17} \\ \Rightarrow\ & J = M. \end{aligned}$$

Hence the theorem. ∎

Theorem 4.14. *Let M be an ordered Γ-semiring. If I is an ideal of M and J is a strongly irreducible ideal of M with $I \subseteq J$, then J/I is a strongly irreducible ideal of M/I.*

Proof. Let M be an ordered Γ-semiring, I be an ideal of M and J be a strongly irreducible ideal of M with $I \subseteq J$. By Theorem 4.3, J/I is an ideal of the quotient ordered Γ-semiring M/I. Suppose K/I and H/I are ideals of M/I such that $K/I \cap H/I \subseteq J/I \Rightarrow K \cap H \subseteq J$. Since J is a strongly irreducible ideal of M/I.

$$\begin{aligned} &\Rightarrow K \subseteq J \text{ or } H \subseteq J \\ &\Rightarrow K/I \subseteq J/I \text{ or } H/I \subseteq J/I. \end{aligned}$$

Hence J/I is a strongly irreducible ideal of an ordered Γ-semiring M/I. ∎

Theorem 4.15. *Any commutative finite pre-integral quotient ordered* Γ*-semiring with unity is a quotient ordered field* Γ*-semiring.*

Proof. Let M/I be a commutative finite pre-integral quotient ordered Γ-semiring with unity. Suppose $M/I = \{a_1 + I, a_2 + I, \ldots, a_n + I\}$, $I \neq a + I \in M/I$ and $\alpha \in \Gamma$. Then $a\alpha a_1 + I$, $a\alpha a_2 + I, \ldots, a\alpha a_n + I$ are distinct elements in M/I, since $a\alpha a_i + I = a\alpha a_j + I \Rightarrow (a + I)\alpha(a_i + I) = (a + I)\alpha(a_j + I) \Rightarrow a_i + I = a_j + I$.

Since $1+I$ is an unity, there exists $a\alpha a_k + I \in M/I$ such that $a\alpha a_k + I = 1 + I$. Therefore $(a + I)\alpha(a_k + I) = 1 + I$. Hence the theorem. ■

Definition 4.16. Let M/I be an ordered Γ-semiring is said to be mono ordered Γ-semiring. If $a + I, c + I \in M/I$, $\alpha \in \Gamma$ and $a + I$ is α idempotent then $(a + I)\alpha(c + I) = (a + I) + (c + I)$.

Theorem 4.17. *If* M/I *is a mono regular ordered* Γ*-semiring in which* Γ*-semigroup* M/I *is negatively ordered and* $(b + I)\alpha(a + I) + (b + I) = (b + I)$ *for all* $b + I$, $a + I \in M/I$, $\alpha \in \Gamma$. *then* M/I *is a pre-integral ordered* Γ*-semiring.*

Proof. Let M/I be a mono regular ordered Γ-semiring and $a + I$, $b + I$, $c + I \in M/I$ and $\gamma \in \Gamma$.

$$\begin{aligned}
\text{Suppose }\ & (b + I)\gamma(a + I) = (c + I)\gamma(a + I),\\
& (b + I)\alpha(b + I) = (b + I), (c + I)\beta(c + I) = (c + I), \alpha, \beta \in \Gamma\\
\Rightarrow\ & \big((b + I)\alpha(b + I)\big)\gamma(a + I) = \big((c + I)\beta(c + I)\big)\gamma(a + I)\\
\Rightarrow\ & (b + I)\alpha\big((b + I)\gamma(a + I)\big) = (c + I)\beta\big((c + I)\gamma(a + I)\big)\\
\Rightarrow\ & (b + I) + (b + I)\gamma(a + I) = (c + I) + (c + I)\gamma(a + I).
\end{aligned}$$

Therefore $(b + I) = (c + I)$. Hence M/I is a pre-integral ordered Γ-semiring. ■

Theorem 4.18. *Every pre-integral ordered* Γ*-semiring* M/I *is an integral ordered* Γ*-semiring.*

Proof. Let M/I be a pre-integral ordered Γ-semiring. Suppose $(a+I)\alpha(b+I) = I$, $a + I, b + I \in M/I$, $\alpha \in \Gamma$ and $b + I \neq I$.

$$\begin{aligned}
&\Rightarrow (a + I)\alpha(b + I) = I\alpha(b + I)\\
&\Rightarrow a + I = I,\ \text{since } M/I \text{ is a pre-integral ordered } \Gamma\text{-semiring.}
\end{aligned}$$

Hence the theorem. ■

Theorem 4.19. *Let* M *be an ordered* Γ*-semiring with unity* 1 *and zero element* 0. *Then* P *is a prime ideal of an ordered* Γ*-semiring* M *if and only if quotient ordered* Γ*-semiring* M/P *is an integral ordered* Γ*-semiring.*

Proof. Suppose M/P is a quotient integral ordered Γ-semiring, P is a prime ideal of an ordered Γ-semiring M, $a+P, b+P \in M/P$, $\alpha \in \Gamma$ and $(a+P)\alpha(b+P) = P$.

$$\begin{aligned} &\Rightarrow a\alpha b + P = P \\ &\Rightarrow a\alpha b \in P \\ &\Rightarrow a \in P \text{ or } b \in P \\ &\Rightarrow a + P = P \text{ or } b + P = P. \end{aligned}$$

Therefore M/P is a quotient integral ordered Γ-semiring.

Conversely suppose that M/P is a quotient integral ordered Γ-semiring and $a\alpha b \in P$, $a, b \in M$, $\alpha \in \Gamma$.

$$\begin{aligned} &\Rightarrow a\alpha b + P = P \\ &\Rightarrow (a + P)\alpha(b + P) = P \\ &\Rightarrow a + P = P \text{ or } b + P = P \\ &\Rightarrow a \in P \text{ or } b \in P. \end{aligned}$$

Hence P is a prime ideal of an integral ordered Γ-semiring M. ■

Theorem 4.20. *Every ideal in a mono regular ordered* Γ*-semiring in which* Γ*-semigroup* M *is negatively ordered and* $b\alpha a + b = b$ *for all* $b, a \in M$, $\alpha \in \Gamma$, *is a prime ideal.*

Proof. Let M be a mono regular ordered Γ-semiring with unity 1 and zero element 0 and I be an ideal of M. Obviously M/I is a mono regular ordered Γ-semiring. By Theorem 4.17, M/I is a pre-integral ordered Γ-semiring. By Theorem 4.18, M/I is an integral ordered Γ-semiring. By Theorem 4.19, I is a prime ideal of an ordered Γ-semiring M. Hence every ideal of mono regular ordered Γ-semiring M is a prime ideal. ■

5. Conclusion

In this paper, we introduced the notion of k-ideal, $m - k$ ideal, prime ideal, maximal ideal, filter, irreducible ideal, strongly irreducible ideal in ordered Γ-semirings. We studied the properties of ideals in ordered Γ-semirings and the relations between them. We characterized $m-k$ ideals using derivation of ordered Γ-semiring and proved that every ideal in a mono regular ordered Γ-semiring is a prime ideal and field ordered Γ-semiring is simple.

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