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COMPLETELY ARCHIMEDEAN SEMIRINGS

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Abstract

In this paper we give a structural description of completely Archimedean semirings which is an extension of the structure theorem of completely Archimedean semigroups.

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1. Introduction

In 1984, Galbiati and Veronesi [1] studied completely π-regular semigroups in which every regular element is completely regular. The semigroups are named after them as GV-semigroups (semigroup of Galbiati-Veronesi). A GV-semigroup is characterized as a semilattice of completely Archimedean semigroups. In [8], Bogdanović and Milić studied nil-extensions of completely simple semigroups. They proved that a semigroup is a completely Archimedean semigroup if and only if it is a nil-extension of a completely simple semigroup. Again, a completely Archimedean semigroup is Archimedean and completely π-regular.

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From an algebraic point of view, semirings provide the most natural generalization of the theory of rings. The properties of semirings and their structural representations have been studied by many authors, for example, by Pastijn, Guo, Sen, Shum, Grillet and others. A special class of semirings, namely completely regular semirings play a very important role in semiring theory. The concept of completely regular semiring has been first introduced by Sen, Maity and Shum [4]. The authors have characterized a completely regular semiring as a b-lattice of completely simple semirings. In [9], Maity and Ghosh obtained that the idea of GV-semigroups and completely π-regular semigroups coincide when extended under semirings. The semirings are named as quasi completely regular semirings. In a quasi completely regular semiring, every additively regular element is completely regular. Naturally, a quasi completely regular semiring is a generalization of a completely regular semiring. Again, a quasi completely regular semiring is characterized as a b-lattice of completely Archimedean semirings. In [10], Maity and Ghosh proved that a semiring is a completely Archimedean semiring if and only if it is a nil-extension of a completely simple semiring. In [5], Sen, Maity and Weinert established that a semiring is a completely simple semiring if and only if it is isomorphic to a Rees matrix semiring. Thus a semiring is a completely Archimedean semiring if and only if it is a nil-extension of a Rees matrix semiring. In this paper, we give structural description of a completely Archimedean semiring using the structure theorem of a Rees matrix semiring. Structure theorem of a completely Archimedean semiring is an extension of the structure theorem of completely Archimedean semigroup. Structure theorem of aforesaid semigroup was given by Milić and Pavlović [11]. The preliminaries and prerequisites we need for this article are discussed in section 2. In section 3, we discuss our main result.

2. Preliminaries

A *semiring* $(S, +, \cdot)$ is a non-empty set S together with two binary operations '+' and '$\cdot$', respectively called addition and multiplication, such that the semigroup reducts $(S, +)$ and $(S, \cdot)$ are connected by ring like distributivity, that is, $a(b+c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in S$. An element a in a semiring S is said to be *infinite* [3] if and only if $a + x = a = x + a$ for all $x \in S$. Infinite element in a semiring is unique and is denoted by ∞. An infinite element ∞ in a semiring S having the property that $x \cdot \infty = \infty = \infty \cdot x$ for all $x(\neq 0) \in S$ is called *strongly infinite* [3]. An element a in a semiring $(S, +, \cdot)$ is said to be *additively regular* if there exists an element $x \in S$ such that $a = a + x + a$. Let a be an additively regular element in a semiring S. An element $y \in S$ satisfying $a + y + a = a$ and $y + a + y = y$ is called *additive inverse* of the element a. An element a in a semiring $(S, +, \cdot)$ is called *completely regular* [4] if there exists an element $x \in S$ such that $a = a+x+a$, $a+x = x+a$ and $a(a+x) = a+x$. We call a semiring $(S,+,\cdot)$ *completely regular* if every element of S is completely regular. A semiring $(S,+,\cdot)$ is called a *skew-ring* if its additive reduct $(S,+)$ is a group. An element a in a semiring $(S,+,\cdot)$ is said to be *additively quasi regular* if there exists a positive integer n such that na is additively regular. An element a in a semiring $(S,+,\cdot)$ is said to be *quasi completely regular* [9] if there exists a positive integer n such that na is completely regular. Naturally, a semiring $(S,+,\cdot)$ is said to be *quasi completely regular* if every element of S is quasi completely regular. An element a in a semigroup $(S,\cdot)$ is called an *idempotent* if $a^2 = a$. A semigroup is said to be a *band* if its every element is idempotent. A commutative band is called a *semilattice*. A semiring $(S,+,\cdot)$ is said to be a *band semilattice* (in short *b-lattice*) if $(S,\cdot)$ is a band and $(S,+)$ is a semilattice. A semiring is called an *idempotent semiring* if both $(S,\cdot)$ and $(S,+)$ are bands. An equivalence relation ρ on a semiring S is said to be a *congruence* on S if ρ is a congruence on both the semigroup reducts $(S,+)$ and $(S,\cdot)$, i.e., for $a,b,c \in S$, $a\,\rho\,b$ implies $(c+a)\,\rho\,(c+b)$, $(a+c)\,\rho\,(b+c)$, $ca\,\rho\,cb$ and $ac\,\rho\,bc$. A congruence ρ on a semiring S is called a *b-lattice congruence (idempotent semiring congruence)* if S/ρ is a b-lattice (resp. an idempotent semiring). A semiring S is called a *b-lattice (idempotent semiring) Y of semirings S_α* $(\alpha \in Y)$ if S admits a b-lattice congruence (resp. an idempotent semiring congruence) ρ on S such that $Y = S/\rho$ and each S_α is a ρ-class.

Throughout this paper, we always let $E^+(S)$ be the set of all additive idempotents of the semiring S. Also we denote the set of all additive inverse elements of an additively regular element a in a semiring $(S,+,\cdot)$ by $V^+(a)$. As usual, we denote the Green's relations on the semiring $(S,+,\cdot)$ by $\mathscr{L}$, $\mathscr{R}$, $\mathscr{D}$, $\mathscr{J}$ and $\mathscr{H}$ and correspondingly, the $\mathscr{L}$-relation, $\mathscr{R}$-relation, $\mathscr{D}$-relation, $\mathscr{J}$-relation and $\mathscr{H}$-relation on $(S,+)$ are denoted by $\mathscr{L}^+$, $\mathscr{R}^+$, $\mathscr{D}^+$, $\mathscr{J}^+$ and $\mathscr{H}^+$, respectively. In fact, the relations $\mathscr{L}^+$, $\mathscr{R}^+$, $\mathscr{D}^+$, $\mathscr{J}^+$ and $\mathscr{H}^+$ are all congruence relations on the multiplicative reduct $(S,\cdot)$. Thus if any one of these happens to be a congruence on $(S,+)$, it will be a congruence on the semiring $(S,+,\cdot)$. For any $a \in S$, we let H_a^+ be the $\mathscr{H}^+$-class in S containing a. We further denote the Green's relations on a quasi regular semigroup $(S,\cdot)$ by $\mathscr{L}^*, \mathscr{R}^*, \mathscr{D}^*, \mathscr{J}^*$ and $\mathscr{H}^*$. For other notations and terminologies not given in this paper, the reader is referred to [2] and [9].

Definition 2.1 [4]**.** A completely regular semiring $(S,+,\cdot)$ is called a completely simple semiring if any two elements of S are $\mathscr{J}^+$-related, i.e., $\mathscr{J}^+ = S \times S$.

Theorem 2.2 [5]**.** *Let R be a skew-ring, $(I,\cdot)$ and $(\Lambda,\cdot)$ are bands, such that $I \cap \Lambda = \{o\}$. Let $P = (p_{\lambda,i})$ be a matrix over R, $i \in I$, $\lambda \in \Lambda$ and assume*

A1. $p_{\lambda,o} = p_{o,i} = 0$,

A2. $p_{\lambda\mu,kj} = p_{\lambda\mu,ij} - p_{\nu\mu,ij} + p_{\nu\mu,kj}$,

A3. $p_{\mu\lambda,jk} = p_{\mu\lambda,ji} - p_{\mu\nu,ji} + p_{\mu\nu,jk}$, A4. $ap_{\lambda,i} = p_{\lambda,i}a = 0$,

A5. $ab + p_{o\mu,io} = p_{o\mu,io} + ab$,

A6. $ab + p_{\lambda o,oj} = p_{\lambda o,oj} + ab$, *for all* $i, j, k \in I,\ \lambda, \mu, \nu \in \Lambda$ *and* $a, b \in R$.

Let $\mathscr{M}$ *consist of the elements of* $I \times R \times \Lambda$ *and defined operations '+' and '·' on* $\mathscr{M}$ *by*

$$(i, a, \lambda) + (j, b, \mu) = (i, a + p_{\lambda,j} + b, \mu)$$

and

$$(i, a, \lambda) \cdot (j, b, \mu) = (ij, -p_{\lambda\mu,ij} + ab, \lambda\mu).$$

Then $(\mathscr{M}, +, \cdot)$ *is a completely simple semiring. Conversely, every completely simple semiring is isomorphic to such a semiring.*

The semiring constructed in Theorem 2.2 is denoted by $\mathscr{M}(I, R, \Lambda; P)$ and is called the *Rees matrix semiring.*

Corollary 2.3 [5]**.** *Let* $\mathscr{M}(I, R, \Lambda; P)$ *be a Rees matrix semiring. Then* $p_{\lambda\mu,ij} = p_{\lambda 0,0j} + p_{0\mu,i0}$ *holds for all* $i, j \in I$*;* $\lambda, \mu \in \Lambda$*. This yields* $p_{\lambda,i} = p_{\lambda 0,0i} + p_{0\lambda,i0}$ *and hence by assumption* (A5) *and* (A6) *stated in the above Theorem* 2.2, $ab + p_{\lambda,i} = p_{\lambda,i} + ab$ *for all* $i \in I$*;* $\lambda \in \Lambda$ *and* $a, b \in R$.

Definition 2.4 [9]**.** Let $(S, +, \cdot)$ be an additively quasi regular semiring. Then the relations $\mathscr{L}^{*^+}$, $\mathscr{R}^{*^+}$, $\mathscr{J}^{*^+}$, $\mathscr{H}^{*^+}$ and $\mathscr{D}^{*^+}$ on S are defined by : for $a, b \in S$,

$$a\,\mathscr{L}^{*^+}\,b \text{ if and only if } ma\,\mathscr{L}^+\,nb,$$
$$a\,\mathscr{R}^{*^+}\,b \text{ if and only if } ma\,\mathscr{R}^+\,nb,$$
$$a\,\mathscr{J}^{*^+}\,b \text{ if and only if } ma\,\mathscr{J}^+\,nb,$$
$$\mathscr{H}^{*^+} = \mathscr{L}^{*^+} \cap \mathscr{R}^{*^+} \text{ and } \mathscr{D}^{*^+} = \mathscr{L}^{*^+} o\,\mathscr{R}^{*^+},$$

where m and n are the smallest positive integers such that ma and nb are respectively additively regular.

Definition 2.5 [9]**.** A quasi completely regular semiring $(S, +, \cdot)$ is said to be a completely Archimedean semiring if any two elements of S are $\mathscr{J}^{*+}$-related, i.e., $\mathscr{J}^{*+} = S \times S$.

Definition 2.6 [9]**.** Let R be subskew-ring of a semiring S. If for every $a \in S$ there exists a positive integer n such that $na \in R$, then S is said to be a quasi skew-ring.

Theorem 2.7 [9]**.** *The following conditions on a semiring* $(S, +, \cdot)$ *are equivalent.*

1. S *is a quasi completely regular semiring.*
2. *Every* $\mathscr{H}^{*^+}$*-class is a quasi skew-ring.*
3. S *is* (*disjoint*) *union of quasi skew-rings.*

4. *S is a b-lattice of completely Archimedean semirings.*
5. *S is an idempotent semiring of quasi skew-rings.*

Definition 2.8 [10]**.** Let $(S, +, \cdot)$ be a semiring. A nonempty subset I of S is said to be a bi-ideal of S if $a \in I$ and $x \in S$ imply that $a + x$, $x + a$, ax, $xa \in I$.

Definition 2.9 [10]**.** Let I be a bi-ideal of a semiring S. We define a relation ρ_I on S in the following way :

$$a\rho_I b \text{ if and only if either } a, b \in I \text{ or } a = b; \text{ where } a, b \in S.$$

It is easy to verify that ρ_I is a congruence on S. This congruence is said to be *Rees congruence* on S and the quotient semiring S/ρ_I contains a strongly infinite element, namely I. This quotient semiring S/ρ_I is said to be the *Rees quotient semiring* and is denoted by S/I. In this case the semiring S is said to be an *ideal extension* or simply an extension of I by the semiring S/I. An ideal extension S of a semiring I is said to be a *nil-extension* of I if for any $a \in S$ there exists a positive integer n such that $na \in I$.

Theorem 2.10 [10]**.** *A semiring S is a quasi skew-ring if and only if S is a nil-extension of a skew-ring.*

Theorem 2.11 [10]**.** *The following conditions on a semiring are equivalent:*

1. *S is a completely Archimedean semiring.*
2. *S is a nil-extension of a completely simple semiring.*
3. *S is Archimedean and quasi completely regular.*

Theorem 2.12 [10]**.** *Let $(S, +, \cdot)$ be a completely Archimedean semiring. Then the subskew-rings are given by $H_e^+ = e + S + e$, where $e \in E^+(S)$.*

3. MAIN RESULTS

A semiring $(S, +, \cdot)$ is a completely Archimedean semiring if and only if it is nil-extension of a completely simple semiring [10, Theorem 3.19]. In this section we establish the structure theorem of a completely Archimedean semiring.

Definition 3.1. A partial semiring S is a nonempty set together with two binary operations '+' and '$\cdot$' defined for some elements of S, such that for all $x, y, z \in S$

1. if $x + (y + z)$ and $(x + y) + z$ exist, then $x + (y + z) = (x + y) + z$;
2. if $x \cdot (y \cdot z)$ and $(x \cdot y) \cdot z$ exist, then $x \cdot (y \cdot z) = (x \cdot y) \cdot z$;
3. if $x \cdot (y + z)$, $(x + y) \cdot z$, $(x \cdot y) + (x \cdot z)$ and $(x \cdot z) + (y \cdot z)$ exist, then $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$.

Example 3.2. The set of all irrational numbers with respect to usual addition and usual multiplication is a partial semiring.

Definition 3.3. Let $\mathscr{M}(I, R, \Lambda; P)$ be a Rees matrix semiring, where R is a skew-ring, $(I, \cdot)$ and $(\Lambda, \cdot)$ are bands and Q is a partial semiring, such that $(I \times R \times \Lambda) \cap Q = \emptyset$. Let $\xi : Q \to I$ and $\eta : Q \to \Lambda$ be two mappings, such that $\xi : q \mapsto \xi_q$ and $\eta : q \mapsto \eta_q$. Also let $\varphi : Q \times I \to R$ be a mapping, such that for all $q, r \in Q$; $i, j, k \in I$; $\lambda, \mu \in \Lambda$ the following conditions hold :

(C1) if $q + r \in Q$, then $\xi_{q+r} = \xi_q$ and $\eta_{q+r} = \eta_r$.

(C2) if $q + r \in Q$, then $\varphi(q + r, i) = \varphi(q, \xi_r) + \varphi(r, i)$.

(C3) $p_{\lambda,\xi_q} + \varphi(q, i) - p_{\eta_q,i}$ does not depend on $i \in I$ and is denoted by $\psi(q, \lambda)$; $\lambda \in \Lambda$.

(C4) if $qr \in Q$, then $\xi_{qr} = \xi_q\xi_r$ and $\eta_{qr} = \eta_q\eta_r$.

(C5) if $qr \in Q$, then $p_{\lambda,\xi_{qr}} + \varphi(qr, i) = p_{\lambda,i} + \varphi(q, j)\varphi(r, k)$.

Let $S = (I \times R \times \Lambda) \cup Q$. Define addition '+' and multiplication '·' on S by

$$(i, a, \lambda) + (j, b, \mu) = (i, a + p_{\lambda,j} + b, \mu) \tag{1}$$

$$q + (i, a, \lambda) = \Big(\xi_q, \varphi(q, i) + a, \lambda\Big) \tag{2}$$

$$(i, a, \lambda) + q = \Big(i, a + \psi(q, \lambda), \eta_q\Big) \tag{3}$$

$$\text{if}\, q + r = s \in Q \,\text{then}\, q + r = s \in (I \times R \times \Lambda) \cup Q \tag{4}$$

$$\text{if } q + r \notin Q \text{ then } q + r = \Big(\xi_q, \varphi(q, \xi_r) + \varphi(r, i) - p_{\eta_r,i}, \eta_r\Big) \tag{5}$$

$$(i, a, \lambda) \cdot (j, b, \mu) = (ij, -p_{\lambda\mu,ij} + ab, \lambda\mu) \tag{6}$$

$$q \cdot (i, a, \lambda) = \Big(\xi_q i, -p_{\eta_q\lambda,\xi_q i} + \varphi(q, i)a, \eta_q\lambda\Big) \tag{7}$$

$$(i, a, \lambda) \cdot q = \Big(i\xi_q, -p_{\lambda\eta_q,i\xi_q} + a\varphi(q, i), \lambda\eta_q\Big) \tag{8}$$

$$\text{if } qr = t \in Q \text{ then } qr = t \in (I \times R \times \Lambda) \cup Q \tag{9}$$

$$\text{if } qr \notin Q \text{ then } qr = \Big(\xi_q\xi_r, -p_{\eta_q\eta_r,\xi_q\xi_r} + \varphi(q, i)\varphi(r, i), \eta_q\eta_r\Big) \tag{10}$$

for all $q, r, s, t \in Q$; $a, b \in R$; $i, j \in I$; $\lambda, \mu \in \Lambda$. We denote the above system by $S = \mathscr{M}(I, R, \Lambda; P, Q, \varphi, \psi, \xi, \eta)$.

Theorem 3.4. *The system $S = \mathscr{M}(I, R, \Lambda; P, Q, \varphi, \psi, \xi, \eta)$, as defined in Definition* 3.3, *is a semiring.*

Proof. We consider the set S and two binary operations '+' and '·' on S as defined by formulae (1) to (10) in Definition 3.3. The mapping $\xi : Q \to I$ can be considered as a mapping $\xi : Q \to \mathscr{T}(I)$. Similarly, considering $\eta : Q \to \Lambda$ as a mapping $\eta : Q \to \mathscr{T}(\Lambda)$, where $\mathscr{T}(I)$ and $\mathscr{T}(\Lambda)$ are semigroups of all mappings of I in I and Λ in Λ, respectively, we may conclude, by [11, Lemma 1.1], $(S,+)$ is a semigroup. Now by (C3), it follows that the multiplication defined by the formula (10) is well-defined. To prove $(S,\cdot)$ is a semigroup we have left to show that for any $q,r,s\in Q$ and $(i,a,\lambda),(j,b,\mu),(k,c,\nu)\in(I\times R\times\Lambda)$

(i) $q\cdot\Big((i,a,\lambda)\cdot(j,b,\mu)\Big)=\Big(q\cdot(i,a,\lambda)\Big)\cdot(j,b,\mu)$

(ii) $\Big((i,a,\lambda)\cdot q\Big)\cdot(j,b,\mu)=(i,a,\lambda)\cdot\Big(q\cdot(j,b,\mu)\Big)$

(iii) $(i,a,\lambda)\cdot\Big((j,b,\mu)\cdot q\Big)=\Big((i,a,\lambda)\cdot(j,b,\mu)\Big)\cdot q$

(iv) $(q\cdot r)\cdot(i,a,\lambda)=q\cdot\Big(r\cdot(i,a,\lambda)\Big)$ for both $qr\in Q$ and $qr\notin Q$

(v) $(i,a,\lambda)\cdot(q\cdot r)=\Big((i,a,\lambda)\cdot q\Big)\cdot r$ for both $qr\in Q$ and $qr\notin Q$

(vi) $\Big(q\cdot(i,a,\lambda)\Big)\cdot r=q\cdot\Big((i,a,\lambda)\cdot r\Big)$

(vii) $(q\cdot r)\cdot s=q\cdot(r\cdot s)$

(viii) $(i,a,\lambda)\Big((j,b,\mu)\cdot(k,c,\nu)\Big)=\Big((i,a,\lambda)\cdot(j,b,\mu)\Big)(k,c,\nu)$.

Now,

$$\begin{aligned} q\cdot\Big((i,a,\lambda)\cdot(j,b,\mu)\Big) &= q(ij,-p_{\lambda\mu,ij}+ab,\lambda\mu)\\ &=\Big(\xi_q ij,-p_{\eta_q\lambda\mu,\xi_q ij}+\varphi(q,ij)ab,\eta_q\lambda\mu\Big)\end{aligned}$$

and

$$\begin{aligned} \Big(q\cdot(i,a,\lambda)\Big)\cdot(j,b,\mu) &= \Big(\xi_q i,-p_{\eta_q\lambda,\xi_q i}+\varphi(q,i)a,\eta_q\lambda\Big)(j,b,\mu)\\ &=\Big(\xi_q ij,-p_{\eta_q\lambda\mu,\xi_q ij}+\varphi(q,i)ab,\eta_q\lambda\mu\Big).\end{aligned}$$

By using $ap_{\lambda,i}=p_{\lambda,i}a=0$ in (C3), we have $\varphi(q,ij)ab=\varphi(q,i)ab$. Hence from the above two (i) follows. Similarly, (ii) and (iii) can be proved easily.
Let $q,r\in Q$, such that $qr\in Q$. Now,

$$\begin{aligned} (q\cdot r)\cdot(i,a,\lambda) &= \Big(\xi_{qr}i,-p_{\eta_{qr}\lambda,\xi_{qr}i}+\varphi(qr,i)a,\eta_{qr}\lambda\Big)\\ &=\Big(\xi_q\xi_r i,-p_{\eta_q\eta_r\lambda,\xi_q\xi_r i}+\varphi(qr,i)a,\eta_q\eta_r\lambda\Big).\end{aligned}$$

Let $qr\notin Q$. Then

$$\begin{aligned} (q\cdot r)\cdot(i,a,\lambda) &= \Big(\xi_q\xi_r,-p_{\eta_q\eta_r,\xi_q\xi_r}+\varphi(q,i)\varphi(r,i),\eta_q\eta_r\Big)(i,a,\lambda)\\ &=\Big(\xi_q\xi_r i,-p_{\eta_q\eta_r\lambda,\xi_q\xi_r i}+\varphi(q,i)\varphi(r,i)a,\eta_q\eta_r\lambda\Big).\end{aligned}$$

Again,

$$\begin{aligned} q\cdot\Big(r\cdot(i,a,\lambda)\Big) &= q\cdot\Big(\xi_r i, -p_{\eta_r\lambda,\xi_r i}+\varphi(r,i)a, \eta_r\lambda\Big)\\ &= \Big(\xi_q\xi_r i, -p_{\eta_q\eta_r\lambda,\xi_q\xi_r i}+\varphi(q,\xi_r i)\varphi(r,i)a, \eta_q\eta_r\lambda\Big). \end{aligned}$$

By using $ap_{\lambda,i}=p_{\lambda,i}a=0$ in (C3) and (C5), we have $\varphi(qr,i)a=\varphi(q,\xi_r i)\varphi(r,i)a$ and $\varphi(q,i)\varphi(r,i)a=\varphi(q,\xi_r i)\varphi(r,i)a$, respectively. Hence from the above three we have $(q\cdot r)\cdot(i,a,\lambda)=q\cdot\big(r\cdot(i,a,\lambda)\big)$ for both $qr\in Q$ and $qr\notin Q$ and (iv) follows. Similarly, (v) can be proved easily. Also, we can easily prove (vi). As Q is a partial semiring, so (vii) follows obviously and (viii) follows immediately by Theorem 2.2. Hence $(S,\cdot)$ is a semigroup.

Now to prove $(S,+,\cdot)$ is a semiring we have left to verify distributive properties which are as follows:

(ix) $(i,a,\lambda)\Big((j,b,\mu)+(k,c,\nu)\Big)=(i,a,\lambda)(j,b,\mu)+(i,a,\lambda)(k,c,\nu)$

(x) $\Big((i,a,\lambda)+(j,b,\mu)\Big)(k,c,\nu)=(i,a,\lambda)(k,c,\nu)+(j,b,\mu)(k,c,\nu)$

(xi) $q\Big((i,a,\lambda)+(j,b,\mu)\Big)=q(i,a,\lambda)+q(j,b,\mu)$

(xii) $\Big(q+(i,a,\lambda)\Big)(j,b,\mu)=q(j,b,\mu)+(i,a,\lambda)(j,b,\mu)$

(xiii) $(i,a,\lambda)\Big(q+(j,b,\mu)\Big)=(i,a,\lambda)q+(i,a,\lambda)(j,b,\mu)$

(xiv) $\Big((i,a,\lambda)+q\Big)(j,b,\mu)=(i,a,\lambda)(j,b,\mu)+q(j,b,\mu)$

(xv) $(i,a,\lambda)\Big((j,b,\mu)+q\Big)=(i,a,\lambda)(j,b,\mu)+(i,a,\lambda)q$

(xvi) $\Big((i,a,\lambda)+(j,b,\mu)\Big)q=(i,a,\lambda)q+(j,b,\mu)q$

(xvii) $(q+r)(i,a,\lambda)=q(i,a,\lambda)+r(i,a,\lambda)$ for both $q+r\in Q$ and $q+r\notin Q$

(xviii) $(i,a,\lambda)(q+r)=(i,a,\lambda)q+(i,a,\lambda)r$ for both $q+r\in Q$ and $q+r\notin Q$

(xix) $q\Big(r+(i,a,\lambda)\Big)=qr+q(i,a,\lambda)$ for both $qr\in Q$ and $qr\notin Q$

(xx) $q\Big((i,a,\lambda)+r\Big)=q(i,a,\lambda)+qr$ for both $qr\in Q$ and $qr\notin Q$

(xxi) $\Big(q+(i,a,\lambda)\Big)r=qr+(i,a,\lambda)r$ for both $qr\in Q$ and $qr\notin Q$

(xxii) $\Big((i,a,\lambda)+q\Big)r=(i,a,\lambda)r+qr$ for both $qr\in Q$ and $qr\notin Q$

(xxiii) $q(r+s)=qr+qs$

(xxiv) $(q+r)s=qs+rs.$

The cases (ix) and (x) follow immediately by Theorem 2.2.

Now,

$$q\Big((i,a,\lambda)+(j,b,\mu)\Big) = q(i,a+p_{\lambda,j}+b,\mu) \\ = \Big(\xi_q i, -p_{\eta_q\mu,\xi_q i}+\varphi(q,i)a+\varphi(q,i)b,\eta_q\mu\Big).$$

On the other hand,

$$\begin{aligned} & q(i,a,\lambda)+q(j,b,\mu) \\ &= \Big(\xi_q i, -p_{\eta_q\lambda,\xi_p i}+\varphi(q,i)a,\eta_p\lambda\Big)+\Big(\xi_q j,-p_{\eta_q\mu,\xi_q j}+\varphi(q,j)b,\eta_q\mu\Big) \\ &= \Big(\xi_q i,-p_{\eta_q\lambda,\xi_q i}+\varphi(q,i)a+p_{\eta_q\lambda,\xi_q j}-p_{\eta_q\mu,\xi_q j}+\varphi(q,j)b,\eta_q\mu\Big) \\ &= \Big(\xi_q i,-p_{\eta_q\mu,\xi_q i}+\varphi(q,i)a+\varphi(q,i)b,\eta_q\mu\Big) \quad \text{[by Corollary 2.3].} \end{aligned}$$

Hence (xi) follows immediately. Similarly, (xvi) can be proved easily.

Now,

$$\Big(q+(i,a,\lambda)\Big)(j,b,\mu) = \Big(\xi_q,\varphi(q,i)+a,\lambda\Big)(j,b,\mu) \\ = \Big(\xi_q j,-p_{\lambda\mu,\xi_q j}+\varphi(q,i)b+ab,\lambda\mu\Big).$$

Again,

$$\begin{aligned} & q(j,b,\mu)+(i,a,\lambda)(j,b,\mu) \\ &= \Big(\xi_q j,-p_{\eta_q\mu,\xi_q j}+\varphi(q,j)b,\eta_q\mu\Big)+\Big(ij,-p_{\lambda\mu,ij}+ab,\lambda\mu\Big) \\ &= \Big(\xi_q j,-p_{\eta_q\mu,\xi_q j}+\varphi(q,j)b+p_{\eta_q\mu,ij}-p_{\lambda\mu,ij}+ab,\lambda\mu\Big) \\ &= \Big(\xi_q j,-p_{\lambda\mu,\xi_q j}+\varphi(q,j)b+ab,\lambda\mu\Big) \quad \text{[by Corollary 2.3].} \end{aligned}$$

Hence (xii) follows. Similarly, (xiii), (xiv) and (xv) can be proved easily.

Now,

$$\begin{aligned} & q(i,a,\lambda)+r(i,a,\lambda) \\ &= \Big(\xi_q i,-p_{\eta_q\lambda,\xi_q i}+\varphi(q,i)a,\eta_q\lambda\Big)+\Big(\xi_r i,-p_{\eta_r\lambda,\xi_r i}+\varphi(r,i)a,\eta_r\lambda\Big) \\ &= \Big(\xi_q i,-p_{\eta_q\lambda,\xi_q i}+\varphi(q,i)a+p_{\eta_q\lambda,\xi_r i}-p_{\eta_r\lambda,\xi_r i}+\varphi(r,i)a,\eta_r\lambda\Big) \\ &= \Big(\xi_q i,-p_{\eta_r\lambda,\xi_q i}+\varphi(q,i)a+\varphi(r,i)a,\eta_r\lambda\Big) \quad \text{[by Corollary 2.3].} \end{aligned}$$

Let $q+r \in Q$. Then

$$\begin{aligned}(q+r)(i,a,\lambda) &= \Big(\xi_{q+r}i, -p_{\eta_{q+r}\lambda,\xi_{q+r}i} + \varphi(q+r,i)a, \eta_{q+r}\lambda\Big)\\ &= \Big(\xi_q i, -p_{\eta_r\lambda,\xi_q i} + \varphi(q,\xi_r)a + \varphi(r,i)a, \eta_r\lambda\Big) \quad \text{[by (C2)]}.\end{aligned}$$

Let $q+r \notin Q$. Then

$$\begin{aligned}(q+r)(i,a,\lambda) &= \Big(\xi_q, \varphi(q,\xi_r) + \varphi(r,i) - p_{\eta_r,i}, \eta_r\Big)(i,a,\lambda)\\ &= \Big(\xi_q i, -p_{\eta_r\lambda,\xi_q i} + \varphi(q,\xi_r)a + \varphi(r,i)a, \eta_r\lambda\Big).\end{aligned}$$

Using (A3), we have $\varphi(q,\xi_r)a = \varphi(q,i)a$. Hence from the above three (xvii) follows. Similarly, (xviii) can be proved easily.

Now,

$$\begin{aligned}q\Big(r + (i,a,\lambda)\Big) &= q\Big(\xi_r, \varphi(r,i) + a, \lambda\Big)\\ &= \Big(\xi_q\xi_r, -p_{\eta_q\lambda,\xi_q\xi_r} + \varphi(q,\xi_r)\varphi(r,i) + \varphi(q,\xi_r)a, \eta_q\lambda\Big)\\ &= \Big(\xi_q\xi_r, -p_{\eta_q\lambda,\xi_q\xi_r} + \varphi(q,i)\varphi(r,i) + \varphi(q,i)a, \eta_q\lambda\Big).\end{aligned}$$

When $qr \in Q$, then

$$\begin{aligned}qr + q(i,a,\lambda) &= qr + \Big(\xi_q i, -p_{\eta_q\lambda,\xi_q i} + \varphi(q,i)a, \eta_q\lambda\Big)\\ &= \Big(\xi_{qr}, \varphi(qr,\xi_q i) - p_{\eta_q\lambda,\xi_q i} + \varphi(q,i)a, \eta_q\lambda\Big)\\ &= \Big(\xi_{qr}, -p_{\eta_q\lambda,\xi_q i} + \varphi(qr,\xi_q i) + \varphi(q,i)a, \eta_q\lambda\Big) \quad \text{[by Corollary 2.3]}\\ &= \Big(\xi_{qr}, -p_{\eta_q\lambda,\xi_{qr}} + \varphi(q,i)\varphi(r,i) + \varphi(q,i)a, \eta_q\lambda\Big) \quad \text{[by (C5)]}\\ &= \Big(\xi_q\xi_r, -p_{\eta_q\lambda,\xi_q\xi_r} + \varphi(q,i)\varphi(r,i) + \varphi(q,i)a, \eta_q\lambda\Big) \quad \text{[by (C4)]}.\end{aligned}$$

Also, when $qr \notin Q$, then

$$\begin{aligned}&qr + q(i,a,\lambda)\\ &= \Big(\xi_q\xi_r, -p_{\eta_q\eta_r,\xi_q\xi_r} + \varphi(q,i)\varphi(r,i), \eta_q\eta_r\Big) + \Big(\xi_q i, -p_{\eta_q\lambda,\xi_q i} + \varphi(q,i)a, \eta_q\lambda\Big)\\ &= \Big(\xi_q\xi_r, -p_{\eta_q\eta_r,\xi_q\xi_r} + \varphi(q,i)\varphi(r,i) + p_{\eta_q\eta_r,\xi_q i} - p_{\eta_q\lambda,\xi_q i} + \varphi(q,i)a, \eta_q\lambda\Big)\\ &= \Big(\xi_q\xi_r, -p_{\eta_q\eta_r,\xi_q\xi_r} + p_{\eta_q\eta_r,\xi_q i} - p_{\eta_q\lambda,\xi_q i} + \varphi(q,i)\varphi(r,i) + \varphi(q,i)a, \eta_q\lambda\Big)\\ &= \Big(\xi_q\xi_r, -p_{\eta_q\lambda,\xi_q\xi_r} + \varphi(q,i)\varphi(r,i) + \varphi(q,i)a, \eta_q\lambda\Big).\end{aligned}$$

Hence (xix) follows from above three. Similarly, (xx), (xxi) and (xxii) can be proved easily. Also (xxiii) and (xxiv) follow obviously as Q is a partial semiring. Hence $(S,+,\cdot)$ is a semiring. ∎

Theorem 3.5. *A semiring $(S,+,\cdot)$ is a completely Archimedean semiring if and only if it is isomorphic to a semiring $\mathscr{M}(I,R,\Lambda;P,Q,\varphi,\psi,\xi,\eta)$.*

Proof. Let $(S,+,\cdot)$ be a completely Archimedean semiring. By Theorem 2.11, S is nil-extension of a completely simple semiring. Therefore, there exists a bi-ideal K of S such that S is a nil-extension of K, where K is a completely simple semiring. Again by Theorem 2.2, K is isomorphic to a Rees matrix semiring and let $K=\mathscr{M}(I,R,\Lambda;P)$. Clearly, $(K,+)$ is a completely simple semigroup. Let $Q=S\setminus K$. Then Q is a partial semiring and $S=K\cup Q$, that is, S consists of the elements of $(I\times R\times\Lambda)\cup Q$. By [11, Theorem 1.1], $(S,+)$ is isomorphic to the semigroup $\big(\mathscr{M}(I,R,\Lambda;P,Q,\varphi,\psi,\xi,\eta),+\big)$, where $\xi:Q\to\mathscr{T}(I)$ and $\eta:Q\to\mathscr{T}(\Lambda)$ are two mappings, such that $\xi:p\mapsto\xi_p,\eta:p\mapsto\eta_p$ and $\mathscr{T}(I)$, $\mathscr{T}(\Lambda)$ are semigroups of all mappings of I in I and Λ in Λ, respectively.

Let $e=(i,-p_{\lambda,i},\lambda)$. Then $e\in E^+(S)$. Also let $q\in Q$ and n be the smallest positive integer such that $nq\in H_f^+$ where $f\in E^+(S)$. As $e\in E^+(S)$ then $e+e=e$. Again, $H_e^+=\{(i,a,\lambda):a\in R\}$. Now, $qe+qe=qe$ and so qe is an additive idempotent. Let $q(i,-p_{\lambda,i},\lambda)=(k,-p_{\nu,k},\nu)$.
Now, $\big(q+(i,a,\lambda)\big)(i,-p_{\lambda,i},\lambda)=\big(i\xi_q,\varphi(q,i)+a,\lambda\big)(i,-p_{\lambda,i},\lambda)=\big(i\xi_qi,-p_{\lambda,i\xi_qi},\lambda\big)$ and

$$\begin{aligned}q(i,-p_{\lambda,i},\lambda)+(i,a,\lambda)(i,-p_{\lambda,i},\lambda)&=q(i,-p_{\lambda,i},\lambda)+(i,-p_{\lambda,i},\lambda)\\&=(k,-p_{\nu,k},\nu)+(i,-p_{\lambda,i},\lambda)\\&=(k,-p_{\nu,k}+p_{\nu,i}-p_{\lambda,i},\lambda).\end{aligned}$$

Hence $k=i\xi_qi$. Similarly, from $\big((i,a,\lambda)+q\big)(i,-p_{\lambda,i},\lambda)=(i,a,\lambda)(i,-p_{\lambda,i},\lambda)+q(i,-p_{\lambda,i},\lambda)$, we have $\nu=\lambda\eta_q\lambda$.

Hence $q(i,-p_{\lambda,i},\lambda)=\big(i\xi_qi,-p_{\lambda\eta_q\lambda,i\xi_qi},\lambda\eta_q\lambda\big)$ and similarly,

$$\big(i,-p_{\lambda,i},\lambda\big)q=\big(ii\xi_q,-p_{\lambda\lambda\eta_q,ii\xi_q},\lambda\lambda\eta_q\big).$$

Let $q(i,a,\lambda)=(u,v,w)$. Now,

$$\begin{aligned}\Big(q+(i,-p_{\lambda,i},\lambda)\Big)(i,a,\lambda)&=\Big(i\xi_q,\varphi(q,i)-p_{\lambda,i},\lambda\Big)(i,a,\lambda)\\&=\Big(i\xi_qi,-p_{\lambda,i\xi_qi}+\varphi(q,i)a,\lambda\Big)\end{aligned}$$

and

$$\begin{aligned}q(i,a,\lambda)+(i,-p_{\lambda,i},\lambda)(i,a,\lambda)&=(u,v,w)+(i,-p_{\lambda,i},\lambda)\\&=(u,v+p_{w,i}-p_{\lambda,i},\lambda).\end{aligned}$$

From the above two we have $u = i\xi_q i$. Similarly, from

$$\Big((i, -p_{\lambda,i}, \lambda) + q\Big)(i, a, \lambda) = (i, -p_{\lambda,i}, \lambda)(i, a, \lambda) + q(i, a, \lambda),$$

we have $w = \lambda\eta_q\lambda$.

Also from $v + p_{w,i} - p_{\lambda,i} = -p_{\lambda,i\xi_q i} + \varphi(q, i)a$, we have

$$\begin{aligned} v &= -p_{\lambda,i\xi_q i} + \varphi(q, i)a + p_{\lambda,i} - p_{w,i} \\ &= -p_{\lambda,i\xi_q i} + p_{\lambda,i} - p_{\lambda\eta_q\lambda,i} + \varphi(q, i)a \\ &= -p_{\lambda\eta_q\lambda,i\xi_q i} + \varphi(q, i)a. \end{aligned}$$

Hence,

$$q(i, a, \lambda) = \Big(i\xi_q i, -p_{\lambda\eta_q\lambda,i\xi_q i} + \varphi(q, i)a, \lambda\eta_q\lambda\Big) \quad \ldots (*)$$

and similarly,

$$(i, a, \lambda)q = \Big(ii\xi_q, -p_{\lambda\lambda\eta_q,ii\xi_q} + a\varphi(q, i), \lambda\lambda\eta_q\Big) \quad \ldots (**).$$

Now,

$$\begin{aligned} \Big(q + (i, a, \lambda)\Big)(j, b, \mu) &= \Big(i\xi_q, \varphi(q, i) + a, \lambda\Big)(j, b, \mu) \\ &= \Big(i\xi_q j, -p_{\lambda\mu,i\xi_q j} + \varphi(q, i)b + ab, \lambda\mu\Big). \end{aligned}$$

On the other hand,

$$\begin{aligned} &q(j, b, \mu) + (i, a, \lambda)(j, b, \mu) \\ &= \Big(j\xi_q j, -p_{\mu\eta_q\mu,j\xi_q j} + \varphi(q, j)b, \mu\eta_q\mu\Big) + (ij, -p_{\lambda\mu,ij} + ab, \lambda\mu) \\ &= \Big(j\xi_q j, -p_{\mu\eta_q\mu,j\xi_q j} + \varphi(q, j)b + p_{\mu\eta_q\mu,ij} - p_{\lambda\mu,ij} + ab, \lambda\mu\Big) \\ &= \Big(j\xi_q j, -p_{\mu\eta_q\mu,j\xi_q j} + p_{\mu\eta_q\mu,ij} - p_{\lambda\mu,ij} + \varphi(q, j)b + ab, \lambda\mu\Big) \\ &= \Big(j\xi_q j, -p_{\lambda\mu,j\xi_q j} + \varphi(q, j)b + ab, \lambda\mu\Big). \end{aligned}$$

Hence $i\xi_q j = j\xi_q j$ and similarly, $ji\xi_q = jj\xi_q$, $\lambda\eta_q\lambda = \mu\eta_q\lambda$, $\lambda\lambda\eta_q = \lambda\mu\eta_q$.

As I is a band and $i\xi_q \in I$, using above equations, we have for any $i, j \in I$, $i\xi_q = i\xi_q i\xi_q = i\xi_q j\xi_q = j\xi_q j\xi_q = j\xi_q$, i.e., ξ_q is a constant function for all $q \in Q$. Therefore, ξ_q represents a unique element of I. Thus without any loss of generality we can assume ξ as a function from Q to I. Similarly, η can be considered as a function from Q to Λ.

Using these in [11, Lemma 1.1], (C1)–(C3) and formulae (1) to (5) of the Definition 3.3 follow automatically. Formula (6) follows immediately by the Theorem 2.2.

Now, the equations (*) and (**) become

$$q \cdot (i, a, \lambda) = \Big(\xi_q i, -p_{\eta_q \lambda, \xi_q i} + \varphi(q, i)a, \eta_q \lambda\Big)$$

and

$$(i, a, \lambda) \cdot q = \Big(i\xi_q, -p_{\lambda\eta_q, i\xi_q} + a\varphi(q, i), \lambda\eta_q\Big).$$

Thus the formulae (7) and (8) follow. Now we have that $\psi(q, \lambda)$; $\lambda \in \Lambda$ does not depend on $i \in I$ and $\psi(q, \lambda) = p_{\lambda, \xi_q} + \varphi(q, i) - p_{\eta_q, i}$. Also $\psi(q, \lambda) = p_{\lambda, \xi_q} + \varphi(q, j) - p_{\eta_q, j}$.

Using $ap_{\lambda, i} = p_{\lambda, i}a = 0$, from the above two we have for any $i, j \in I$, $\lambda, \mu \in \Lambda$ and $a \in R$, $\psi(q, \lambda)a = \varphi(q, i)a$ and $\psi(q, \lambda)a = \varphi(q, j)a$. Hence $\varphi(q, i)a = \varphi(q, j)a$ and similarly, $a\varphi(q, i) = a\varphi(q, j)$.

Now, for any $q, r \in Q$ we have,

$$\begin{aligned}
\Big(q + (i, a, \lambda)\Big)r &= \Big(\xi_q, \varphi(q, i) + a, \lambda\Big)r \\
&= \Big(\xi_q\xi_r, -p_{\lambda\eta_r, \xi_q\xi_r} + \big(\varphi(q, i) + a\big)\varphi(r, \xi_q), \lambda\eta_r\Big) \\
&= \Big(\xi_q\xi_r, -p_{\lambda\eta_r, \xi_q\xi_r} + \varphi(q, i)\varphi(r, \xi_q) + a\varphi(r, \xi_q), \lambda\eta_r\Big).
\end{aligned}$$

Let $qr \in Q$. Then

$$\begin{aligned}
qr + (i, a, \lambda)r &= qr + \Big(i\xi_r, -p_{\lambda\eta_r, i\xi_r} + a\varphi(r, i), \lambda\eta_r\Big) \\
&= \Big(\xi_{qr}, \varphi(qr, i\xi_r) - p_{\lambda\eta_r, i\xi_r} + a\varphi(r, i), \lambda\eta_r\Big).
\end{aligned}$$

From above two equations we have $\xi_{qr} = \xi_q\xi_r$.

Also

$$\varphi(qr, i\xi_r) - p_{\lambda\eta_r, i\xi_r} + a\varphi(r, i) = -p_{\lambda\eta_r, \xi_q\xi_r} + \varphi(q, i)\varphi(r, \xi_q) + a\varphi(r, \xi_q),$$

i.e.,

$$\varphi(qr, i\xi_r) - p_{\lambda\eta_r, i\xi_r} = -p_{\lambda\eta_r, \xi_{qr}} + \varphi(q, i)\varphi(r, \xi_q),$$

i.e.,

$$\varphi(qr, j) - p_{\mu, j} = -p_{\mu, \xi_{qr}} + \varphi(q, i)\varphi(r, \xi_q),$$

i.e.,

$$p_{\mu, \xi_{qr}} + \varphi(qr, j) = p_{\mu, j} + \varphi(q, i)\varphi(r, k)$$

[assuming $j = i\xi_r$, $\mu = \lambda\eta_r$ and $k = \xi_q$].

Similarly, we have $\eta_{qr} = \eta_q\eta_r$.

Let $qr \notin Q$ and $qr = (k, g, l)$. Then

$$\begin{aligned} qr + (i, a, \lambda)r &= (k, g, l) + \Big(i\xi_r, -p_{\lambda\eta_r, i\xi_r} + a\varphi(r, i), \lambda\eta_r\Big) \\ &= \Big(k, g + p_{l, i\xi_r} - p_{\lambda\eta_r, i\xi_r} + a\varphi(r, i), \lambda\eta_r\Big). \end{aligned}$$

From $qr + (i, a, \lambda)r = \Big(q + (i, a, \lambda)\Big)r$, we have $k = \xi_q\xi_r$. Similarly, $l = \eta_q\eta_r$.

Also

$$g + p_{l, i\xi_r} - p_{\lambda\eta_r, i\xi_r} + a\varphi(r, i) = -p_{\lambda\eta_r, \xi_q\xi_r} + \varphi(q, i)\varphi(r, \xi_q) + a\varphi(r, \xi_q)$$

implies

$$g + p_{\eta_q\eta_r, i\xi_r} - p_{\lambda\eta_r, i\xi_r} = -p_{\lambda\eta_r, \xi_q\xi_r} + \varphi(q, i)\varphi(r, i),$$

i.e.,

$$g + p_{\eta_q\eta_r, i\xi_r} - p_{\lambda\eta_r, i\xi_r} = \varphi(q, i)\varphi(r, i) - p_{\lambda\eta_r, \xi_q\xi_r},$$

i.e.,

$$g + p_{\eta_q\eta_r, i\xi_r} - p_{\lambda\eta_r, i\xi_r} + p_{\lambda\eta_r, \xi_q\xi_r} = \varphi(q, i)\varphi(r, i),$$

i.e.,

$$g + p_{\eta_q\eta_r, \xi_q\xi_r} = \varphi(q, i)\varphi(r, i),$$

i.e.,

$$g = \varphi(q, i)\varphi(r, i) - p_{\eta_q\eta_r, \xi_q\xi_r},$$

i.e.,

$$g = -p_{\eta_q\eta_r, \xi_q\xi_r} + \varphi(q, i)\varphi(r, i).$$

Hence if $qr \notin Q$, then $qr = \big(\xi_q\xi_r, -p_{\eta_q\eta_r, \xi_q\xi_r} + \varphi(q, i)\varphi(r, i), \eta_q\eta_r\big)$. Consequently, $S \cong \mathscr{M}(I, R, \Lambda; P, Q, \varphi, \psi, \xi, \eta)$.

Conversely, let $S = \mathscr{M}(I, R, \Lambda; P, Q, \varphi, \psi, \xi, \eta)$. By Theorem 3.4, it follows that S is a semiring. Let $K = \mathscr{M}(I, R, \Lambda; P)$. Then K is a bi-ideal of S and K is a completely simple semiring. $(S, +)$ is a completely Archimedean semigroup and is nil-extension of a completely simple semigroup $(K, +)$. Let $a \in S$. Then there exists a positive integer n such that $na \in K$. Then na is completely regular, as $(K, +, \cdot)$ is completely regular semiring. So a is quasi completely regular. Hence $(S, +, \cdot)$ is quasi completely regular semiring. For any $a, b \in S$, there exist positive integers m and n such that $ma, nb \in K$. As $(K, +, \cdot)$ is completely simple semiring, it follows that $ma \mathscr{J}^+ nb$, i.e., $a \mathscr{J}^{*^+} b$. Hence $(S, +, \cdot)$ is completely Archimedean semiring. ∎

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