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CONGRUENCES AND TRAJECTORIES IN PLANAR SEMIMODULAR LATTICES

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Abstract

A 1955 result of J. Jakubík states that for the prime intervals $\mathfrak{p}$ and $\mathfrak{q}$ of a finite lattice, $\operatorname{con}(\mathfrak{p}) \geq \operatorname{con}(\mathfrak{q})$ iff $\mathfrak{p}$ is congruence-projective to $\mathfrak{q}$ (*via* intervals of arbitrary size). The problem is how to determine whether $\operatorname{con}(\mathfrak{p}) \geq \operatorname{con}(\mathfrak{q})$ involving only prime intervals.

Two recent papers approached this problem in different ways. G. Czédli's used trajectories for slim rectangular lattices—a special subclass of slim, planar, semimodular lattices. I used the concept of prime-projectivity for arbitrary finite lattices. In this note I show how my approach can be used to reprove Czédli's result and generalize it to arbitrary slim, planar, semimodular lattices.

Keywords: semimodular lattice, planar lattice, slim lattice, rectangular lattice, congruence, trajectory, prime interval.

2010 Mathematics Subject Classification: Primary: 06C10; Secondary: 06B10.

1. Introduction

To describe the congruence lattice, $\operatorname{Con} L$, of a finite lattice L, note that a prime interval $\mathfrak{p}$ of L generates a join-irreducible congruence $\operatorname{con}(\mathfrak{p})$, and conversely. See the discussion on pages 213 and 214 of LTF (reference [15]). So if we can determine when $\operatorname{con}(\mathfrak{p}) \geq \operatorname{con}(\mathfrak{q})$ holds for the prime intervals $\mathfrak{p}$ and $\mathfrak{q}$ of L, then we know the lattice $\operatorname{Con} L$ up to isomorphism.

The following result of Jakubík [32] (see Lemma 238 in LTF) accomplishes this goal, where $\Rightarrow$ is congruence-projectivity; see Section 2.

Lemma 1. *Let L be a finite lattice and let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals in L. Then* $\operatorname{con}(\mathfrak{p}) \geq \operatorname{con}(\mathfrak{q})$ *iff* $\mathfrak{p} \Rightarrow \mathfrak{q}$.

Jakubík's condition is easy to visualize; see Figure 1. Even though $\mathfrak{p}$ and $\mathfrak{q}$ are prime intervals, congruence-projectivity goes through arbitrary large intervals.

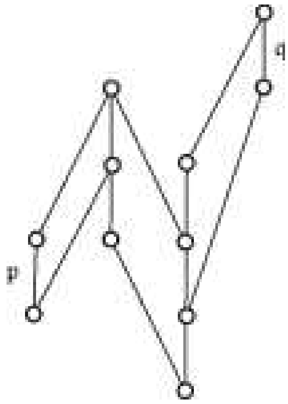

Figure 1. Illustrating Jakubík's condition for $\operatorname{con}(\mathfrak{p}) \geq \operatorname{con}(\mathfrak{q})$.

A *rectangular lattice* is a planar semimodular lattice L with exactly two doubly-irreducible elements on the boundary of L that are complementary and distinct from 0 and 1, see Grätzer and Knapp [26]. Although rectangular lattices are very special, from the point of view of congruence lattices they are quite general. Every finite distributive lattice can be represented as the congruence lattice of a rectangular lattice, see Grätzer and Knapp [26].

A rectangular lattice is *slim* if it contains no M_3 as a sublattice.

For slim rectangular lattices, Czédli [1] approached the problem of having to use arbitrary large intervals in the congruence-projectivities through the use of trajectories.

In a planar semimodular lattice L, two prime intervals of L are *consecutive* if they are opposite sides of a 4-cell (a covering C_2^2 sublattice with no interior element). As in Czédli and Schmidt [11], maximal sequences of consecutive prime intervals form a *trajectory*, see Section 4. Any prime interval $\mathfrak{p}$ in a trajectory $\mathcal{T}$ defines the same congruence $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\mathcal{T})$, but not all prime intervals $\mathfrak{p}$ with $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\mathcal{T})$ are necessarily in $\mathcal{T}$. So Czédli defines a quasi-ordering $\leq_C$ of the trajectories utilizing only prime intervals, see Section 4.

The reflexive and transitive extension of $\leq_C$ defines a quasiordering $\leq_T$ on the set of trajectories, which in turn, defines an ordering $\leq$. For a trajectory $\mathcal{T}$, let $\widehat{\mathcal{T}}$ denote the equivalence class containing $\mathcal{T}$. By definition, $\mathcal{T}$ and $\mathcal{T}'$ are in the same equivalence class, $\widehat{\mathcal{T}} = \widehat{\mathcal{T}'}$, iff $\mathcal{T} \leq_C \mathcal{T}'$ and $\mathcal{T}' \leq_C \mathcal{T}$. Let $\widehat{\operatorname{Traj}}\,(L)$ denote the set of equivalence classes of trajectories of L. The set $\widehat{\operatorname{Traj}}(L)$ under the ordering $\leq_T$ forms an ordered set.

Czédli [1] proves the following result:

Theorem 2 (Trajectory Theorem for Slim Rectangular Lattices)**.** *Let L be a slim rectangular lattice. The ordered set* $\widehat{\mathrm{Traj}}\, L$ *is isomorphic to* $\mathrm{J}(\mathrm{Con}\, L)$*, the ordered set of join-irreducible congruences of L, under the isomorphism* $\widehat{\mathcal{T}} \mapsto \mathrm{con}(\widehat{\mathcal{T}})$.

Since $\leq_C$ deals with prime intervals only, this resolves the problem for slim rectangular lattices of determining when $\mathrm{con}(\mathfrak{p}) \geq \mathrm{con}(\mathfrak{q})$ holds using prime intervals only.

My paper [19] took a more elementary approach. For the prime intervals $\mathfrak{p}$ and $\mathfrak{q}$, it introduces the concept of prime-perspectivity, involving only the two prime intervals. Prime-projectivity is the transitive extension of prime-perspectivity. The Prime-projectivity Lemma in [19] states that $\mathrm{con}(\mathfrak{p}) \geq \mathrm{con}(\mathfrak{q})$ iff $\mathfrak{p}$ is prime-projective to $\mathfrak{q}$, which involves only prime intervals. A stronger form of this lemma for slim, planar, semimodular lattices, the Swing Lemma, is also stated in [19] and verified in Grätzer [21]. Czédli applies in [2] the Trajectory Theorem for Slim Rectangular Lattices to prove the Swing Lemma for rectangular lattices.

In this paper, I show how the Swing Lemma can be used to verify the Trajectory Theorem for Slim Rectangular Lattices and generalize it to slim, planar, semimodular lattices.

1.1. References

Grätzer and Knapp [23]–[27] started the theory of slim planar semimodular lattices; the work was continued in Czédli and Schmidt [12] and [13]. There has been a lot of activity in this field, see an overview in Czédli and Grätzer [9] (Chapter 4 of the volume [30], Grätzer and Wehrung eds.) and Grätzer [17] (Chapter 5 of the volume [30]).

In the Bibliography we list the more recent contributions to this topic that did not make it into [30].

1.2. Outline

In Section 2, we introduce and illustrate the basic concepts. Then we define the swing relation and state the Swing Lemma. In Section 3, we analyze the Swing Lemma, making a number of easy observations and deriving some elementary consequences. We introduce trajectories in Section 4. The Trajectory Theorem is proved for slim, planar, semimodular lattices in Section 5.

1.3. Chronology

This paper was originally written in 2014, a part of a sequence of papers (see Section 1.1) by Gábor Czédli and myself, studying how congruences spread in planar semimodular lattices.

The final version was submitted to arXiv in Aug. 2014 (arXiv:1406.0439v3). It was revised as guided by a detailed referee's report in Oct. 2014. The revised version was submitted to Research Gate.

The final version was not submitted for publication until 2018. I believe that this paper is as relevant today as it was a little over three years ago.

2. The Swing Lemma

2.1. Notation and terminology

We use the concepts and notation of LTF.

For an ideal I, we use the notation $I = [0_I, 1_I]$.

We recall that $[a, b] \sim [c, d]$ denotes *perspectivity*, $[a, b] \overset{\text{up}}{\sim} [c, d]$ and $[a, b] \overset{\text{dn}}{\sim} [c, d]$ perspectivity up and down, see Figure 2; $[a, b] \approx [c, d]$ denotes *projectivity*, the transitive closure of perspectivity.

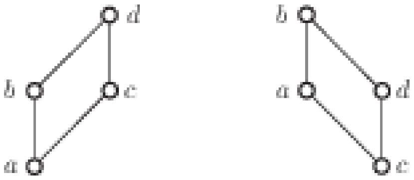

Figure 2. Perspectivity: $[a, b] \sim [c, d]$ ($[a, b] \overset{\text{up}}{\sim} [c, d]$ on the left, $[a, b] \overset{\text{dn}}{\sim} [c, d]$ on the right).

$[a, b] \twoheadrightarrow [c, d]$ denotes *congruence-perspectivity*, $[a, b] \overset{\text{up}}{\twoheadrightarrow} [c, d]$ and $[a, b] \overset{\text{dn}}{\twoheadrightarrow} [c, d]$ denote congruence-perspectivity up and down, see Figure 3; $[a, b] \Rrightarrow [c, d]$ denotes *congruence-projectivity*, the transitive closure of congruence-perspectivity.

A planar semimodular lattice is called *slim* if it contains no M_3 as a sublattice (Grätzer and Knapp [23]–[27] and Czédli and Schmidt [11]). An *SPS lattice* is a slim, planar, semimodular lattice.

Let L be an SPS lattice. For an element $a \in L$, the *multifork* at a is the set of all prime intervals $\mathfrak{p}$ with $1_{\mathfrak{p}} = a$, at least three in number. The prime intervals in the multifork on the left and right are the *exterior prime intervals*; the others are the *interior* prime intervals. We know that if $\mathfrak{p}$ and $\mathfrak{q}$ are interior prime intervals of a multifork, then $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\mathfrak{q})$.

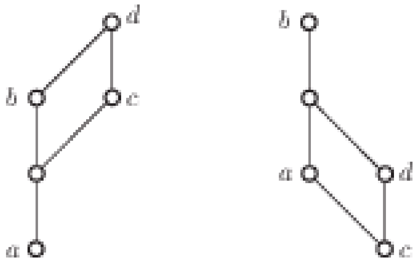

Figure 3. Congruence-perspectivity: $[a,b] \twoheadrightarrow [c,d]$ ($[a,b] \overset{\text{up}}{\twoheadrightarrow} [c,d]$ on the left, $[a,b] \overset{\text{dn}}{\twoheadrightarrow} [c,d]$ on the right).

2.2. The swing relation

Let L be an SPS lattice. For the prime intervals $\mathfrak{p}$ and $\mathfrak{q}$ of L, we define a binary relation: $\mathfrak{p}$ *swings* to $\mathfrak{q}$, in formula, $\mathfrak{p} \curvearrowright \mathfrak{q}$, if $\mathfrak{p}$ and $\mathfrak{q}$ are in a multifork and $\mathfrak{q}$ is an interior prime interval. See Figure 4 for two examples. Let $\mathfrak{p} \curvearrowright \mathfrak{q}$; if $\mathfrak{p}$ is an exterior prime interval of the multifork, we write $\mathfrak{p} \overset{\text{ex}}{\curvearrowright} \mathfrak{q}$ for an *external swing* and if $\mathfrak{p}$ is an interior prime interval of the multifork, we write $\mathfrak{p} \overset{\text{in}}{\curvearrowright} \mathfrak{q}$ for an *interior swing.*

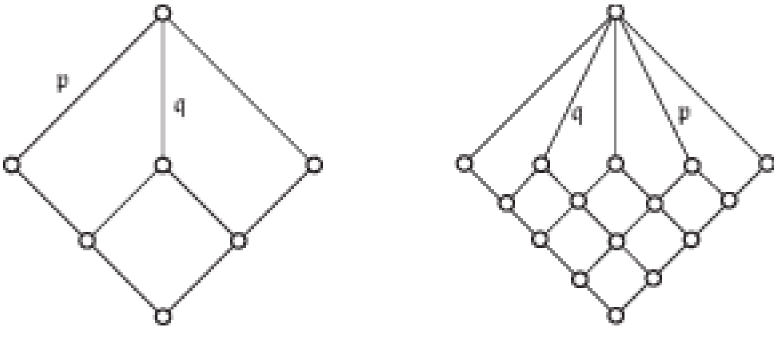

Figure 4. Swing: $\mathfrak{p}$ to $\mathfrak{q}$; two examples: an external swing and an interior swing.

Observation 3. *If* $\mathfrak{p} \overset{\text{in}}{\curvearrowright} \mathfrak{q}$*, then* $\mathfrak{q} \overset{\text{in}}{\curvearrowright} \mathfrak{p}$*.*

For the following result, see Grätzer [19, Lemma 15], [21], and Czédli [2].

Lemma 4 (Swing Lemma)**.** *Let L be an SPS lattice and let $\mathfrak{p}$ and $\mathfrak{q}$ be prime intervals in L. Then* $\operatorname{con}\mathfrak{p} \geq \operatorname{con}\mathfrak{q}$ *iff there exists a prime interval $\mathfrak{r}$ and sequence of prime intervals*

$$\mathfrak{r} = \mathfrak{r}_0, \mathfrak{r}_1, \ldots, \mathfrak{r}_n = \mathfrak{q} \tag{1}$$

such that $\mathfrak{p}$ *is up perspective to* $\mathfrak{r}$*, and* $\mathfrak{r}_i$ *is down perspective to or swings to* $\mathfrak{r}_{i+1}$ *for* $i = 0, \dots, n-1$*. In addition, the sequence* (1) *also satisfies*

$$1_{\mathfrak{r}_0} \geq 1_{\mathfrak{r}_1} \geq \cdots \geq 1_{\mathfrak{r}_n}. \tag{2}$$

See Figure 5 for an illustration with $n = 4$.

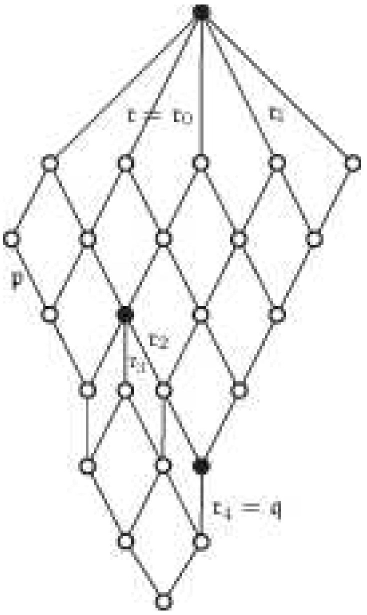

Figure 5. $\mathrm{con}(\mathfrak{p}) \geq \mathrm{con}(\mathfrak{q})$ in five steps. Step 1. $\mathfrak{p}$ is up perspective to $\mathfrak{r} = \mathfrak{r}_0$. Step 2. $\mathfrak{r}_0$ swings to $\mathfrak{r}_1$. Step 3. $\mathfrak{r}_1$ is down perspective to $\mathfrak{r}_2$. Step 4. $\mathfrak{r}_2$ swings to $\mathfrak{r}_3$. Step 5. $\mathfrak{r}_3$ is down perspective to $\mathfrak{r}_4 = \mathfrak{q}$.

Czédli, Grätzer, and Lakser [10] generalizes the Swing Lemma to planar semimodular lattices.

3. ANALYZING THE SWING LEMMA

We now make a number of elementary observations about the Swing Lemma.

Observation 5. *We associate with the sequence* (1) *of prime intervals, the sequence of binary relations* $\varrho_1, \dots, \varrho_{n-1}$ *such that*

$$\mathfrak{r} = \mathfrak{r}_0 \, \varrho_1 \, \mathfrak{r}_1 \, \varrho_2 \cdots \varrho_n \, \mathfrak{r}_n = \mathfrak{q}, \tag{3}$$

where each binary relation is one of $\overset{\text{dn}}{\sim}$, $\overset{\text{ex}}{\curvearrowright}$, $\overset{\text{in}}{\curvearrowright}$ *and* (*here and in the subsequent discussions*) *the relations* $\overset{\text{dn}}{\sim}$ *and* $\overset{\text{in}}{\curvearrowright}$ *are* proper*, that is, they relate two distinct prime intervals.*

Observation 6. *We can assume that down perspectivities and swings alternate.*

Indeed, the relations $\overset{\text{dn}}{\sim}$ and $\overset{\text{in}}{\curvearrowright}$ are transitive, so $\overset{\text{dn}}{\sim} \circ \overset{\text{dn}}{\sim} = \overset{\text{dn}}{\sim}$ and $\overset{\text{in}}{\curvearrowright} \circ \overset{\text{in}}{\curvearrowright} = \overset{\text{in}}{\curvearrowright}$.

Observation 7. If $\varrho_i = \overset{\text{dn}}{\sim}$ for $i < n$, then $\varrho_{i+1} = \overset{\text{ex}}{\curvearrowright}$.

Observation 8. *Let us assume that down perspectivities and swings alternate, see Observation* 6. *Then* ϱ_1 *may be an interior swing. All the other swings in* (3) *are exterior swings.*

The last two observations follow from the fact that there is no down perspectivity to an interior prime interval of a multifork in an SPS lattice.

If $\mathfrak{p} \overset{\text{in}}{\curvearrowright} \mathfrak{q}$ (as in the second diagram of Figure 4), then $\operatorname{con}\mathfrak{p} = \operatorname{con}\mathfrak{q}$; nevertheless, interior swings play an important role, see the example in Figure 5.

In view of these observations, we derive some simple consequences of the Swing Lemma.

Corollary 9. *Let L be an SPS lattice. If* $\mathfrak{q}$ *is an exterior and* $\mathfrak{p}$ *is an interior prime interval of a multifork, then* $\operatorname{con}(\mathfrak{q}) > \operatorname{con}(\mathfrak{p})$.

Proof. We know that $\operatorname{con}(\mathfrak{q}) \geq \operatorname{con}(\mathfrak{p})$. Let us assume that $\operatorname{con}(\mathfrak{q}) = \operatorname{con}(\mathfrak{p})$. Then $\operatorname{con}(\mathfrak{p}) \geq \operatorname{con}(\mathfrak{q})$ and by Observation 5 there is a sequence (3). We must have $\mathfrak{p} = \mathfrak{r}$, because $\mathfrak{p}$ is an interior prime interval. If the first step is a swing, it is to another interior prime interval. So the next step is a down perspectivity. By (2), none of the $\mathfrak{r}_i$ can reach the height of $\mathfrak{q}$ for $i = 2, \ldots, n$. This proves the statement. ■

Corollary 10. *Let* $\mathfrak{p}$ *and* $\mathfrak{q}$ *be prime intervals in an SPS lattice L. If* $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\mathfrak{q})$*, then there is a prime interval* $\mathfrak{r}$ *such that one of the following two conditions hold* (*see Figure* 6):

(i) $\mathfrak{p}$ *is up perspective to* $\mathfrak{q}$ *and* $\mathfrak{q}$ *is down perspective to* $\mathfrak{r}$*; in formula,*

$$\mathfrak{p} \overset{\text{up}}{\sim} \mathfrak{r} \overset{\text{dn}}{\sim} \mathfrak{q}.$$

(ii) $\mathfrak{p}$ *swings interiorly to* $\mathfrak{r}$ *and* $\mathfrak{r}$ *is down perspective to* $\mathfrak{q}$*; in formula,*

$$\mathfrak{p} \overset{\text{in}}{\curvearrowright} \mathfrak{r} \overset{\text{dn}}{\sim} \mathfrak{q}.$$

Proof. If there are no swings in (1), we get (i).

For the sequence (3), by Corollary 9, there can be no external swings. By Observation 7, a perspectivity cannot be followed by an interior swing. So we are left with (ii). ■

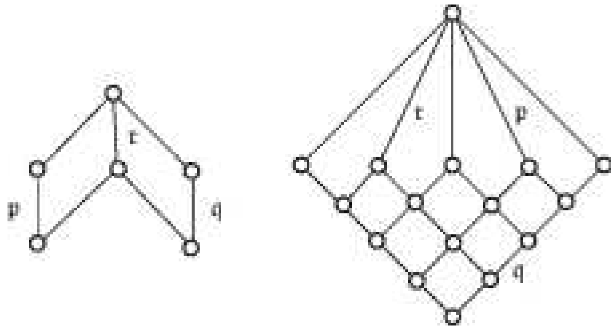

Figure 6. $\operatorname{con}(\mathfrak{p}) = \operatorname{con}(\mathfrak{q})$.

Corollary 11. *Let L be an SPS lattice. If $\mathfrak{s}$ is an exterior prime interval and $\mathfrak{t}$ is an interior prime interval of a multifork, then* $\operatorname{con}(\mathfrak{s}) \succ \operatorname{con}(\mathfrak{t})$ *in the order of join-irreducible congruences of L.*

Proof. Let $\mathfrak{s}'$ denote the other external prime interval. If $\mathfrak{t}$ is a prime interval with $\operatorname{con}(\mathfrak{t}) > \operatorname{con}(\mathfrak{p})$, then we can take a sequence as in (3). We can assume that $\mathfrak{t} = \mathfrak{r}$. Working our way back from $\mathfrak{r}_n = \mathfrak{p}$, the last step cannot be a down perspectivity, because $\mathfrak{r}_n = \mathfrak{p}$ is an interior prime interval. So it must be a swing. If it is an external swing, we get $\operatorname{con}(\mathfrak{t}) \geq \operatorname{con}(\mathfrak{q})$ or $\operatorname{con}(\mathfrak{t}) \geq \operatorname{con}(\mathfrak{q}')$. This proves the statement. ■

4. TRAJECTORIES

Let L be an SPS lattice. The prime intervals $\mathfrak{p}$ and $\mathfrak{q}$ of L are *consecutive*, if they are opposite sides of a 4-cell. A maximal sequence of consecutive prime intervals form a *trajectory*, see, for example, the trajectories in Figure 7. This concept originated in Czédli and Schmidt [12]. See also Czédli and Grätzer [9] for an overview.

A trajectory is a *straight-trajectory*, which goes straight up or straight down or a *hat-trajectory*, which goes up and then it goes down (at least one step each). A trajectory does not branch out. Note that the left and right ends of a trajectory are on the boundary of L. A trajectory $\mathcal{T}$ has a *top prime interval*, $\operatorname{top}(\mathcal{T})$, with the property that $0_{\operatorname{top}(\mathcal{T})} \geq 0_{\mathfrak{q}}$ and $1_{\operatorname{top}(\mathcal{T})} \geq 1_{\mathfrak{q}}$ for any $\mathfrak{q} \in \mathcal{T}$. A trajectory $\mathcal{P}$ *swings* to the trajectory $\mathcal{Q}$, in formula $\mathcal{P} \curvearrowright \mathcal{Q}$, if there is a $\mathfrak{p} \in \mathcal{P}$ and $\mathfrak{q} \in \mathcal{Q}$ such that $\mathfrak{p}$ swings to $\mathfrak{q}$. We define $\operatorname{con}(\mathcal{T}) = \operatorname{con}(\mathfrak{p})$ for any $\mathfrak{p} \in \mathcal{P}$.

Now we state the crucial definition of Czédli [1].

For the trajectories $\mathcal{P} \neq \mathcal{Q}$, let $\mathcal{P} \leq_C \mathcal{Q}$ if $\mathcal{P}$ is a hat trajectory, $1_{\operatorname{top}(\mathcal{P})} \leq 1_{\operatorname{top}(\mathcal{Q})}$, and $0_{\operatorname{top}(\mathcal{P})} \nleq 0_{\operatorname{top}(\mathcal{Q})}$, see Figure 8. Slightly changing Czédli's approach, we define $\leq_T$ as the reflexive and transitive closure of $\leq_C$. (The notation in

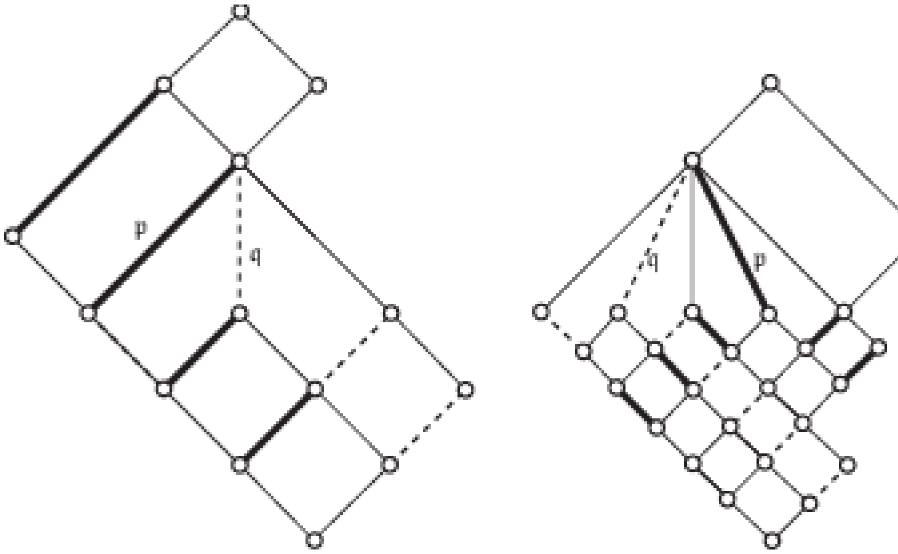

Figure 7. Two trajectories.

Czédli [1] is different.) So for a trajectory $\mathcal{P}$, we can define the closure, $\widehat{\mathcal{P}}$, of $\mathcal{P}$: $\mathcal{Q} \in \widehat{\mathcal{P}}$ iff $\mathcal{P} \leq_C \mathcal{Q}$ and $\mathcal{Q} \leq_C \mathcal{P}$, equivalently, iff $P \leq_T Q$ and $Q \leq_T P$.

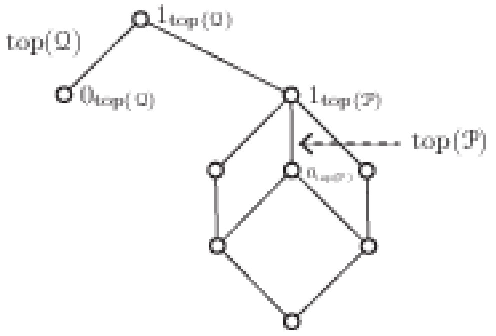

Figure 8. $\mathcal{P} \leq_C \mathcal{Q}$.

Observe that if $\mathcal{P}, \mathcal{P}' \in \widehat{\mathcal{T}}$, then $\mathcal{P} \leq_C \mathcal{Q}$ iff $\mathcal{P}' \leq_C \mathcal{Q}$; similarly, if $\mathcal{Q}, \mathcal{Q}' \in \widehat{\mathcal{T}}$, then $\mathcal{P} \leq_C \mathcal{Q}$ iff $\mathcal{P} \leq_C \mathcal{Q}'$. It follows that, by a slight abuse of terminology, we can use $\leq_T$ as an ordering on $\widehat{\operatorname{Traj}} L$.

For a trajectory $\mathcal{T}$, we can define $\operatorname{con}(\widehat{\mathcal{T}}) = \operatorname{con}(\mathcal{T})$. Indeed, let $\mathcal{P}, \mathcal{Q} \in \widehat{\mathcal{T}}$. Then $\mathcal{P} \leq_C \mathcal{Q}$ and $\mathcal{Q} \leq_C \mathcal{P}$, therefore, $1_{\operatorname{top}(\mathcal{P})} \leq 1_{\operatorname{top}(\mathcal{Q})}$ and $1_{\operatorname{top}(\mathcal{Q})} \leq 1_{\operatorname{top}(\mathcal{P})}$, and so $1_{\operatorname{top}(\mathcal{P})} = 1_{\operatorname{top}(\mathcal{Q})}$. Hence, $\operatorname{top}(\mathcal{P})$ and $\operatorname{top}(\mathcal{Q})$ are interior edges of the multifork at $1_{\operatorname{top}(\mathcal{P})} = 1_{\operatorname{top}(\mathcal{Q})}$ and so $\operatorname{con}(\operatorname{top}(\mathcal{P})) = \operatorname{con}(\operatorname{top}(\mathcal{Q}))$, from which $\operatorname{con}(\mathcal{P}) = \operatorname{con}(\mathcal{Q})$ follows.

5. THE TRAJECTORY THEOREM FOR SPS LATTICES

We have seen that $\widehat{\mathrm{Traj}}\, L$ is an ordered set under the ordering $\leq_T$ and that all the prime intervals $\mathfrak{p}$ in a trajectory $\mathcal{P} \in \widehat{\mathcal{T}}$ generate the same join-irreducible congruence $\mathrm{con}(\mathfrak{p})$ of L. The join-irreducible congruences of L form an ordered set $J(\mathrm{Con}\, L)$. It is the main result of Czédli [1] that these two ordered sets are isomorphic.

Theorem 12 (Trajectory Theorem for SPS Lattices)**.** *Let L be an SPS lattice. Then the ordered set $\widehat{\mathrm{Traj}}\, L$ is isomorphic to the ordered set $J(\mathrm{Con}\, L)$ under the isomorphism $\widehat{\mathcal{T}} \mapsto \mathrm{con}(\widehat{\mathcal{T}})$.*

Proof. First, we prove that

$$\mathcal{P} \leq_T \mathcal{Q} \text{ implies that } \mathrm{con}(\mathcal{P}) \leq \mathrm{con}(\mathcal{Q}). \tag{4}$$

Since $\leq_T$ is the reflexive and transitive closure of $\leq_C$, it is sufficient to prove (4) for $\mathcal{P} \leq_C \mathcal{Q}$. So assume the following: $\mathcal{P} \neq \mathcal{Q}$, $\mathcal{P}$ is a hat trajectory, $1_{\mathrm{top}(\mathcal{P})} \leq 1_{\mathrm{top}(\mathcal{Q})}$, and $0_{\mathrm{top}(\mathcal{P})} \nleq 0_{\mathrm{top}(\mathcal{Q})}$, see Figure 8. Then

$$0_{\mathrm{top}(\mathcal{Q})} \equiv 1_{\mathrm{top}(\mathcal{Q})} \pmod{\mathrm{con}\mathcal{Q}},$$

so

$$0_{\mathrm{top}(\mathcal{Q})} \wedge 1_{\mathrm{top}(\mathcal{Q})} \equiv 1_{\mathrm{top}(\mathcal{Q})} \wedge 1_{\mathrm{top}(\mathcal{Q})} = 1_{\mathrm{top}(\mathcal{Q})} \pmod{\mathrm{con}\mathcal{Q}}.$$

Let $0_{\mathrm{top}(\mathcal{Q})} \wedge 1_{\mathrm{top}(\mathcal{Q})} \leq a \prec 1_{\mathrm{top}(\mathcal{Q})}$. We conclude that

$$\mathrm{con}(\mathcal{Q}) = \mathrm{con}(\mathrm{top}(\mathcal{Q})) \geq \mathrm{con}\big([a, 1_{\mathrm{top}(\mathcal{Q})}]\big) \geq \mathrm{con}(\mathrm{top}(\mathcal{Q})) = \mathrm{con}(\mathcal{P}),$$

verifying (4).

Let $a = 0_{\mathrm{top}(\mathcal{Q})} \wedge 1_{\mathrm{top}(\mathcal{P})}$, and remember that $\mathcal{P}$ is a hat trajectory by definition. Since $a < 1_{\mathrm{top}(\mathcal{P})}$, there is a prime interval $\mathfrak{r}$ in the multifork with top $1_{\mathrm{top}(\mathcal{P})}$ such that $a \leq 0_{\mathfrak{r}}$. Hence, $\mathrm{top}(\mathcal{Q})$ is down-congruence perspective to $\mathfrak{r}$, and we have $\mathrm{con}(\mathcal{Q}) \geq \mathrm{con}(\mathfrak{r})$. Since $\mathrm{top}(\mathcal{P})$ is an interior prime interval of our multifork, it follows that $\mathrm{con}(\mathfrak{r}) \geq \mathrm{con}(\mathrm{top}(\mathcal{P})) = \mathrm{con}(\mathcal{P})$. Thus, $\mathrm{con}(\mathcal{Q}) \geq \mathrm{con}(\mathcal{P})$, verifying (4).

Second, we prove the converse of (4):

$$\mathrm{con}(\mathcal{P}) \leq \mathrm{con}(\mathcal{Q}) \text{ implies that } \mathcal{P} \leq_T \mathcal{Q}. \tag{5}$$

Let $\mathfrak{r} = \mathrm{top}(\mathcal{P})$ and $\mathfrak{q} = \mathrm{top}(\mathcal{Q})$. By the Swing Lemma and Observation 5, we get the sequence (3) of binary relations. Note that

(a) trajectories are closed with respect to up and down perspectivities;

(b) the equivalence class $\widehat{P}$ of a trajectory P is closed with respect to interior swings;

(c) whenever r_{i-1} externally swings to r_i, then r_i is the top of a hat trajectory R_i and (denoting the trajectory of r_{i-1} by R_{i-1}), we clearly have that $R_{i-1} \geq_C R_i$.

This completes the proof of (5). ∎

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