

GENERALIZED DERIVATIONS WITH LEFT ANNIHILATOR CONDITIONS IN PRIME AND SEMIPRIME RINGS

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Abstract

Let R be a prime ring with its Utumi ring of quotients U , $C = Z(U)$ be the extended centroid of R , H and G two generalized derivations of R , L a noncentral Lie ideal of R , I a nonzero ideal of R . The left annihilator of $S \subseteq R$ is denoted by $l_R(S)$ and defined by $l_R(S) = \{x \in R \mid xS = 0\}$. Suppose that $S = \{H(u^n)u^n + u^nG(u^n) \mid u \in L\}$ and $T = \{H(x^n)x^n + x^nG(x^n) \mid x \in I\}$, where $n \geq 1$ is a fixed integer. In the paper, we investigate the cases when the sets $l_R(S)$ and $l_R(T)$ are nonzero.

Keywords: prime ring, derivation, Lie ideal, generalized derivation, extended centroid, Utumi quotient ring.

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1. INTRODUCTION

Let R be an associative ring with center $Z(R)$. For $x, y \in R$, the commutator of x, y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$. By d we mean a derivation of R . An additive mapping F from R to R is called a generalized derivation if there exists a derivation d from R to R such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Throughout this paper, R will always present a prime ring with center $Z(R)$, extended centroid C and U is its Utumi quotient ring. A well known result proved by Posner [20], states that if the commutators $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. Then result of Posner was generalized in many

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directions by a number of authors. Posner's theorem was extended to Lie ideals in prime rings by Lee [17] and then by Lanski [12].

On the other hand, authors generalized Posner's theorem by considering two derivations. In [3], Brešar proved that if d and δ are two derivations of R such that $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$, then either $d = \delta = 0$ or R is commutative. Later Lee and Wong [18] consider the situation $d(x)x - x\delta(x) \in Z(R)$ for all x in some noncentral Lie ideal L of R and they proved that either $d = \delta = 0$ or R satisfies s_4 .

Recently in [22] Vukman proves that if d and δ are derivations on a $2mn(m+n-1)!$ -torsion free semiprime rings R such that $d(x^m)x^n + x^n\delta(x^m) = 0$ for all $x \in R$, where $m, n \geq 1$ are fixed integers, then both derivations d and δ map R into its center and $d = -\delta$.

In [23], Wei and Xiao studied the similar situation replacing derivations d and δ by generalized derivations G and H . More precisely they proved the following:

Let m, n be fixed positive integers, R be a noncommutative $2(m+n)!$ -torsion free prime ring and G, H be a pair of generalized Jordan derivations on R . If $G(x^m)x^n + x^nH(x^m) \in Z(R)$ for all $x \in R$, then G and H both are right (or left) multipliers.

In [14], Lee and Zhou studied the same situation of above result without considering torsion free restriction on R . In this paper, Lee and Zhou [14] proved the following:

Let R be a prime ring that is not commutative and such that $R \not\cong M_2(GF(2))$, let G, H be two generalized derivations of R , and let m, n be two fixed positive integers. Then $G(x^m)x^n - x^nH(x^m) = 0$ for all $x \in R$ iff the following two conditions hold:

- (1) *There exists $w \in Q$ such that $G(x) = xw$ and $H(x) = wx$ for all $x \in R$;*
- (2) *either $w \in C$, or x^m and x^n are C -dependent for all $x \in R$.*

There are many papers in the literature which studied the identities of generalized derivations with left annihilator conditions.

For any subset S of R , denote by $r_R(S)$ the right annihilator of S in R , that is, $r_R(S) = \{x \in R \mid Sx = 0\}$ and $l_R(S)$ the left annihilator of S in R that is, $l_R(S) = \{x \in R \mid xS = 0\}$. If $r_R(S) = l_R(S)$, then $r_R(S)$ is called an annihilator ideal of R and is written as $ann_R(S)$.

In [4], Carini *et al.* studied the left annihilator of the set $\{H(u)u - uG(u) \mid u \in L\}$, where L is a noncentral Lie ideal of R and H, G two non-zero generalized derivations of R . In case the annihilator is not zero, the conclusion is one of the following:

- (1) *there exist $b', c' \in U$ such that $H(x) = b'x + xc'$, $G(x) = c'x$ with $ab' = 0$;*
- (2) *R satisfies s_4 and there exist $b', c', q' \in U$ such that $H(x) = b'x + xc'$, $G(x) = c'x + xq'$, with $a(b' - q') = 0$.*

Recently, Carini and De Filippis proved the following theorem:

Let R be a prime ring, U the Utumi quotient ring of R , $C = Z(U)$ the extended centroid of R , L a non-central Lie ideal of R , H and G non-zero generalized derivations of R . Suppose that there exists an integer $n \geq 1$ such that $H(u^n)u^n + u^nG(u^n) \in C$, for all $u \in L$, then either there exists $a \in U$ such that $H(x) = xa, G(x) = -ax$, or R satisfies the standard identity s_4 . Moreover, in the last case the structures of the maps G, H are obtained.

In the present paper, we shall investigate the left annihilator of the sets $\{H(u^n)u^n + u^nG(u^n) \mid u \in L\}$ and $\{H(x^n)x^n + x^nG(x^n) \mid x \in I\}$, where L is a noncentral Lie ideal of R , I is a nonzero ideal of R , $n \geq 1$ is a fixed integer and H, G two non-zero generalized derivations of R . More precisely, we prove the following theorems:

Theorem 1.1. *Let R be a prime ring with its Utumi ring of quotients U , $C = Z(U)$ be the extended centroid of R , H and G two generalized derivations of R , L a noncentral Lie ideal of R and $S = \{H(u^n)u^n + u^nG(u^n) \mid u \in L\}$, where $n \geq 1$ is a fixed integer. If $l_R(S) \neq \{0\}$, then either there exist $b', p \in U$ such that $H(x) = b'x - xp$ and $G(x) = px$ for all $x \in R$ with $ab' = 0$ or R satisfies s_4 . Moreover, in the last case, if R satisfies s_4 , then one of the following holds:*

- (1) $\text{char}(R) = 2$;
- (2) n is even, there exist $b, p \in U$ and derivations d, δ of R such that $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$ for all $x \in R$, with $a(b + p) = 0$;
- (3) n is odd, there exist $b, p \in U$ and derivations d, δ of R such that $H(x) = bx + d(x)$ and $G(x) = xp + \delta(x)$ for all $x \in R$, with $a(b + p) = 0$.

Theorem 1.2. *Let R be a noncommutative prime ring with $\text{char}(R) \neq 2$, U its Utumi ring of quotients, $C = Z(U)$ be the extended centroid of R , H and G two generalized derivations of R , I a nonzero ideal of R and $S = \{H(x^n)x^n + x^nG(x^n) \mid x \in I\}$, where $n \geq 1$ is a fixed integer. If $l_R(S) \neq \{0\}$, then there exist $b', p \in U$ such that $H(x) = b'x - xp$ and $G(x) = px$ for all $x \in R$ with $ab' = 0$.*

As an immediate application of the Theorem 1.1, in particular when $G = -H$, then we have the following result which gives a particular result of Theorem 1.1 in [6].

Corollary 1.3. *Let R be a prime ring with its Utumi ring of quotients U , $C = Z(U)$ be the extended centroid of R , H a generalized derivation of R and L a noncentral Lie ideal of R . Suppose that there exists $0 \neq a \in R$ such that $a[H(u^n), u^n] = 0$ for all $u \in L$, where $n \geq 1$ is a fixed integer. Then either there exists $\lambda \in C$ such that $H(x) = \lambda x$ for all $x \in R$ or R satisfies s_4 .*

As an application of the Theorem 1.1, in particular when $G = 0$, then using

Theorem 2.2 in [8], we have the following result which gives a generalization of Theorem 1.1 in [21].

Corollary 1.4. *Let R be a prime ring of char $(R) \neq 2$ with its Utumi ring of quotients U , $C = Z(U)$ be the extended centroid of R , H a generalized derivation of R and L a noncentral Lie ideal of R . Suppose that there exists $0 \neq a \in R$ such that $aH(u^n)u^n = 0$ for all $u \in L$, where $n \geq 1$ is a fixed integer. Then either there exist $b', p \in U$ such that $H(x) = b'x$ for all $x \in R$ with $ab' = 0$.*

2. PROOF OF MAIN RESULTS IN PRIME RINGS

Let R be a prime ring with extended centroid C . Let $H(x) = bx + xc$ and $G(x) = px + xq$ for all $x \in R$ and for some $b, c, p, q \in U$, be two inner generalized derivations of R and L be a noncentral Lie ideal of R . Then $a(H(x^n)x^n + x^nG(x^n)) = 0$ implies $a(bx^{2n} + x^n(c+p)x^n + x^{2n}q) = 0$ for all $x \in L$. We know that if char $(R) \neq 2$, by [2, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. If char $(R) = 2$ and $\dim_C RC > 4$ i.e., char $(R) = 2$ and R does not satisfy s_4 , then by [13, Theorem 13] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. We assume that R does not satisfy s_4 . Then in any case of char $(R) = 2$ or char $(R) \neq 2$, we can conclude that there exists a nonzero ideal I of R such that $0 \neq [I, I] \subseteq L$. By hypothesis, we have

$$(1) \quad a(b[x_1, x_2]^{2n} + [x_1, x_2]^n(c+p)[x_1, x_2]^n + [x_1, x_2]^{2n}q) = 0$$

for all $x_1, x_2 \in I$. Then following lemmas are immediate consequences:

Lemma 2.1. *R satisfies a nontrivial generalized polynomial identity (GPI) or $c, p, q \in C$ such that $a(b+c+p+q) = 0$.*

Proof. Assume that R does not satisfy any nontrivial GPI. Then R must be noncommutative. Let $T = U *_C C\{x_1, x_2\}$, the free product of U and $C\{x_1, x_2\}$, the free C -algebra in noncommuting indeterminates x_1 and x_2 .

Then,

$$a(b[x_1, x_2]^{2n} + [x_1, x_2]^n(c+p)[x_1, x_2]^n + [x_1, x_2]^{2n}q)$$

is zero element in T . If $q \notin C$, then q and 1 are linearly independent over C . Then from above

$$a[x_1, x_2]^{2n}q = 0 \in T,$$

implying $q = 0$, since $a \neq 0$, a contradiction. Therefore, we conclude that $q \in C$.

Then by hypothesis

$$(2) \quad a((b + q)[x_1, x_2]^n + [x_1, x_2]^n(c + p))[x_1, x_2]^n = 0 \in T.$$

If $c + p \notin C$, then by (2)

$$a([x_1, x_2]^n(c + p)) [x_1, x_2]^n = 0 \in T,$$

implying $c + p = 0$, since $a \neq 0$, a contradiction. Therefore, we have $c + p \in C$ and hence

$$a(b + q + c + p)[x_1, x_2]^{2n} = 0 \in T.$$

This implies $a(b + q + c + p) = 0$. ■

Lemma 2.2. $c + p, q \in C$ with $a(b + c + p + q) = 0$, unless R satisfies s_4 .

Proof. By hypothesis, R satisfies GPI

$$(3) \quad f(x_1, x_2) = a(b[x_1, x_2]^{2n} + [x_1, x_2]^n(c + p)[x_1, x_2]^n + [x_1, x_2]^{2n}q).$$

If R does not satisfy any nontrivial GPI, by Lemma 2.1, we obtain $c, p, q \in C$ with $a(b + c + p + q) = 0$ which gives the conclusion. So, we assume that R satisfies a nontrivial GPI. Since R and U satisfy the same generalized polynomial identities (see [5]), U satisfies $f(x_1, x_2)$. In case C is infinite, we have $f(x_1, x_2) = 0$ for all $x_1, x_2 \in U \otimes_C \overline{C}$, where \overline{C} is the algebraic closure of C . Moreover, both U and $U \otimes_C \overline{C}$ are prime and centrally closed algebras [9]. Hence, replacing R by U or $U \otimes_C \overline{C}$ according to C finite or infinite, without loss of generality we may assume that $C = Z(R)$ and R is C -algebra centrally closed. By Martindale's theorem [19], R is then a primitive ring having nonzero socle $\text{soc}(R)$ with C as the associated division ring. Hence, by Jacobson's theorem [10, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C .

If $\dim_C V = 2$, then $R \cong M_2(C)$, that is, R satisfies s_4 , a contradiction. So, let $\dim_C V \geq 3$.

We show that for any $v \in V$, v and qv are linearly C -dependent. Suppose that v and qv are linearly independent for some $v \in V$. Since $\dim_C V \geq 3$, there exists $u \in V$ such that v, qv, u are linearly C -independent set of vectors. By density, there exists $x_1, x_2 \in R$ such that

$$x_1v = v, \quad x_1qv = 0, \quad x_1u = qv; \quad x_2v = 0, \quad x_2qv = u, \quad x_2u = 0.$$

Then $0 = a(b[x_1, x_2]^{2n} + [x_1, x_2]^n(c + p)[x_1, x_2]^n + [x_1, x_2]^{2n}q)v = aqv$.

This implies that if for some $v \in V$, $aqv \neq 0$, then by contradiction, v and qv are linearly C -dependent.

Now choose $v \in V$ such that v and qv are linearly C -independent. Then $aqv = 0$. Let us consider a subspace $W = \{\alpha v + \beta qv \mid \alpha, \beta \in C\}$ of V . Let $aq \neq 0$. Then, there exists $w \in V$ such that $aqw \neq 0$. Then $aq(v-w) = aqw \neq 0$. Then by the above argument, w, qw are linearly C -dependent and $(v-w), q(v-w)$ too. Thus there exist $\alpha, \beta \in C$ such that $qw = \alpha w$ and $q(v-w) = \beta(v-w)$. Then $qv = \beta(v-w) + qw = \beta(v-w) + \alpha w$ i.e., $(\alpha - \beta)w = qv - \beta v \in W$. Now $\alpha = \beta$ implies that $qv = \beta v$, a contradiction. Hence $\alpha \neq \beta$ and so $w \in W$.

Next assume that $u \in V$ such that $aqu = 0$. Then $aq(w+u) = aqw \neq 0$. By above argument, $aq(w+u) \neq 0$ implies $w+u \in W$. Since $w \in W$, we have $u \in W$. Thus it is observed that for any $v \in V$, $aqv \neq 0$ implies $v \in W$ and $aqv = 0$ implies $v \in W$. This implies that $V = W$ i.e., $\dim_C V = 2$, a contradiction.

Thus up to now we have proved that v and qv are linearly C -dependent for all $v \in V$, unless $aq = 0$. If $aq \neq 0$, by standard argument, it follows that $qv = \lambda v$ for all $v \in V$ and $\lambda \in C$ fixed. Then $(q - \lambda)V = 0$, implying $q = \lambda \in C$.

Now let $aq = 0$. Since $\dim_C V \geq 3$, there exists $w \in V$ such that v, qv, w are linearly C -independent set of vectors. By density, there exists $x_1, x_2 \in R$ such that

$$x_1v = v, \quad x_1qv = 0, \quad x_1w = v + qv; \quad x_2v = 0, \quad x_2qv = w, \quad x_2w = 0.$$

Then $0 = a(b[x_1, x_2]^{2n} + [x_1, x_2]^n(c+p)[x_1, x_2]^n + [x_1, x_2]^{2n}q)v = av$. Then by above argument, since $a \neq 0$, $q \in C$.

Therefore, we have proved that in any case $q \in C$. Hence our identity reduces to

$$a(b'[x_1, x_2]^{2n} + [x_1, x_2]^n c'[x_1, x_2]^n) = 0,$$

where $b' = b + q$ and $c' = c + p$.

Now we prove that v and $c'v$ are linearly C -dependent. If possible let v and $c'v$ be linearly independent for some $v \in V$. Then there exists $w \in V$ such that $v, c'v$ and w are linearly independent over C . By density there exist $x_1, x_2 \in R$ such that

$$x_1v = 0, \quad x_1c'v = v, \quad x_1w = 2c'v; \quad x_2v = c'v, \quad x_2c'v = w, \quad x_2w = 0.$$

Then $0 = a(b'[x_1, x_2]^{2n} + [x_1, x_2]^n c'[x_1, x_2]^n)v = a(b' + c')v$. As above, this implies either $a(b' + c') = 0$ or $c' \in C$. Let $a(b' + c') = 0$. Then we have that R satisfies $0 = a[c', [x_1, x_2]^n][x_1, x_2]^n$. By density there exist $x_1, x_2 \in R$ such that

$$x_1v = 0, \quad x_1c'v = v, \quad x_1w = c'v; \quad x_2v = c'v, \quad x_2c'v = w, \quad x_2w = 0.$$

Thus $0 = a[c', [x_1, x_2]^n][x_1, x_2]^n v = ac'v$. This implies either $ac' = 0$ or $c' \in C$. Let $ac' = 0$. Then we have that R satisfies $0 = a[x_1, x_2]^n c'[x_1, x_2]^n$. Again by density there exist $x_1, x_2 \in R$ such that

$$x_1v = 0, \quad x_1c'v = v, \quad x_1w = v + c'v; \quad x_2v = c'v, \quad x_2c'v = w, \quad x_2w = 0.$$

Thus $0 = a[x_1, x_2]^n c' [x_1, x_2]^n v = av$. Since $a \neq 0$, this implies $c' \in C$. Thus in any case, we have $c' \in C$. Hence R satisfies $0 = a(b' + c')[x_1, x_2]^{2n}$, which implies $a(b' + c') = 0$. ■

Proof of Theorem 1.1. Let $0 \neq a \in l_R(S)$. Then $a(H(u^n)u^n + u^nG(u^n)) = 0$ for all $u \in L$. If $\text{char}(R) = 2$ and R satisfies s_4 , then we obtain our conclusion (1). So we assume that either $\text{char}(R) \neq 2$ or R does not satisfy s_4 . Then by [2, Lemma 1] and [13, Theorem 13], since L is a noncentral Lie ideal of R , there exists a nonzero ideal I of R such that $[I, I] \subseteq L$. Hence, by our hypothesis, we have,

$$a(H([x_1, x_2]^n)[x_1, x_2]^n + [x_1, x_2]^nG([x_1, x_2]^n)) = 0$$

for all $x_1, x_2 \in I$. Since I, R and U satisfy the same generalized polynomial identities (see [5]) as well as the same differential identities (see [16]), they also satisfy the same generalized differential identities. Hence, by [15], U satisfies

$$a(H([x_1, x_2]^n)[x_1, x_2]^n + [x_1, x_2]^nG([x_1, x_2]^n)) = 0$$

for all $x_1, x_2 \in U$, where $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$, for some $b, p \in U$ and derivations d and δ of U , that is, U satisfies

$$(4) \quad a(b[x_1, x_2]^{2n} + d([x_1, x_2]^n)[x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n \delta([x_1, x_2]^n)) = 0.$$

Now we divide the proof into two cases:

Case I. Let $d(x) = [c, x]$ for all $x \in U$ and $\delta(x) = [q, x]$ for all $x \in U$ i.e., d and δ be inner derivations of U . Then from (4), we obtain that U satisfies

$$(5) \quad a((b + c)[x_1, x_2]^{2n} + [x_1, x_2]^n(p - c + q)[x_1, x_2]^n - [x_1, x_2]^{2n}q) = 0.$$

By Lemma 2.2, when R does not satisfy s_4 , we have $q, p - c + q \in C$ with $a(b + p) = 0$. This implies $p - c \in C$. Hence $H(x) = bx + [c, x] = bx + [p, x] = b'x - xp$, $G(x) = px$ for all $x \in U$ and so for all $x \in R$ with $ab' = 0$, where $b' = b + p$.

Moreover, when R satisfies s_4 (in this case by assumption $\text{char}(R) \neq 2$), then $R \subseteq M_2(F)$ and, R and $M_2(F)$ satisfy the same GPI, where $M_2(F)$ is a matrix ring over a field F . Hence $M_2(F)$ satisfies $a((b + c)[x_1, x_2]^{2n} + [x_1, x_2]^n(p - c + q)[x_1, x_2]^n - [x_1, x_2]^{2n}q) = 0$. Since $[x, y]^2 \in Z(M_2(F))$ for all $x, y \in M_2(F)$, $M_2(F)$ satisfies

$$(6) \quad a((b + c - q)[x_1, x_2]^{2n} + [x_1, x_2]^n(p - c + q)[x_1, x_2]^n) = 0.$$

If n is even, then by choosing $x_1 = e_{12}, x_2 = e_{21}$, we have $0 = a(b + p)$.

If n is odd, then $M_2(F)$ satisfies $a((b + c - q)[x_1, x_2] + [x_1, x_2](p - c + q))[x_1, x_2]^{2n-1} = 0$. By Lemma 2.7 in [7], we conclude that $p - c + q \in Z(R)$ and $a(b + p) = 0$.

Thus when R satisfies s_4 , one of the following holds:

- (i) n is even and $a(b + p) = 0$. In this case, $H(x) = bx + [c, x]$ and $G(x) = px + [q, x]$ for all $x \in R$, with $a(b + p) = 0$. This is our conclusion (2).
- (ii) n is odd and $p - c + q \in C$ and $a(b + p) = 0$. Hence $H(x) = bx + [c, x]$ and $G(x) = px + [q, x] = px - [p - c, x] = xp + [c, x]$ for all $x \in R$, with $a(b + p) = 0$. This is our conclusion (3).

Case II. Next assume that d and δ are not both inner derivations of U , but they are C -dependent modulo inner derivations of U . Suppose $d = \lambda\delta + ad_c$, that is, $d(x) = \lambda\delta(x) + [c, x]$ for all $x \in U$, where $\lambda \in C$, $c \in U$. Then d can not be inner derivation of U . From (4), we have that U satisfies

$$a \left(b[x_1, x_2]^{2n} + \lambda\delta([x_1, x_2]^n)[x_1, x_2]^n + [c, [x_1, x_2]^n][x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n \delta([x_1, x_2]^n) \right) = 0.$$

This gives

$$a \left(b[x_1, x_2]^{2n} + \lambda \sum_{i=0}^{n-1} [x_1, x_2]^i \delta([x_1, x_2])[x_1, x_2]^{n-1-i} [x_1, x_2]^n + [c, [x_1, x_2]^n][x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i \delta([x_1, x_2])[x_1, x_2]^{n-1-i} \right) = 0.$$

Then by Kharchenko's theorem [11], we have that U satisfies

$$(7) \quad a \left(b[x_1, x_2]^{2n} + \lambda \sum_{i=0}^{n-1} [x_1, x_2]^i ([y_1, x_2] + [x_1, y_2])[x_1, x_2]^{n-1-i} [x_1, x_2]^n + [c, [x_1, x_2]^n][x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i ([y_1, x_2] + [x_1, y_2])[x_1, x_2]^{n-1-i} \right) = 0.$$

In particular U satisfies blended component

$$(8) \quad a \left(b[x_1, x_2]^{2n} + [c, [x_1, x_2]^n][x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n \right) = 0$$

and

$$(9) \quad a \left(\lambda \sum_{i=0}^{n-1} [x_1, x_2]^i ([y_1, x_2] + [x_1, y_2])[x_1, x_2]^{n-1-i} [x_1, x_2]^n + [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i ([y_1, x_2] + [x_1, y_2])[x_1, x_2]^{n-1-i} \right) = 0.$$

For $y_1 = [q, x_1]$ and $y_2 = [q, x_2]$, where $q \notin C$ we have that U satisfies

$$(10) \quad a([\lambda q, [x_1, x_2]^n][x_1, x_2]^n + [x_1, x_2]^n[q, [x_1, x_2]^n]) = 0.$$

By Lemma 2.2, if R does not satisfy s_4 , then $q \in C$, a contradiction. Hence we conclude that R satisfies s_4 . Now the relations (8) and (10) are similar to the relation (5). Thus by same argument as given in Case I, when R satisfies s_4 (in this case $\text{char}(R)$ must be not equal to 2), one of the following holds:

(i) Let n be even. Then by (8), $a(b + p) = 0$. Thus $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$ for all $x \in R$, with $a(b + p) = 0$. This is our conclusion (2).

(ii) Let n be odd. Then by (8), $p - c \in C$ and $a(b + p) = 0$. Again by (10), $q - \lambda q = q(1 - \lambda) \in C$. Since $q \notin C$, we have $\lambda = 1$. Then replacing $y_1 = x_1$ and $y_2 = 0$, (9) gives $na(\lambda + 1)[x_1, x_2]^{2n} = 0$, implying $2na = 0$. Since $\text{char}(R) \neq 2$, $na = 0$. Hence $H(x) = bx + \lambda\delta(x) + [c, x] = bx + \delta(x) + [c, x]$ and $G(x) = px + \delta(x) = (p - c)x + cx + \delta(x) = x(p - c) + cx + \delta(x) = xp + \delta(x) + [c, x]$ for all $x \in R$. This is our conclusion (3).

The situation when $\delta = \lambda d + ad_c$ is similar.

Next assume that d and δ are C -independent modulo inner derivations of U . Since neither d nor δ is inner, by Kharchenko's Theorem [11], we have from (4) that U satisfies

$$(11) \quad a\left(b[x_1, x_2]^{2n} + \sum_{i=0}^{n-1} [x_1, x_2]^i([u_1, x_2] + [x_1, u_2])[x_1, x_2]^{n-1-i}[x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i([v_1, x_2] + [x_1, v_2])[x_1, x_2]^{n-1-i}\right) = 0.$$

Then U satisfies blended component

$$(12) \quad a\left(b[x_1, x_2]^{2n} + [x_1, x_2]^n p[x_1, x_2]^n\right) = 0$$

and

$$(13) \quad a\left([x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i([v_1, x_2] + [x_1, v_2])[x_1, x_2]^{n-1-i}\right) = 0.$$

Replacing v_1 with $[q, x_1]$ and v_2 with $[q, x_2]$ for some $q \notin C$ in (13), we obtain that U satisfies

$$(14) \quad a([x_1, x_2]^n[q, [x_1, x_2]^n]) = 0.$$

By Lemma 2.2, we have $q \in C$, a contradiction, unless R satisfies s_4 . So we consider the case when R satisfies s_4 . In this case by same argument of Case I, (12) and (14) together implies that n is even and $a(b + p) = 0$. This gives our conclusion (2). Hence the theorem is proved. ■

Corollary 2.3. *Let R be a prime ring with its Utumi ring of quotients U , $C = Z(U)$ be the extended centroid of R , H and G two generalized derivations of R and L a noncentral Lie ideal of R . Suppose that there exists $0 \neq a \in R$ such that $a(H(u^2)u^2 + u^2G(u^2)) = 0$ for all $u \in L$. Then either there exist $b', p \in U$ such that $H(x) = b'x - xp$ and $G(x) = px$ for all $x \in R$ with $ab' = 0$ or R satisfies s_4 . Moreover, if R satisfies s_4 , then one of the following holds:*

- (1) $\text{char}(R) = 2$;
- (2) there exist $b, p \in U$ and derivations d, δ of R such that $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$ for all $x \in R$, with $a(b + p) = 0$.

Proof of Theorem 1.2. Let $0 \neq a \in l_R(S)$. Then $a(H(x^n)x^n + x^nG(x^n)) = 0$ for all $x \in I$. By Theorem 1.1, we have only to consider the case when R satisfies s_4 . In this case R is a PI-ring, and so there exists a field K such that $R \subseteq M_2(K)$ and, R and $M_2(K)$ satisfy the same GPI. First we assume that H and G are inner generalized derivations of R , that is, $H(x) = bx + xc$ for all $x \in R$ and $G(x) = px + xq$ for all $x \in R$, for some $b, c, p, q \in R$. Since $M_2(F)$ is a simple ring, by our hypothesis, $M_2(F)$ satisfies

$$(15) \quad a(bx^{2n} + x^n(c + p)x^n + x^{2n}q) = 0.$$

Moreover, R is a dense ring of K -linear transformations over a vector space V . Let $aq \neq 0$. Assume there exists $v \neq 0$, such that $\{v, qv\}$ is linear K -independent. By the density of R , there exists $r \in R$ such that

$$rv = 0; \quad r(qv) = qv.$$

Hence

$$0 = a(br^{2n} + r^n(c + p)r^n + r^{2n}q)v = aqv.$$

Of course for any $w \in V$ such that $\{w, v\}$ are linearly K -dependent implies $aqw = 0$. Since $aq \neq 0$, there exists $w \in V$ such that $aqw \neq 0$. Then $\{w, v\}$ must be linearly K -independent. By the above argument it follows that w and qw are linearly K -dependent, as are $\{w + v, q(w + v)\}$ and $\{w - v, q(w - v)\}$. Therefore there exist $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in K$ such that

$$qw = \alpha_w w, \quad q(w + v) = \alpha_{w+v}(w + v), \quad q(w - v) = \alpha_{w-v}(w - v).$$

In other words we have

$$(16) \quad \alpha_w w + qv = \alpha_{w+v} w + \alpha_{w+v} v$$

and

$$(17) \quad \alpha_w w - qv = \alpha_{w-v} w - \alpha_{w-v} v.$$

By comparing (16) with (17) we get both

$$(18) \quad (2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0$$

and

$$(19) \quad 2qv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v.$$

By (18) and since $\{w, v\}$ is K -independent and $\text{char}(K) \neq 2$, we have $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$. Thus by (19) it follows $2qv = 2\alpha_w v$. Since $\{qv, v\}$ is K -independent, the conclusion $\alpha_w = \alpha_{w+v} = 0$ follows, that is $qw = 0$ and $q(w+v) = 0$, which implies the contradiction $qv = 0$.

Hence we conclude that for any $v \in V$, $\{v, qv\}$ is linearly K -dependent. Thus there exists a suitable $\alpha_v \in K$ such that $qv = \alpha_v v$, and standard argument shows that there is $\alpha \in K$ such that $qv = \alpha v$ for all $v \in V$. Now let $r \in R, v \in V$. Since $qv = \alpha v$,

$$(20) \quad [q, r]v = (qr)v - (rq)v = q(rv) - r(qv) = \alpha(rv) - r(\alpha v) = 0.$$

Thus $[q, r]v = 0$ for all $v \in V$ i.e., $[q, r]V = 0$. Since $[q, r]$ acts faithfully as a linear transformation on the vector space V , $[q, r] = 0$ for all $r \in R$. Therefore, $q \in C$.

Thus up to now, we have proved that either $aq = 0$ or $q \in C$.

Let $aq = 0$. In this case, assume that there exists $v \neq 0$, such that $\{v, qv\}$ is linear K -independent. By the density of R , there exists $r \in R$ such that

$$rv = 0; \quad r(qv) = v + qv.$$

Hence

$$0 = a(br^{2n} + r^n(c+p)r^n + r^{2n}q)v = av.$$

Thus by the same argument as above, this implies either $a = 0$ or $q \in C$. Since $a \neq 0, q \in C$.

Thus in any case we conclude that $q \in C$.

Then (15) reduces to

$$(21) \quad a((b+q)x^n + x^n(c+p))x^n = 0.$$

Let there exists $v \neq 0$, such that $\{v, (c+p)v\}$ is linear K -independent. By the density of R , there exists $r \in R$ such that

$$rv = 0; \quad r((c+p)v) = (c+p)v.$$

Hence

$$0 = a((b+q)r^n + r^n(c+p))r^n v = a(c+p)v.$$

Then again by same argument, $c + p \in C$. Then (21) reduces to

$$(22) \quad a(b + c + p + q)x^{2n} = 0$$

for all $x \in R$. This implies $a(b + c + p + q) = 0$, where $q, c + p \in C$. Hence $H(x) = bx + xc = bx + x(c + p) - xp = (b + c + p)x - xp = (b + c + p + q)x - x(p + q)$ for all $x \in R$ and $G(x) = (p + q)x$ for all $x \in R$. This gives our conclusion.

Next assume that $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$, where d, δ are not both inner derivations of R . In this case by our hypothesis, R satisfies

$$(23) \quad a(bx^{2n} + d(x^n)x^n + x^npx^n + x^n\delta(x^n)) = 0.$$

If d and δ are C -dependent modulo inner derivations of R , then $d = \lambda\delta + ad_c$ for some $\lambda \in C$. In this case (23) reduces to

$$(24) \quad a(bx^{2n} + \lambda\delta(x^n)x^n + [c, x^n]x^n + x^npx^n + x^n\delta(x^n)) = 0.$$

By Kharchenko's Theorem [11], R satisfies

$$(25) \quad a\left(bx^{2n} + \lambda \sum_i x^i y x^{n-i-1} x^n + [c, x^n]x^n + x^n p x^n + x^n \sum_i x^i y x^{n-i-1}\right) = 0.$$

Replacing y with $[p, x]$ for some $p \notin C$, we have from (25) that

$$(26) \quad a(bx^{2n} + \lambda[p, x^n]x^n + [c, x^n]x^n + x^n p x^n + x^n[p, x^n]) = 0.$$

Then this implies as above (for inner derivation case) that $p \in C$, a contradiction.

The case when $\delta = \lambda d + ad_{c'}$ for some $\lambda \in C$, is similar.

Next assume that d and δ are C -independent modulo inner derivations of R . Then by Kharchenko's Theorem [11], R satisfies

$$(27) \quad a\left(bx^{2n} + \sum_i x^i y x^{n-i-1} x^n + x^n p x^n + x^n \sum_i x^i z x^{n-i-1}\right) = 0.$$

Replacing y with $[p, x]$ and z with $[p', x]$ for some $p, p' \notin C$, we have

$$(28) \quad a(bx^{2n} + [p, x^n]x^n + x^n p x^n + x^n[p', x^n]) = 0.$$

Then by same argument as above, it yields that $p' \in C$, a contradiction. ■

In particular, when H and G are two derivations of R , we have the following:

Corollary 2.4. *Let R be a noncommutative prime ring with $\text{char}(R) \neq 2$ and C the extended centroid of R . Let d and δ be two derivations of R . If there exists $0 \neq a \in R$ such that $a(d(x^n)x^n + x^n\delta(x^n)) = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer, then $d = \delta = 0$.*

3. RESULTS ON SEMIPRIME RINGS

In this section we extend the Corollary 2.4 to semiprime rings. Let R be a semiprime ring and U the left Utumi ring of quotients of R . Then $C = Z(U)$, center of U , is called extended centroid of R . It is well known that C is a Von Neumann regular ring. It is known that C is a field if and only if R is a prime ring. The set of all idempotents of C is denoted by E . The elements of E are called central idempotents.

We know that any derivation of R can be uniquely extended to a derivation of U (see [16, Lemma 2]).

By using the standard theory of orthogonal completions for semiprime rings, we prove the following:

Theorem 3.1. *Let R be a noncommutative 2-torsion free semiprime ring, U the left Utumi quotient ring of R and d, δ be two derivations of R . If there exists $0 \neq a \in R$ such that $a(d(x^n)x^n + x^n\delta(x^n)) = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer, then there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $(d + \delta)(e_1U) = 0$, $e_2a = 0$, and e_3U is commutative.*

Proof. Since any derivation d can be uniquely extended to a derivation in U , and U and R satisfy the same differential identities (see [16]), $a(d(x^n)x^n + x^n\delta(x^n)) = 0$ for all $x \in U$.

Let B be the complete Boolean algebra of E . We choose a maximal ideal P of B such that U/PU is 2-torsion free. Then PU is a prime ideal of U , which is d -invariant. Denote $\bar{U} = U/PU$ and $\bar{d}, \bar{\delta}$ be the canonical pair of derivations on \bar{U} induced by d and δ respectively. Then by hypothesis, $\bar{a}(\bar{d}(\bar{x}^n)\bar{x}^n + \bar{x}^n\bar{\delta}(\bar{x}^n)) = 0$ for all $\bar{x} \in \bar{U}$. Since \bar{U} is a prime ring, by Corollary 2.4, either $\bar{d} = \bar{\delta} = 0$ or $[\bar{U}, \bar{U}] = 0$ or $\bar{a} = 0$. In any case, we have $a d(U)[U, U] \subseteq PU$ and $a \delta(U)[U, U] \subseteq PU$ for all P , that is, $a D(U)[U, U] \subseteq PU$ for all P , where $D = d + \delta$. Since $\bigcap \{PU : P \text{ is any maximal ideal in } B \text{ with } U/PU \text{ 2-torsion free}\} = 0$, we have $a D(U)[U, U] = 0$.

By using the theory of orthogonal completion for semiprime rings (see, [1, Chapter 3]), it follows that there exist orthogonal central idempotents $e_1, e_2, e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $D(e_1U) = 0$, $e_2a = 0$, and e_3U is commutative. ■

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