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GENERALIZED DERIVATIONS WITH LEFT ANNIHILATOR CONDITIONS IN PRIME AND SEMIPRIME RINGS

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Abstract

Let R be a prime ring with its Utumi ring of quotients U, $C = Z(U)$ be the extended centroid of R, H and G two generalized derivations of R, L a noncentral Lie ideal of R, I a nonzero ideal of R. The left annihilator of $S \subseteq R$ is denoted by $l_R(S)$ and defined by $l_R(S) = \{x \in R \mid xS = 0\}$. Suppose that $S = \{H(u^n)u^n + u^nG(u^n) \mid u \in L\}$ and $T = \{H(x^n)x^n + x^nG(x^n) \mid x \in I\}$, where $n \geq 1$ is a fixed integer. In the paper, we investigate the cases when the sets $l_R(S)$ and $l_R(T)$ are nonzero.

Keywords: prime ring, derivation, Lie ideal, generalized derivation, extended centroid, Utumi quotient ring.

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1. Introduction

Let R be an associative ring with center $Z(R)$. For $x, y \in R$, the commutator of x, y is denoted by $[x, y]$ and defined by $[x, y] = xy - yx$. By d we mean a derivation of R. An additive mapping F from R to R is called a generalized derivation if there exists a derivation d from R to R such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$.

Throughout this paper, R will always present a prime ring with center $Z(R)$, extended centroid C and U is its Utumi quotient ring. A well known result proved by Posner [20], states that if the commutators $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or R is commutative. Then result of Posner was generalized in many

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directions by a number of authors. Posner's theorem was extended to Lie ideals in prime rings by Lee [17] and then by Lanski [12].

On the other hand, authors generalized Posner's theorem by considering two derivations. In [3], Brešar proved that if d and δ are two derivations of R such that $d(x)x - x\delta(x) \in Z(R)$ for all $x \in R$, then either $d = \delta = 0$ or R is commutative. Later Lee and Wong [18] consider the situation $d(x)x - x\delta(x) \in Z(R)$ for all x in some noncentral Lie ideal L of R and they proved that either $d = \delta = 0$ or R satisfies s_4.

Recently in [22] Vukman proves that if d and δ are derivations on a $2mn(m+n-1)!$-torsion free semiprime rings R such that $d(x^m)x^n + x^n\delta(x^m) = 0$ for all $x \in R$, where $m, n \geq 1$ are fixed integers, then both derivations d and δ map R into its center and $d = -\delta$.

In [23], Wei and Xiao studied the similar situation replacing derivations d and δ by generalized derivations G and H. More precisely they proved the following:

Let m, n be fixed positive integers, R be a noncommutative $2(m+n)!$-torsion free prime ring and G, H be a pair of generalized Jordan derivations on R. If $G(x^m)x^n + x^nH(x^m) \in Z(R)$ for all $x \in R$, then G and H both are right (or left) multipliers.

In [14], Lee and Zhou studied the same situation of above result without considering torsion free restriction on R. In this paper, Lee and Zhou [14] proved the following:

Let R be a prime ring that is not commutative and such that $R \not\cong M_2(GF(2))$, let G, H be two generalized derivations of R, and let m, n be two fixed positive integers. Then $G(x^m)x^n - x^nH(x^m) = 0$ for all $x \in R$ iff the following two conditions hold:

(1) *There exists $w \in Q$ such that $G(x) = xw$ and $H(x) = wx$ for all $x \in R$;*
(2) *either $w \in C$, or x^m and x^n are C-dependent for all $x \in R$.*

There are many papers in the literature which studied the identities of generalized derivations with left annihilator conditions.

For any subset S of R, denote by $r_R(S)$ the right annihilator of S in R, that is, $r_R(S) = \{x \in R \mid Sx = 0\}$ and $l_R(S)$ the left annihilator of S in R that is, $l_R(S) = \{x \in R \mid xS = 0\}$. If $r_R(S) = l_R(S)$, then $r_R(S)$ is called an annihilator ideal of R and is written as $ann_R(S)$.

In [4], Carini *et al.* studied the left annihilator of the set $\{H(u)u - uG(u) \mid u \in L\}$, where L is a noncentral Lie ideal of R and H, G two non-zero generalized derivations of R. In case the annihilator is not zero, the conclusion is one of the following:

(1) there exist $b', c' \in U$ such that $H(x) = b'x + xc', G(x) = c'x$ with $ab' = 0$;
(2) R satisfies s_4 and there exist $b', c', q' \in U$ such that $H(x) = b'x + xc', G(x) = c'x + xq'$, with $a(b' - q') = 0$.

Recently, Carini and De Filippis proved the following theorem:

Let R be a prime ring, U the Utumi quotient ring of R, $C = Z(U)$ the extended centroid of R, L a non-central Lie ideal of R, H and G non-zero generalized derivations of R. Suppose that there exists an integer $n \geq 1$ such that $H(u^n)u^n + u^nG(u^n) \in C$, for all $u \in L$, then either there exists $a \in U$ such that $H(x) = xa, G(x) = -ax$, or R satisfies the standard identity s_4. Moreover, in the last case the structures of the maps G, H are obtained.

In the present paper, we shall investigate the left annihilator of the sets $\{H(u^n)u^n + u^nG(u^n) \mid u \in L\}$ and $\{H(x^n)x^n + x^nG(x^n) \mid x \in I\}$, where L is a noncentral Lie ideal of R, I is a nonzero ideal of R, $n \geq 1$ is a fixed integer and H, G two non-zero generalized derivations of R. More precisely, we prove the following theorems:

Theorem 1.1. *Let R be a prime ring with its Utumi ring of quotients U, $C = Z(U)$ be the extended centroid of R, H and G two generalized derivations of R, L a noncentral Lie ideal of R and $S = \{H(u^n)u^n + u^nG(u^n) \mid u \in L\}$, where $n \geq 1$ is a fixed integer. If $l_R(S) \neq \{0\}$, then either there exist $b', p \in U$ such that $H(x) = b'x - xp$ and $G(x) = px$ for all $x \in R$ with $ab' = 0$ or R satisfies s_4. Moreover, in the last case, if R satisfies s_4, then one of the following holds:*

(1) *char $(R) = 2$;*
(2) *n is even, there exist $b, p \in U$ and derivations d, δ of R such that $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$ for all $x \in R$, with $a(b + p) = 0$;*
(3) *n is odd, there exist $b, p \in U$ and derivations d, δ of R such that $H(x) = bx + d(x)$ and $G(x) = xp + \delta(x)$ for all $x \in R$, with $a(b + p) = 0$.*

Theorem 1.2. *Let R be a noncommutative prime ring with char $(R) \neq 2$, U its Utumi ring of quotients, $C = Z(U)$ be the extended centroid of R, H and G two generalized derivations of R, I a nonzero ideal of R and $S = \{H(x^n)x^n + x^nG(x^n) \mid x \in I\}$, where $n \geq 1$ is a fixed integer. If $l_R(S) \neq \{0\}$, then there exist $b', p \in U$ such that $H(x) = b'x - xp$ and $G(x) = px$ for all $x \in R$ with $ab' = 0$.*

As an immediate application of the Theorem 1.1, in particular when $G = -H$, then we have the following result which gives a particular result of Theorem 1.1 in [6].

Corollary 1.3. *Let R be a prime ring with its Utumi ring of quotients U, $C = Z(U)$ be the extended centroid of R, H a generalized derivation of R and L a noncentral Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a[H(u^n), u^n] = 0$ for all $u \in L$, where $n \geq 1$ is a fixed integer. Then either there exists $\lambda \in C$ such that $H(x) = \lambda x$ for all $x \in R$ or R satisfies s_4.*

As an application of the Theorem 1.1, in particular when $G = 0$, then using

Theorem 2.2 in [8], we have the following result which gives a generalization of Theorem 1.1 in [21].

Corollary 1.4. *Let R be a prime ring of char $(R) \neq 2$ with its Utumi ring of quotients U, $C = Z(U)$ be the extended centroid of R, H a generalized derivation of R and L a noncentral Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $aH(u^n)u^n = 0$ for all $u \in L$, where $n \geq 1$ is a fixed integer. Then either there exist $b', p \in U$ such that $H(x) = b'x$ for all $x \in R$ with $ab' = 0$.*

2. Proof of main results in prime rings

Let R be a prime ring with extended centroid C. Let $H(x) = bx + xc$ and $G(x) = px + xq$ for all $x \in R$ and for some $b, c, p, q \in U$, be two inner generalized derivations of R and L be a noncentral Lie ideal of R. Then $a(H(x^n)x^n + x^nG(x^n)) = 0$ implies $a(bx^{2n} + x^n(c+p)x^n + x^{2n}q) = 0$ for all $x \in L$. We know that if char $(R) \neq 2$, by [2, Lemma 1] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. If char $(R) = 2$ and $dim_C RC > 4$ i.e., char $(R) = 2$ and R does not satisfy s_4, then by [13, Theorem 13] there exists a nonzero ideal I of R such that $0 \neq [I, R] \subseteq L$. We assume that R does not satisfy s_4. Then in any case of char $(R) = 2$ or char $(R) \neq 2$, we can conclude that there exists a nonzero ideal I of R such that $0 \neq [I, I] \subseteq L$. By hypothesis, we have

$$a(b[x_1, x_2]^{2n} + [x_1, x_2]^n(c+p)[x_1, x_2]^n + [x_1, x_2]^{2n}q) = 0 \tag{1}$$

for all $x_1, x_2 \in I$. Then following lemmas are immediate consequences:

Lemma 2.1. *R satisfies a nontrivial generalized polynomial identity* (GPI) *or $c, p, q \in C$ such that $a(b + c + p + q) = 0$.*

Proof. Assume that R does not satisfy any nontrivial GPI. Then R must be noncommutative. Let $T = U *_C C\{x_1, x_2\}$, the free product of U and $C\{x_1, x_2\}$, the free C-algebra in noncommuting indeterminates x_1 and x_2.

Then,

$$a\left(b[x_1, x_2]^{2n} + [x_1, x_2]^n(c+p)[x_1, x_2]^n + [x_1, x_2]^{2n}q\right)$$

is zero element in T. If $q \notin C$, then q and 1 are linearly independent over C. Then from above

$$a[x_1, x_2]^{2n}q = 0 \in T,$$

implying $q = 0$, since $a \neq 0$, a contradiction. Therefore, we conclude that $q \in C$.

Then by hypothesis

$$a((b+q)[x_1,x_2]^n+[x_1,x_2]^n(c+p))[x_1,x_2]^n=0\in T. \tag{2}$$

If $c+p\notin C$, then by (2)

$$a\left([x_1,x_2]^n(c+p)\right)[x_1,x_2]^n=0\in T,$$

implying $c+p=0$, since $a\neq 0$, a contradiction. Therefore, we have $c+p\in C$ and hence

$$a(b+q+c+p)[x_1,x_2]^{2n}=0\in T.$$

This implies $a(b+q+c+p)=0$. ■

Lemma 2.2. *$c+p,q\in C$ with $a(b+c+p+q)=0$, unless R satisfies s_4.*

Proof. By hypothesis, R satisfies GPI

$$f(x_1,x_2)=a(b[x_1,x_2]^{2n}+[x_1,x_2]^n(c+p)[x_1,x_2]^n+[x_1,x_2]^{2n}q). \tag{3}$$

If R does not satisfy any nontrivial GPI, by Lemma 2.1, we obtain $c,p,q\in C$ with $a(b+c+p+q)=0$ which gives the conclusion. So, we assume that R satisfies a nontrivial GPI. Since R and U satisfy the same generalized polynomial identities (see [5]), U satisfies $f(x_1,x_2)$. In case C is infinite, we have $f(x_1,x_2)=0$ for all $x_1,x_2\in U\otimes_C\overline{C}$, where $\overline{C}$ is the algebraic closure of C. Moreover, both U and $U\otimes_C\overline{C}$ are prime and centrally closed algebras [9]. Hence, replacing R by U or $U\otimes_C\overline{C}$ according to C finite or infinite, without loss of generality we may assume that $C=Z(R)$ and R is C-algebra centrally closed. By Martindale's theorem [19], R is then a primitive ring having nonzero socle $soc(R)$ with C as the associated division ring. Hence, by Jacobson's theorem [10, p.75], R is isomorphic to a dense ring of linear transformations of a vector space V over C.

If $\dim_C V=2$, then $R\cong M_2(C)$, that is, R satisfies s_4, a contradiction. So, let $\dim_C V\geq 3$.

We show that for any $v\in V$, v and qv are linearly C-dependent. Suppose that v and qv are linearly independent for some $v\in V$. Since $\dim_C V\geq 3$, there exists $u\in V$ such that v,qv,u are linearly C-independent set of vectors. By density, there exists $x_1,x_2\in R$ such that

$$x_1v=v,\quad x_1qv=0,\quad x_1u=qv;\quad x_2v=0,\quad x_2qv=u,\quad x_2u=0.$$

Then $0=a(b[x_1,x_2]^{2n}+[x_1,x_2]^n(c+p)[x_1,x_2]^n+[x_1,x_2]^{2n}q)v=aqv$.

This implies that if for some $v\in V$, $aqv\neq 0$, then by contradiction, v and qv are linearly C-dependent.

Now choose $v \in V$ such that v and qv are linearly C-independent. Then $aqv = 0$. Let us consider a subspace $W = \{\alpha v + \beta qv \mid \alpha, \beta \in C\}$ of V. Let $aq \neq 0$. Then, there exists $w \in V$ such that $aqw \neq 0$. Then $aq(v - w) = aqw \neq 0$. Then by the above argument, w, qw are linearly C-dependent and $(v-w), q(v-w)$ too. Thus there exist $\alpha, \beta \in C$ such that $qw = \alpha w$ and $q(v - w) = \beta(v - w)$. Then $qv = \beta(v - w) + qw = \beta(v - w) + \alpha w$ i.e., $(\alpha - \beta)w = qv - \beta v \in W$. Now $\alpha = \beta$ implies that $qv = \beta v$, a contradiction. Hence $\alpha \neq \beta$ and so $w \in W$.

Next assume that $u \in V$ such that $aqu = 0$. Then $aq(w + u) = aqw \neq 0$. By above argument, $aq(w+u) \neq 0$ implies $w+u \in W$. Since $w \in W$, we have $u \in W$. Thus it is observed that for any $v \in V$, $aqv \neq 0$ implies $v \in W$ and $aqv = 0$ implies $v \in W$. This implies that $V = W$ i.e., $\dim_C V = 2$, a contradiction.

Thus up to now we have proved that v and qv are linearly C-dependent for all $v \in V$, unless $aq = 0$. If $aq \neq 0$, by standard argument, it follows that $qv = \lambda v$ for all $v \in V$ and $\lambda \in C$ fixed. Then $(q - \lambda)V = 0$, implying $q = \lambda \in C$.

Now let $aq = 0$. Since $\dim_C V \geq 3$, there exists $w \in V$ such that v, qv, w are linearly C-independent set of vectors. By density, there exists $x_1, x_2 \in R$ such that

$$x_1 v = v, \quad x_1 qv = 0, \quad x_1 w = v + qv; \quad x_2 v = 0, \quad x_2 qv = w, \quad x_2 w = 0.$$

Then $0 = a(b[x_1, x_2]^{2n} + [x_1, x_2]^n(c + p)[x_1, x_2]^n + [x_1, x_2]^{2n}q)v = av$. Then by above argument, since $a \neq 0$, $q \in C$.

Therefore, we have proved that in any case $q \in C$. Hence our identity reduces to

$$a(b'[x_1, x_2]^{2n} + [x_1, x_2]^n c'[x_1, x_2]^n) = 0,$$

where $b' = b + q$ and $c' = c + p$.

Now we prove that v and $c'v$ are linearly C-dependent. If possible let v and $c'v$ be linearly independent for some $v \in V$. Then there exists $w \in V$ such that v, $c'v$ and w are linearly independent over C. By density there exist $x_1, x_2 \in R$ such that

$$x_1 v = 0, \quad x_1 c'v = v, \quad x_1 w = 2c'v; \quad x_2 v = c'v, \quad x_2 c'v = w, \quad x_2 w = 0.$$

Then $0 = a(b'[x_1, x_2]^{2n} + [x_1, x_2]^n c'[x_1, x_2]^n)v = a(b' + c')v$. As above, this implies either $a(b' + c') = 0$ or $c' \in C$. Let $a(b' + c') = 0$. Then we have that R satisfies $0 = a[c', [x_1, x_2]^n][x_1, x_2]^n$. By density there exist $x_1, x_2 \in R$ such that

$$x_1 v = 0, \quad x_1 c'v = v, \quad x_1 w = c'v; \quad x_2 v = c'v, \quad x_2 c'v = w, \quad x_2 w = 0.$$

Thus $0 = a[c', [x_1, x_2]^n][x_1, x_2]^n v = ac'v$. This implies either $ac' = 0$ or $c' \in C$. Let $ac' = 0$. Then we have that R satisfies $0 = a[x_1, x_2]^n]c'[x_1, x_2]^n$. Again by density there exist $x_1, x_2 \in R$ such that

$$x_1 v = 0, \quad x_1 c'v = v, \quad x_1 w = v + c'v; \quad x_2 v = c'v, \quad x_2 c'v = w, \quad x_2 w = 0.$$

Thus $0 = a[x_1,x_2]^n]c'[x_1,x_2]^n v = av$. Since $a \neq 0$, this implies $c' \in C$. Thus in any case, we have $c' \in C$. Hence R satisfies $0 = a(b'+c')[x_1,x_2]^{2n}$, which implies $a(b'+c') = 0$. ■

***Proof of Theorem* 1.1.** Let $0 \neq a \in l_R(S)$. Then $a(H(u^n)u^n + u^n G(u^n)) = 0$ for all $u \in L$. If char $(R) = 2$ and R satisfies s_4, then we obtain our conclusion (1). So we assume that either char $(R) \neq 2$ or R does not satisfy s_4. Then by [2, Lemma 1] and [13, Theorem 13], since L is a noncentral Lie ideal of R, there exists a nonzero ideal I of R such that $[I,I] \subseteq L$. Hence, by our hypothesis, we have,

$$a(H([x_1,x_2]^n)[x_1,x_2]^n + [x_1,x_2]^n G([x_1,x_2]^n)) = 0$$

for all $x_1,x_2 \in I$. Since I, R and U satisfy the same generalized polynomial identities (see [5]) as well as the same differential identities (see [16]), they also satisfy the same generalized differential identities. Hence, by [15], U satisfies

$$a(H([x_1,x_2]^n)[x_1,x_2]^n + [x_1,x_2]^n G([x_1,x_2]^n)) = 0$$

for all $x_1,x_2 \in U$, where $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$, for some $b,p \in U$ and derivations d and δ of U, that is, U satisfies

$$\begin{aligned} a(b[x_1,x_2]^{2n} + d([x_1,x_2]^n)[x_1,x_2]^n + [x_1,x_2]^n p[x_1,x_2]^n \\ + [x_1,x_2]^n \delta([x_1,x_2]^n)) = 0. \end{aligned} \tag{4}$$

Now we divide the proof into two cases:

Case I. Let $d(x) = [c,x]$ for all $x \in U$ and $\delta(x) = [q,x]$ for all $x \in U$ i.e., d and δ be inner derivations of U. Then from (4), we obtain that U satisfies

$$a((b+c)[x_1,x_2]^{2n} + [x_1,x_2]^n(p-c+q)[x_1,x_2]^n - [x_1,x_2]^{2n}q) = 0. \tag{5}$$

By Lemma 2.2, when R does not satisfy s_4, we have $q, p-c+q \in C$ with $a(b+p) = 0$. This implies $p-c \in C$. Hence $H(x) = bx + [c,x] = bx + [p,x] = b'x - xp$, $G(x) = px$ for all $x \in U$ and so for all $x \in R$ with $ab' = 0$, where $b' = b+p$.

Moreover, when R satisfies s_4 (in this case by assumption char $(R) \neq 2$), then $R \subseteq M_2(F)$ and, R and $M_2(F)$ satisfy the same GPI, where $M_2(F)$ is a matrix ring over a field F. Hence $M_2(F)$ satisfies $a((b+c)[x_1,x_2]^{2n} + [x_1,x_2]^n(p-c+q)[x_1,x_2]^n - [x_1,x_2]^{2n}q) = 0$. Since $[x,y]^2 \in Z(M_2(F))$ for all $x,y \in M_2(F)$, $M_2(F)$ satisfies

$$a((b+c-q)[x_1,x_2]^{2n} + [x_1,x_2]^n(p-c+q)[x_1,x_2]^n) = 0. \tag{6}$$

If n is even, then by choosing $x_1 = e_{12}$, $x_2 = e_{21}$, we have $0 = a(b+p)$.

If n is odd, then $M_2(F)$ satisfies $a((b+c-q)[x_1,x_2] + [x_1,x_2](p-c+q))$ $[x_1,x_2]^{2n-1} = 0$. By Lemma 2.7 in [7], we conclude that $p-c+q \in Z(R)$ and $a(b+p) = 0$.

Thus when R satisfies s_4, one of the following holds:

(i) n is even and $a(b+p)=0$. In this case, $H(x)=bx+[c,x]$ and $G(x)=px+[q,x]$ for all $x\in R$, with $a(b+p)=0$. This is our conclusion (2).

(ii) n is odd and $p-c+q\in C$ and $a(b+p)=0$. Hence $H(x)=bx+[c,x]$ and $G(x)=px+[q,x]=px-[p-c,x]=xp+[c,x]$ for all $x\in R$, with $a(b+p)=0$. This is our conclusion (3).

Case II. Next assume that d and δ are not both inner derivations of U, but they are C-dependent modulo inner derivations of U. Suppose $d=\lambda\delta+ad_c$, that is, $d(x)=\lambda\delta(x)+[c,x]$ for all $x\in U$, where $\lambda\in C$, $c\in U$. Then d can not be inner derivation of U. From (4), we have that U satisfies

$$a\Big(b[x_1,x_2]^{2n}+\lambda\delta([x_1,x_2]^n)[x_1,x_2]^n+[c,[x_1,x_2]^n][x_1,x_2]^n + [x_1,x_2]^n p[x_1,x_2]^n+[x_1,x_2]^n\delta([x_1,x_2]^n)\Big)=0.$$

This gives

$$a\Big(b[x_1,x_2]^{2n}+\lambda\sum_{i=0}^{n-1}[x_1,x_2]^i\delta([x_1,x_2])[x_1,x_2]^{n-1-i}[x_1,x_2]^n+[c,[x_1,x_2]^n][x_1,x_2]^n + [x_1,x_2]^n p[x_1,x_2]^n+[x_1,x_2]^n\sum_{i=0}^{n-1}[x_1,x_2]^i\delta([x_1,x_2])[x_1,x_2]^{n-1-i}\Big)=0.$$

Then by Kharchenko's theorem [11], we have that U satisfies

$$\begin{aligned}(7)\qquad & a\Big(b[x_1,x_2]^{2n}+\lambda\sum_{i=0}^{n-1}[x_1,x_2]^i([y_1,x_2]+[x_1,y_2])[x_1,x_2]^{n-1-i}[x_1,x_2]^n\\ &+[c,[x_1,x_2]^n][x_1,x_2]^n+[x_1,x_2]^n p[x_1,x_2]^n\\ &+[x_1,x_2]^n\sum_{i=0}^{n-1}[x_1,x_2]^i([y_1,x_2]+[x_1,y_2])[x_1,x_2]^{n-1-i}\Big)=0.\end{aligned}$$

In particular U satisfies blended component

$$(8)\qquad a\Big(b[x_1,x_2]^{2n}+[c,[x_1,x_2]^n][x_1,x_2]^n+[x_1,x_2]^n p[x_1,x_2]^n\Big)=0$$

and

$$\begin{aligned}(9)\qquad & a\Big(\lambda\sum_{i=0}^{n-1}[x_1,x_2]^i([y_1,x_2]+[x_1,y_2])[x_1,x_2]^{n-1-i}[x_1,x_2]^n\\ &+[x_1,x_2]^n\sum_{i=0}^{n-1}[x_1,x_2]^i([y_1,x_2]+[x_1,y_2])[x_1,x_2]^{n-1-i}\Big)=0.\end{aligned}$$

For $y_1 = [q, x_1]$ and $y_2 = [q, x_2]$, where $q \notin C$ we have that U satisfies

$$a\big([\lambda q, [x_1, x_2]^n][x_1, x_2]^n + [x_1, x_2]^n[q, [x_1, x_2]^n]\big) = 0. \tag{10}$$

By Lemma 2.2, if R does not satisfy s_4, then $q \in C$, a contradiction. Hence we conclude that R satisfies s_4. Now the relations (8) and (10) are similar to the relation (5). Thus by same argument as given in Case I, when R satisfies s_4 (in this case char (R) must be not equal to 2), one of the following holds:

(i) Let n be even. Then by (8), $a(b + p) = 0$. Thus $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$ for all $x \in R$, with $a(b + p) = 0$. This is our conclusion (2).

(ii) Let n be odd. Then by (8), $p - c \in C$ and $a(b + p) = 0$. Again by (10), $q - \lambda q = q(1 - \lambda) \in C$. Since $q \notin C$, we have $\lambda = 1$. Then replacing $y_1 = x_1$ and $y_2 = 0$, (9) gives $na(\lambda + 1)[x_1, x_2]^{2n} = 0$, implying $2na = 0$. Since char $(R) \neq 2$, $na = 0$. Hence $H(x) = bx + \lambda\delta(x) + [c, x] = bx + \delta(x) + [c, x]$ and $G(x) = px + \delta(x) = (p - c)x + cx + \delta(x) = x(p - c) + cx + \delta(x) = xp + \delta(x) + [c, x]$ for all $x \in R$. This is our conclusion (3).

The situation when $\delta = \lambda d + ad_c$ is similar.

Next assume that d and δ are C-independent modulo inner derivations of U. Since neither d nor δ is inner, by Kharchenko's Theorem [11], we have from (4) that U satisfies

$$a\Bigg(b[x_1, x_2]^{2n} + \sum_{i=0}^{n-1} [x_1, x_2]^i([u_1, x_2] + [x_1, u_2])[x_1, x_2]^{n-1-i}[x_1, x_2]^n + [x_1, x_2]^n p[x_1, x_2]^n + [x_1, x_2]^n \sum_{i=0}^{n-1} [x_1, x_2]^i([v_1, x_2] + [x_1, v_2])[x_1, x_2]^{n-1-i}\Bigg) = 0. \tag{11}$$

Then U satisfies blended component

$$a\Bigg(b[x_1, x_2]^{2n} + [x_1, x_2]^n p[x_1, x_2]^n\Bigg) = 0 \tag{12}$$

and

$$a\Bigg([x_1, x_2]^n \sum_{i=0}^{n-1}[x_1, x_2]^i([v_1, x_2] + [x_1, v_2])[x_1, x_2]^{n-1-i}\Bigg) = 0. \tag{13}$$

Replacing v_1 with $[q, x_1]$ and v_2 with $[q, x_2]$ for some $q \notin C$ in (13), we obtain that U satisfies

$$a([x_1, x_2]^n[q, [x_1, x_2]^n]) = 0. \tag{14}$$

By Lemma 2.2, we have $q \in C$, a contradiction, unless R satisfies s_4. So we consider the case when R satisfies s_4. In this case by same argument of Case I, (12) and (14) together implies that n is even and $a(b + p) = 0$. This gives our conclusion (2). Hence the theorem is proved. ■

Corollary 2.3. *Let R be a prime ring with its Utumi ring of quotients U, $C = Z(U)$ be the extended centroid of R, H and G two generalized derivations of R and L a noncentral Lie ideal of R. Suppose that there exists $0 \neq a \in R$ such that $a(H(u^2)u^2 + u^2G(u^2)) = 0$ for all $u \in L$. Then either there exist $b', p \in U$ such that $H(x) = b'x - xp$ and $G(x) = px$ for all $x \in R$ with $ab' = 0$ or R satisfies s_4. Moreover, if R satisfies s_4, then one of the following holds:*

(1) *char $(R) = 2$;*

(2) *there exist $b, p \in U$ and derivations d, δ of R such that $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$ for all $x \in R$, with $a(b + p) = 0$.*

***Proof of Theorem* 1.2.** Let $0 \neq a \in l_R(S)$. Then $a(H(x^n)x^n + x^nG(x^n)) = 0$ for all $x \in I$. By Theorem 1.1, we have only to consider the case when R satisfies s_4. In this case R is a PI-ring, and so there exists a field K such that $R \subseteq M_2(K)$ and, R and $M_2(K)$ satisfy the same GPI. First we assume that H and G are inner generalized derivations of R, that is, $H(x) = bx + xc$ for all $x \in R$ and $G(x) = px + xq$ for all $x \in R$, for some $b, c, p, q \in R$. Since $M_2(F)$ is a simple ring, by our hypothesis, $M_2(F)$ satisfies

$$a(bx^{2n} + x^n(c + p)x^n + x^{2n}q) = 0. \tag{15}$$

Moreover, R is a dense ring of K-linear transformations over a vector space V. Let $aq \neq 0$. Assume there exists $v \neq 0$, such that $\{v, qv\}$ is linear K-independent. By the density of R, there exists $r \in R$ such that

$$rv = 0; \quad r(qv) = qv.$$

Hence

$$0 = a(br^{2n} + r^n(c + p)r^n + r^{2n}q)v = aqv.$$

Of course for any $w \in V$ such that $\{w, v\}$ are linearly K-dependent implies $aqw = 0$. Since $aq \neq 0$, there exists $w \in V$ such that $aqw \neq 0$. Then $\{w, v\}$ must be linearly K-independent. By the above argument it follows that w and qw are linearly K-dependent, as are $\{w + v, q(w + v)\}$ and $\{w - v, q(w - v)\}$. Therefore there exist $\alpha_w, \alpha_{w+v}, \alpha_{w-v} \in K$ such that

$$qw = \alpha_w w, \quad q(w + v) = \alpha_{w+v}(w + v), \quad q(w - v) = \alpha_{w-v}(w - v).$$

In other words we have

$$\alpha_w w + qv = \alpha_{w+v} w + \alpha_{w+v} v \tag{16}$$

and

$$\alpha_w w - qv = \alpha_{w-v} w - \alpha_{w-v} v. \tag{17}$$

By comparing (16) with (17) we get both

$$(2\alpha_w - \alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w-v} - \alpha_{w+v})v = 0 \tag{18}$$

and

$$2qv = (\alpha_{w+v} - \alpha_{w-v})w + (\alpha_{w+v} + \alpha_{w-v})v. \tag{19}$$

By (18) and since $\{w, v\}$ is K-independent and $char(K) \neq 2$, we have $\alpha_w = \alpha_{w+v} = \alpha_{w-v}$. Thus by (19) it follows $2qv = 2\alpha_w v$. Since $\{qv, v\}$ is K-independent, the conclusion $\alpha_w = \alpha_{w+v} = 0$ follows, that is $qw = 0$ and $q(w+v) = 0$, which implies the contradiction $qv = 0$.

Hence we conclude that for any $v \in V$, $\{v, qv\}$ is linearly K-dependent. Thus there exists a suitable $\alpha_v \in K$ such that $qv = \alpha_v v$, and standard argument shows that there is $\alpha \in K$ such that $qv = \alpha v$ for all $v \in V$. Now let $r \in R$, $v \in V$. Since $qv = \alpha v$,

$$[q, r]v = (qr)v - (rq)v = q(rv) - r(qv) = \alpha(rv) - r(\alpha v) = 0. \tag{20}$$

Thus $[q, r]v = 0$ for all $v \in V$ i.e., $[q, r]V = 0$. Since $[q, r]$ acts faithfully as a linear transformation on the vector space V, $[q, r] = 0$ for all $r \in R$. Therefore, $q \in C$.

Thus up to now, we have proved that either $aq = 0$ or $q \in C$.

Let $aq = 0$. In this case, assume that there exists $v \neq 0$, such that $\{v, qv\}$ is linear K-independent. By the density of R, there exists $r \in R$ such that

$$rv = 0; \quad r(qv) = v + qv.$$

Hence

$$0 = a(br^{2n} + r^n(c+p)r^n + r^{2n}q)v = av.$$

Thus by the same argument as above, this implies either $a = 0$ or $q \in C$. Since $a \neq 0$, $q \in C$.

Thus in any case we conclude that $q \in C$.

Then (15) reduces to

$$a((b+q)x^n + x^n(c+p))x^n = 0. \tag{21}$$

Let there exists $v \neq 0$, such that $\{v, (c+p)v\}$ is linear K-independent. By the density of R, there exists $r \in R$ such that

$$rv = 0; \quad r((c+p)v) = (c+p)v.$$

Hence

$$0 = a((b+q)r^n + r^n(c+p))r^n v = a(c+p)v.$$

Then again by same argument, $c+p \in C$. Then (21) reduces to

$$a(b+c+p+q)x^{2n} = 0 \tag{22}$$

for all $x \in R$. This implies $a(b+c+p+q) = 0$, where $q, c+p \in C$. Hence $H(x) = bx + xc = bx + x(c+p) - xp = (b+c+p)x - xp = (b+c+p+q)x - x(p+q)$ for all $x \in R$ and $G(x) = (p+q)x$ for all $x \in R$. This gives our conclusion.

Next assume that $H(x) = bx + d(x)$ and $G(x) = px + \delta(x)$, where d, δ are not both inner derivations of R. In this case by our hypothesis, R satisfies

$$a\left(bx^{2n} + d(x^n)x^n + x^n px^n + x^n \delta(x^n)\right) = 0. \tag{23}$$

If d and δ are C-dependent modulo inner derivations of R, then $d = \lambda\delta + ad_c$ for some $\lambda \in C$. In this case (23) reduces to

$$a\left(bx^{2n} + \lambda\delta(x^n)x^n + [c, x^n]x^n + x^n px^n + x^n \delta(x^n)\right) = 0. \tag{24}$$

By Kharchenko's Theorem [11], R satisfies

$$a\left(bx^{2n} + \lambda\sum_i x^i y x^{n-i-1} x^n + [c, x^n]x^n + x^n px^n + x^n \sum_i x^i y x^{n-i-1}\right) = 0. \tag{25}$$

Replacing y with $[p, x]$ for some $p \notin C$, we have from (25) that

$$a\left(bx^{2n} + \lambda[p, x^n]x^n + [c, x^n]x^n + x^n px^n + x^n[p, x^n]\right) = 0. \tag{26}$$

Then this implies as above (for inner derivation case) that $p \in C$, a contradiction.

The case when $\delta = \lambda d + ad_{c'}$ for some $\lambda \in C$, is similar.

Next assume that d and δ are C-independent modulo inner derivations of R. Then by Kharchenko's Theorem [11], R satisfies

$$a\left(bx^{2n} + \sum_i x^i y x^{n-i-1} x^n + x^n px^n + x^n \sum_i x^i z x^{n-i-1}\right) = 0. \tag{27}$$

Replacing y with $[p, x]$ and z with $[p', x]$ for some $p, p' \notin C$, we have

$$a\left(bx^{2n} + [p, x^n]x^n + x^n px^n + x^n[p', x^n]\right) = 0. \tag{28}$$

Then by same argument as above, it yields that $p' \in C$, a contradiction. ■

In particular, when H and G are two derivations of R, we have the following:

Corollary 2.4. *Let R be a noncommutative prime ring with char $(R) \neq 2$ and C the extended centroid of R. Let d and δ be two derivations of R. If there exists $0 \neq a \in R$ such that $a(d(x^n)x^n + x^n\delta(x^n)) = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer, then $d = \delta = 0$.*

3. Results on semiprime rings

In this section we extend the Corollary 2.4 to semiprime rings. Let R be a semiprime ring and U the left Utumi ring of quotients of R. Then $C = Z(U)$, center of U, is called extended centroid of R. It is well known that C is a Von Neumann regular ring. It is known that C is a field if and only if R is a prime ring. The set of all idempotents of C is denoted by E. The elements of E are called central idempotents.

We know that any derivation of R can be uniquely extended to a derivation of U (see [16, Lemma 2]).

By using the standard theory of orthogonal completions for semiprime rings, we prove the following:

Theorem 3.1. *Let R be a noncommutative 2-torsion free semiprime ring, U the left Utumi quotient ring of R and d, δ be two derivations of R. If there exists $0 \neq a \in R$ such that $a(d(x^n)x^n + x^n\delta(x^n)) = 0$ for all $x \in R$, where $n \geq 1$ is a fixed integer, then there exist orthogonal central idempotents e_1, e_2, $e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $(d+\delta)(e_1U) = 0$, $e_2a = 0$, and e_3U is commutative.*

Proof. Since any derivation d can be uniquely extended to a derivation in U, and U and R satisfy the same differential identities (see [16]), $a(d(x^n)x^n + x^n\delta(x^n)) = 0$ for all $x \in U$.

Let B be the complete Boolean algebra of E. We choose a maximal ideal P of B such that U/PU is 2-torsion free. Then PU is a prime ideal of U, which is d-invariant. Denote $\overline{U} = U/PU$ and $\overline{d}$, $\overline{\delta}$ be the canonical pair of derivations on $\overline{U}$ induced by d and δ respectively. Then by hypothesis, $\overline{a}(\overline{d}(\overline{x}^n)\overline{x}^n + \overline{x}^n\overline{\delta}(\overline{x}^n)) = 0$ for all $\overline{x} \in \overline{U}$. Since $\overline{U}$ is a prime ring, by Corollary 2.4, either $\overline{d} = \overline{\delta} = 0$ or $[\overline{U}, \overline{U}] = 0$ or $\overline{a} = 0$. In any case, we have $ad(U)[U,U] \subseteq PU$ and $a\delta(U)[U,U] \subseteq PU$ for all P, that is, $aD(U)[U,U] \subseteq PU$ for all P, where $D = d+\delta$. Since $\bigcap\{PU : P$ is any maximal ideal in B with U/PU 2-*torsion free*$\} = 0$, we have $aD(U)[U,U] = 0$.

By using the theory of orthogonal completion for semiprime rings (see, [1, Chapter 3]), it follows that there exist orthogonal central idempotents e_1, e_2, $e_3 \in U$ with $e_1 + e_2 + e_3 = 1$ such that $D(e_1U) = 0$, $e_2a = 0$, and e_3U is commutative. ■

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