

## ON THE SECOND SPECTRUM OF LATTICE MODULES

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### Abstract

The second spectrum  $Spec^s(M)$  is the collection of all second elements of  $M$ . In this paper, we study the topology on  $Spec^s(M)$ , which is a generalization of the Zariski topology on the prime spectrum of lattice modules. Besides some properties,  $Spec^s(M)$  is characterized and the interrelations between the topological properties of  $Spec^s(M)$  and the algebraic properties of  $M$ , are studied.

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### 1. INTRODUCTION

The Zariski topology for second spectrum of a module over a commutative ring is being introduced and studied by Ansari-Toroghy, Farshadifar in [1]. As a generalization of most of the results in [1], we introduce the concept of second elements of a lattice module  $M$  over a  $C$ -lattice  $L$  and also study the Zariski topology on  $Spec^s(M)$ , the collection of all second elements of a lattice module  $M$ .

The concept of second element of a comultiplication lattice module was introduced in [10]. A lattice module  $M$  is said to be *comultiplication* if for every element  $N$  of  $M$ , there exists an element  $a \in L$  such that  $N = (0_M : a)$  and an

element  $0_M \neq N \in M$  is said to be *second*, if for each  $a \in L$ , either  $aN = N$  or  $aN = 0_M$ .

There are many generalizations of the Zariski topology over the set of all prime submodules of a  $R$ -module  $M$  (see [1, 5, 8, 9, 15, 17]). In [5], the Zariski topology over the prime spectrum  $\text{Spec}(M)$  of a lattice module  $M$  over a  $C$ -lattice  $L$  has been studied by Sachin Ballal and Villas Kharat. In [20], authors introduced and studied the concept of quasi-prime elements as a generalization of prime elements and also the Zariski topology on the quasi-prime spectrum of a lattice module  $M$  over a  $C$ -lattice  $L$ .

The Zariski topology on the set  $\text{Spec}(L)$  of all prime elements in multiplicative lattices is being studied in [18] by Thakare, Manjarekar and Maeda, and in [19] by Thakare and Manjarekar as a generalization of the Zariski topology of a commutative ring with unity.

A lattice  $L$  is said to be *complete*, if for any subset  $S$  of  $L$ , we have  $\vee S, \wedge S \in L$ . A complete lattice  $L$  is said to be a *multiplicative lattice*, if there is defined a binary operation " ." called multiplication on  $L$  satisfying the following conditions:

- (1)  $a.b = b.a$ , for all  $a, b, c \in L$ ;
- (2)  $a.(b.c) = (a.b).c$ , for all  $a, b, c \in L$ ;
- (3)  $a.(\vee_{\alpha} b_{\alpha}) = \vee_{\alpha} (a.b_{\alpha})$ , for all  $a, b_{\alpha} \in L$ ;
- (4)  $a.1 = a$ , for all  $a \in L$ .

Henceforth,  $a.b$  will be simply denoted by  $ab$ . An element  $e \in L$  is said to be *meet principal* (respectively, *join principal*) if it satisfies the identity  $a \wedge be = ((a : e) \wedge b)e$  (respectively,  $((ae \vee b) : e) = a \vee (b : e)$ ), for all  $a, b \in L$ . An element  $e \in L$  is said to be *principal* if it is both meet as well as join principal. If each element of  $L$  is the join of principal elements of  $L$ , then  $L$  is called *principally generated*.

An element  $a$  in  $L$  is called *compact* if  $a \leq \bigvee_{\alpha \in I} b_{\alpha}$  ( $I$  is an indexed set) implies  $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee \dots \vee b_{\alpha_n}$  for some subset  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  of  $I$ . By a  $C$ -lattice, we mean a multiplicative lattice  $L$ , with least element  $0_L$  and greatest element  $1_L$  which is compact as well as multiplicative identity, that is generated under joins by a multiplicatively closed subset  $C$  of compact elements of  $L$ . Throughout this paper,  $L$  will be a  $C$ -lattice.

An element  $p \in L$  is said to be *proper* if  $p < 1$ . A proper element  $m$  of a multiplicative lattice  $L$  is said to be *maximal* if  $m < x \leq 1$  implies  $x = 1$ ,  $x \in L$ . A proper element  $m$  of a multiplicative lattice  $L$  is said to be *minimal* if  $0 \leq x < m$  implies  $x = 0$ ,  $x \in L$ . A proper element  $p$  of a multiplicative lattice  $L$  is said to be *prime* if  $ab \leq p$  implies either  $a \leq p$  or  $b \leq p$ . A proper element  $p$  of a multiplicative lattice  $L$  is said to be *quasi-prime* if  $a \wedge b \leq p$  implies either  $a \leq p$  or  $b \leq p$ . For any  $a \in L$ , its radical is denoted by  $\sqrt{a}$  and defined as  $\sqrt{a} = \vee \{x \in L \mid x^n \leq a, \text{ for some } n \in \mathbb{Z}^+\} = \wedge \{p \in L \mid a \leq p \text{ and } p \text{ is a prime}\}$ .

An element  $a \in L$  with  $\sqrt{a} = a$  is called *semiprime* or *radical*.

A complete lattice  $M$  is said to be *lattice module* over a multiplicative lattice  $L$ , or  $L$ -module, if there is a multiplication between elements of  $M$  and  $L$ , denoted by  $aN \in M$ , for  $a \in L$  and  $N \in M$ , which satisfies the following properties:

1.  $(ab)N = a(bN)$ ;
2.  $(\bigvee_{\alpha} a_{\alpha})(\bigvee_{\beta} N_{\beta}) = (\bigvee_{\alpha\beta} a_{\alpha}N_{\beta})$ ;
3.  $1_L N = N$ ;
4.  $0_L N = 0_M$ ; for all  $a, b, a_{\alpha} \in L$ , and for all  $N, N_{\beta} \in M$ .

The greatest element of  $M$  will be denoted by  $1_M$  and the smallest element will be denoted by  $0_M$ . For  $N \in M$ ,  $b \in L$ , denote  $(N : b) = \bigvee\{K \in M | bK \leq N\}$ . For  $a, b \in L$ , we write  $(a : b) = \bigvee\{x \in L | bx \leq a\}$  and for  $A, B \in M$ ,  $(A : B) = \bigvee\{x \in L | Bx \leq A\}$ . An element  $A \in M$  is said to be *weak meet principal* if  $(B : A)A = B \wedge A$  for all  $B \in M$ ; *weak join principal* if  $(bA : A) = b \vee (0_M : A)$  for all  $b \in L$ ; and *weak principal* if  $A$  is both weak meet principal and weak join principal. An element  $N \in M$  is said to be *compact* if  $N \leq \bigvee_{\alpha \in I} A_{\alpha}$  ( $I$  is an indexed set) implies  $N \leq A_{\alpha_1} \vee A_{\alpha_2} \vee \cdots \vee A_{\alpha_n}$  for some subset  $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$  of  $I$ . If each element of  $M$  is the join of principal (compact) elements of  $M$ , then  $M$  is called *principally generated* (*compactly generated*).

An element  $N < 1_M$  in  $M$  is said to be *prime* if  $aX \leq N$  implies  $X \leq N$  or  $a1_M \leq N$ , i.e.,  $a \leq (N : 1_M)$  for  $a \in L$  and  $X \in M$ . An element  $N < 1_M$  in  $M$  is said to be *quasi-prime* if  $(N : 1_M)$  is a quasi-prime element of  $L$ . Note that, every prime element in  $M$  is quasi-prime. An element  $N < 1_M$  of  $M$  is said to be *maximal* if  $N \leq B$  implies either  $N = B$  or  $B = 1_M$ ,  $B \in M$ . A non-zero element  $K \neq 1_M$  of  $M$  is said to be *minimal* if  $0_M \leq N < K$  implies  $N = 0_M$ ,  $N \in M$ .

Further, all these concepts and for more information on multiplicative lattices and lattice modules, the reader may refer ([3–7, 10–13, 18, 19]).

## 2. TOPOLOGY ON $Spec^s(M)$

Here, we define the second element for a lattice module  $M$  over a  $C$ -lattice  $L$ .

**Definition 2.1.** Let  $M$  be a lattice module over a  $C$ -lattice  $L$ . A non-zero element  $N \in M$  is said to be *second*, if for  $a \in L$ , either  $aN = N$  or  $aN = 0_M$ .

Note that, every minimal element of  $M$  is second.

**Lemma 2.2.** Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and  $N \in M$ . If  $N$  is second then  $(0_M : N)$  is a prime element of  $L$ .

**Proof.** Suppose that  $N$  is a second element of  $M$  and  $abN = 0_M$  for  $a, b \in L$  with  $bN \neq 0_M$ . Since  $N$  is second,  $bN = N$  and so  $aN = 0_M$ , i.e.,  $a \leq (0_M : N)$ . Consequently,  $(0_M : N)$  is a prime element of  $L$ . ■

Converse of Lemma 2.2 is true for comultiplication lattice module (see [10]).

**Lemma 2.3** [10]. *Let  $M$  be a comultiplication lattice module over a multiplicative lattice  $L$  and  $N \in M$ . Then  $N$  is second if and only if  $(0_M : N)$  is a prime element in  $L$ .*

**Example 2.4.** The lattice depicted in Figure (a) is a multiplicative lattice  $L$  and the lattice depicted in Figure (b) is a lattice module  $M$  over a multiplicative lattice  $L$ . Note that,  $X$  is a second element of  $M$  but  $Y, Z, P$  and  $1_M$  are not second elements of  $M$ .

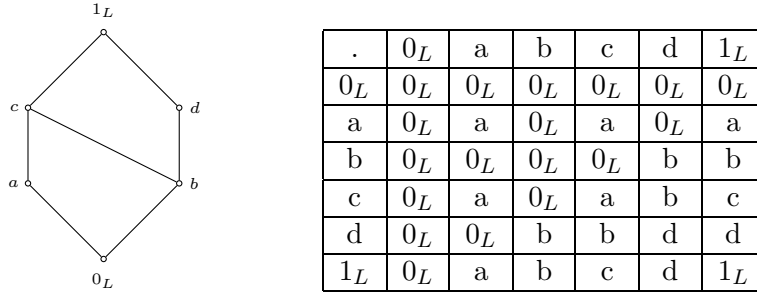


Figure (a). Multiplicative Lattice  $L$ .

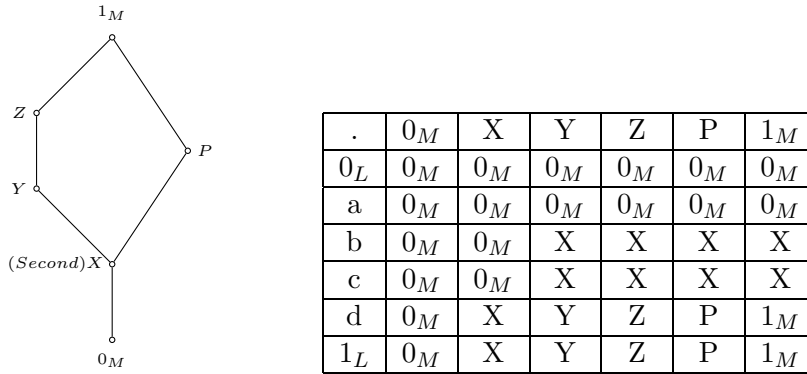


Figure (b). Lattice Module  $M$  over  $L$ .

**Example 2.5.** The lattice depicted in Figure (a) is a multiplicative lattice  $L$  and the lattice depicted in Figure (b) is a Lattice module  $M$  over a multiplicative lattice  $L$ . Note that, all non-zero elements of  $M$  are second elements of  $M$ .

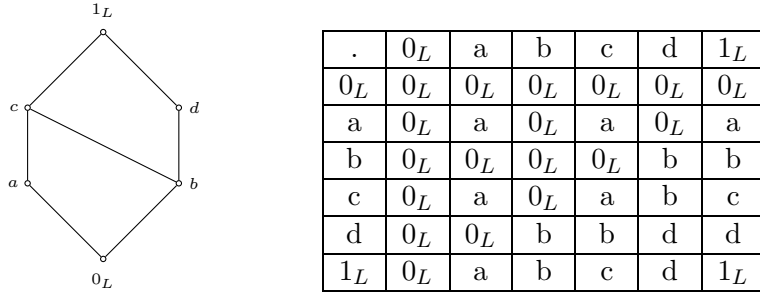


Figure (a). Multiplicative Lattice  $L$ .

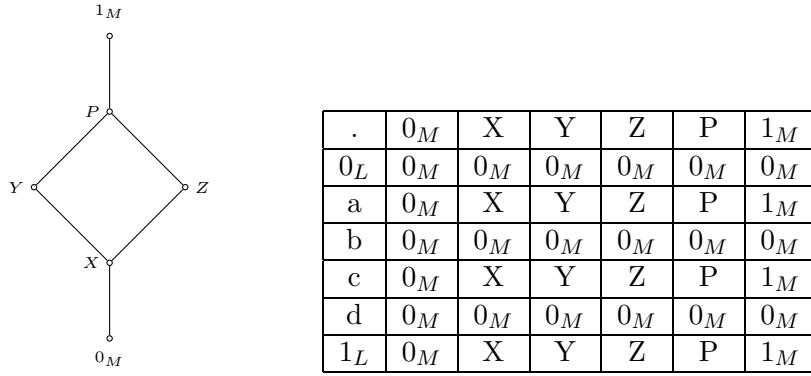


Figure (b). Lattice Module  $M$  over  $L$ .

The following result is useful throughout the paper.

**Lemma 2.6** [14]. *Let  $M$  be a lattice module over a multiplicative lattice  $L$ . Then for  $x \in L$  and  $A, B, C \in M$ , following holds:*

1.  $x \leq (0_M : (0_M : x))$ .
2.  $A \leq (0_M : (0_M : A))$ .
3. If  $A \leq B$  then  $(C : B) \leq (C : A)$ .
4.  $(0_M : A) = (0_M : (0_M : (0_M : A)))$ .
5.  $(A : B \vee C) = (A : B) \wedge (A : C)$ .

Let  $M$  be a lattice module over a  $C$ -lattice  $L$ . Consider the set  $Spec^s(M)$  of second elements of a lattice module  $M$ . Since every minimal element of  $M$  is second,  $Min(M) \subseteq Spec^s(M)$ , where  $Min(M)$  is the set of all minimal elements of  $M$ . Also,  $D^{s*}(N) = \{S \in Spec^s(M) | S \leq N\}$ , for  $N \in M$ . Note that  $D^{s*}(1_M) = Spec^s(M)$ , and  $D^{s*}(0_M)$  is an empty set.

**Proposition 2.7.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and  $N, N_i, K \in M$  ( $i \in I$ ). Then the following statements hold.*

1.  $\cap_{i \in I} D^{s*}(N_i) = D^{s*}(\wedge_{i \in I} N_i)$ .
2.  $D^{s*}(N) \cup D^{s*}(K) \subseteq D^{s*}(N \vee K)$ .

**Proof.** (1) Note that,  $D^{s*}(\wedge_{i \in I} N_i) \subseteq D^{s*}(N_i)$  for each  $i$ , since  $\wedge_{i \in I} N_i \leq N_i$ . Hence  $D^{s*}(\wedge_{i \in I} N_i) \subseteq \cap_{i \in I} D^{s*}(N_i)$ .

Now, suppose that  $K \in \cap_{i \in I} D^{s*}(N_i)$ . Then for each  $i$ ,  $K \in D^{s*}(N_i)$  therefore  $K \leq N_i$ . This implies  $K \leq \wedge_{i \in I} N_i$  and so  $\cap_{i \in I} D^{s*}(N_i) \subseteq D^{s*}(\wedge_{i \in I} N_i)$ . Consequently,  $\cap_{i \in I} D^{s*}(N_i) = D^{s*}(\wedge_{i \in I} N_i)$ .

(2) Since  $N, K \leq N \vee K$ , we have  $D^{s*}(N), D^{s*}(K) \subseteq D^{s*}(N \vee K)$  and so  $D^{s*}(N) \cup D^{s*}(K) \subseteq D^{s*}(N \vee K)$ . ■

We note from Proposition 2.7 that, the set  $\zeta^{s*}(M) = \{D^{s*}(N) | N \in M\}$  forms a topology if and only if it is closed under finite union. In this case,  $\zeta^{s*}(M)$  induces a topology  $\tau^{s*}$  on  $\text{Spec}^s(M)$ , and we call it the *Zariski topology*.

**Proposition 2.8.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and  $a, b \in L$ . Then  $D^{s*}((0_M : a)) \cup D^{s*}((0_M : b)) = D^{s*}((0_M : ab))$ .*

**Proof.** Note that  $D^{s*}((0_M : a)) \cup D^{s*}((0_M : b)) \subseteq D^{s*}((0_M : ab))$  for  $a, b \in L$ .

Now, suppose that  $S \in D^{s*}((0_M : ab))$  with  $S \notin D^{s*}((0_M : b))$ . We claim that  $S \in D^{s*}((0_M : a))$ . By assumption  $S \leq (0_M : ab)$  and  $S \not\leq (0_M : b)$ , therefore  $abS = 0_M$  and  $bS \neq 0_M$ . Since  $S$  is a second element of  $M$  and  $bS \neq 0_M$ , we have  $bS = S$ . Therefore  $abS = aS = 0_M$  and so  $S \leq (0_M : a)$ . Consequently,  $S \in D^{s*}((0_M : a))$ . ■

From Proposition 2.7 and Proposition 2.8, we observe that, the set  $\{D^{s*}((0_M : a)) | a \in L\}$  forms a topology, say  $\tau'^s$  on  $\text{Spec}^s(M)$ .

It clear from Proposition 2.7 that, the collection  $\{D^{s*}(N) | N \in M\}$  need not be closed under finite union. So for  $N \in M$ , we define a new set  $D^s(N) = \{S \in \text{Spec}^s(M) | (0_M : N) \leq (0_M : S)\}$  and we have the following Theorem.

**Theorem 2.9.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and  $N, N_i, K \in M$  ( $i \in I$ ). Then the following statements hold.*

1.  $D^s(1_M) = \text{Spec}^s(M)$ , and  $D^s(0_M)$  is an empty set.
2.  $\cap_{i \in I} D^s(N_i) = D^s(\wedge_{i \in I} (0_M : (0_M : N_i)))$ .
3.  $D^s(N) \cup D^s(K) = D^s(N \vee K)$ .

**Proof.** (1) By definition  $D^s(1_M) = \{S \in \text{Spec}^s(M) | (0_M : 1_M) = 0_L \leq (0_M : S)\} = \text{Spec}^s(M)$  and  $D^s(0_M) = \{S \in \text{Spec}^s(M) | (0_M : 0_M) = 1_L \leq (0_M : S)\}$  is empty.

(2) Suppose that  $S \in \bigcap_{i \in I} D^s(N_i)$ . Then  $S \in D^s(N_i)$ , for each  $i \in I$  therefore  $(0_M : N_i) \leq (0_M : S)$ , for each  $i \in I$  and so  $\bigvee_{i \in I} (0_M : N_i) \leq (0_M : S)$ . Therefore  $(0_M : (0_M : \bigvee_{i \in I} (0_M : N_i))) \leq (0_M : (0_M : (0_M : S))) = (0_M : S)$  by Lemma 2.31(3) and Lemma 2.31(4) and hence  $S \in D^s(\bigwedge_{i \in I} (0_M : (0_M : N_i)))$  by Lemma 2.31(5).

Now, suppose that  $K \in D^s(\bigwedge_{i \in I} (0_M : (0_M : N_i)))$ . Then  $(0_M : \bigwedge_{i \in I} (0_M : (0_M : N_i))) \leq (0_M : K)$ , and hence  $(0_M : (0_M : K)) \leq (0_M : (0_M : \bigwedge_{i \in I} (0_M : (0_M : N_i)))) = \bigwedge_{i \in I} (0_M : (0_M : N_i))$  by Lemma 2.31 (3) and Lemma 2.31(4). Therefore  $(0_M : (0_M : K)) \leq (0_M : (0_M : N_i))$  for each  $i \in I$  and so  $(0_M : N_i) \leq (0_M : K)$ , for each  $i \in I$  by Lemma 2.31(3) and Lemma 2.31(4). Thus  $K \in D^s(N_i)$  for each  $i \in I$  and consequently,  $K \in \bigcap_{i \in I} D^s(N_i)$ .

(3) Note that  $D^s(N) \cup D^s(K) \subseteq D^s(N \vee K)$  for  $N, K \in M$ .

Now, suppose that  $S \in D^s(N \vee K)$ . Then  $(0_M : N \vee K) \leq (0_M : S)$  and so  $(0_M : N) \wedge (0_M : K) \leq (0_M : S)$  by Lemma 2.31(5). Since  $S$  is second,  $(0_M : S)$  is a prime element of  $L$  by Lemma 2.2, and hence quasi-prime. Therefore  $(0_M : N) \leq (0_M : S)$  or  $(0_M : K) \leq (0_M : S)$  by definition of quasi-prime element and so  $S \in D^s(N)$  or  $S \in D^s(K)$ . Consequently,  $S \in D^s(N) \cup D^s(K)$ . ■

Theorem 2.9 shows that, there exists a topology, say  $\tau^s$  on  $\text{Spec}^s(M)$  having  $\{D^s(N) | N \in M\}$  as a family of closed sets.

We denote  $\text{Spec}_p^s(M) = \{N \in M | N \text{ is second and } (0_M : N) = p\}$ , where  $p$  is a prime element of  $L$  and for  $a \in L$ ,  $D^s((0_M : a)) = \{S \in \text{Spec}^s(M) | (0_M : (0_M : a)) \leq (0_M : S)\}$ .

**Lemma 2.10.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and  $N, K \in M$ . Then the following statements hold.*

1. *If  $(0_M : N) = (0_M : K)$ , then  $D^s(N) = D^s(K)$ . Also, the converse is true if  $N, K \in \text{Spec}^s(M)$ .*
2.  *$D^s(N) = \bigcup_{p \in D^s((0_M : N))} \text{Spec}_p^s(M)$ .*
3.  *$D^s(N) = D^s((0_M : (0_M : N))) = D^{s*}((0_M : (0_M : N)))$ . In particular, we have  $D^s((0_M : a)) = D^{s*}((0_M : a))$  for  $a \in L$ .*

**Proof.** (1) Suppose that  $(0_M : N) = (0_M : K)$  and  $S \in D^s(N)$ . Then  $(0_M : N) \leq (0_M : S)$  and so  $(0_M : K) \leq (0_M : S)$ . Therefore  $S \in D^s(K)$  and so  $D^s(N) \subseteq D^s(K)$ . Similarly,  $D^s(K) \subseteq D^s(N)$ .

Conversely, suppose that  $D^s(N) = D^s(K)$  and  $N, K \in \text{Spec}^s(M)$ . Given  $N \in D^s(N)$  and  $D^s(N) = D^s(K)$ , therefore  $(0_M : K) \leq (0_M : N)$  and  $(0_M : N) \leq (0_M : K)$ . Consequently,  $(0_M : N) = (0_M : K)$ .

(2) Suppose that  $P \in D^s(N)$ . Then  $(0_M : N) \leq (0_M : P) = p$ . Therefore  $P \in \bigcup_{p \in D^s((0_M : N))} \text{Spec}_p^s(M)$ . Consequently,  $D^s(N) \subseteq \bigcup_{p \in D^s((0_M : N))} \text{Spec}_p^s(M)$ .

Now, suppose that  $K \in \cup_{p \in D^s((0_M : N))} \text{Spec}_p^s(M)$ . Then there exists  $a \in D^s((0_M : N))$  with  $(0_M : N) \leq a = (0_M : K)$  and hence  $K \in D^s(N)$ , therefore  $\cup_{p \in D^s((0_M : N))} \subseteq D^s(N)$ . Consequently,  $D^s(N) = \cup_{p \in D^s((0_M : N))} \text{Spec}_p^s(M)$ .

(3) Suppose that  $S \in D^s(N)$ . Then  $(0_M : N) \leq (0_M : S)$  and so  $(0_M : (0_M : (0_M : N))) \leq (0_M : (0_M : (0_M : S))) = (0_M : S)$  by Lemma 2.31(3) and Lemma 2.31(4), therefore  $S \in D^s((0_M : (0_M : N)))$ . Thus  $D^s(N) \subseteq D^s((0_M : (0_M : N)))$ .

Now, suppose that  $S \in D^s((0_M : (0_M : N)))$ . Then  $(0_M : (0_M : (0_M : N))) \leq (0_M : S)$ , i.e.,  $(0_M : N) \leq (0_M : S)$  by Lemma 2.31(3) and hence  $S \in D^s(N)$ . Consequently,  $D^s(N) = D^s((0_M : (0_M : N)))$ .

Next, suppose that  $K \in D^s(N)$ . Then  $(0_M : N) \leq (0_M : K)$  and so  $K \leq (0_M : (0_M : K)) \leq (0_M : (0_M : N))$  by Lemma 2.31(2) and Lemma 2.31(3), therefore  $K \in D^{s*}((0_M : (0_M : N)))$ . Thus  $D^s(N) \subseteq D^{s*}((0_M : (0_M : N)))$ .

Now,  $P \in D^{s*}((0_M : (0_M : N)))$  implies  $P \leq (0_M : (0_M : N))$  and hence  $(0_M : (0_M : (0_M : N))) \leq (0_M : P)$  by Lemma 2.31(3). Therefore  $(0_M : N) \leq (0_M : P)$  by Lemma 2.31(4) and so  $P \in D^s(N)$ . Thus  $D^{s*}((0_M : (0_M : N))) \subseteq D^s(N)$ . Consequently,  $D^s(N) = D^{s*}((0_M : (0_M : N)))$ . ■

In what follows and thereafter, the map  $\psi^s : \text{Spec}^s(M) \rightarrow \text{Spec}(L/(0_M : 1_M))$  defined by  $\psi^s(N) = \overline{(0_M : N)}$  is called the *natural map* of  $\text{Spec}^s(M)$ , where  $M$  is a lattice module over a  $C$ -lattice  $L$ .

**Lemma 2.11.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$ . Then the natural map  $\psi^s$  is continuous; more precisely,  $(\psi^s)^{-1}D(\bar{a}) = D^s((0_M : a))$  for  $a \in L$  with  $(0_M : 1_M) \leq a$ .*

**Proof.** Given  $S \in (\psi^s)^{-1}(D(\bar{a}))$ , there exists  $\bar{b} \in D(\bar{a})$  with  $S = (\psi^s)^{-1}(\bar{b})$ . Therefore  $\psi^s(S) = \bar{b}$  and so  $\overline{(0_M : S)} = \bar{b}$ . Thus  $\bar{a} \leq \overline{(0_M : S)} = \bar{b}$  and hence  $a \leq (0_M : S) = b$ . We conclude that,  $(0_M : (0_M : a)) \leq (0_M : S)$  by Lemma 2.31(3) and Lemma 2.31(4). Which implies that  $S \in D^s((0_M : a))$ . Thus  $(\psi^s)^{-1}(D(\bar{a})) \subseteq D^s((0_M : a))$ .

Now, suppose that  $K \in D^s((0_M : a))$ . Then  $(0_M : (0_M : a)) \leq (0_M : K)$ . But by Lemma 2.31(1),  $a \leq (0_M : (0_M : a))$ , therefore  $\bar{a} \leq \overline{(0_M : (0_M : a))} \leq \overline{(0_M : K)}$ . Hence  $K \in (\psi^s)^{-1}D(\bar{a})$ . Consequently,  $(\psi^s)^{-1}D(\bar{a}) = D^s((0_M : a))$ . ■

**Theorem 2.12.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$ . Then the following statements are equivalent.*

1. *The natural map  $\psi^s : \text{Spec}^s(M) \rightarrow \text{Spec}(L/(0_M : 1_M))$  is injective.*
2. *For  $N, K \in \text{Spec}^s(M)$ , if  $D^s(N) = D^s(K)$  then  $N = K$ .*
3.  *$|\text{Spec}_p^s(M)| \leq 1$  for  $p \in \text{Spec}(L)$ .*

**Proof.** (1)  $\Rightarrow$  (2) Suppose that the natural map  $\psi^s$  is injective and  $D^s(N) = D^s(K)$  for  $N, K \in \text{Spec}^s(M)$ . Then  $(0_M : N) = (0_M : K)$ , by Lemma 2.10(1).



Therefore  $\overline{(0_M : N)} = \overline{(0_M : K)}$ , and hence  $\psi^s(N) = \psi^s(K)$ , consequently,  $K = N$ , since  $\psi^s$  is injective.

(2)  $\Rightarrow$  (3) Suppose that  $K, N \in \text{Spec}_p^s(M)$  for some  $p \in \text{Spec}(L)$ . Then  $(0_M : N) = (0_M : K) = p$  and hence  $D^s(N) = D^s(K)$  by Lemma 2.10 (1) and  $N = K$  by (2).

(3)  $\Rightarrow$  (1) Suppose that  $\psi^s(K) = \psi^s(N) = \bar{a}$  for  $K, N \in \text{Spec}^s(M)$ . Then  $\overline{(0_M : K)} = \overline{(0_M : N)} = \bar{a}$ . Therefore  $(0_M : K) = (0_M : N) = a$  and so  $K = N$  by (3). Thus,  $\psi^s$  is injective.  $\blacksquare$

**Theorem 2.13.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$ . If the natural map  $\psi^s$  is surjective, then it is both closed and open. More precisely, for every  $N \in M$ ,  $\psi^s(\overline{D^s(N)}) = \overline{D((0_M : N))}$  and  $\psi^s(\text{Spec}^s(M) - D^s(N)) = \text{Spec}(L/(0_M : 1_M)) - \overline{D((0_M : N))}$ .*

**Proof.** Suppose that  $\psi^s$  is surjective. By Lemma 2.11, we have  $(\psi^s)^{-1}(\overline{D((0_M : N))}) = \overline{D^s((0_M : (0_M : N)))}$  again by Lemma 2.10(3), we have  $D^s((0_M : (0_M : N))) = D^s(N)$ , therefore  $(\psi^s)^{-1}(\overline{D((0_M : N))}) = D^s(N)$ . Since  $\psi^s$  is surjective,  $\psi^s \circ (\psi^s)^{-1}(\overline{D((0_M : N))}) = \psi^s(D^s(N))$ , therefore  $\psi^s(D^s(N)) = \overline{D((0_M : N))}$  and hence  $\psi^s$  is closed. Similarly,  $\psi^s(\text{Spec}^s(M) - D^s(N)) = \text{Spec}(L/(0_M : 1_M)) - \overline{D((0_M : N))}$ , i.e.,  $\psi^s$  open.  $\blacksquare$

**Corollary 2.14.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$ . If the natural map  $\psi^s$  is surjective, then it is bijective if and only if it is homeomorphism.*

Now, we introduce an open base for the Zariski topology on  $\text{Spec}^s(M)$ . For each  $r \in L$ , define  $X^s(r) = \text{Spec}^s(M) - D^s((0_M : r))$ . Then  $X^s(r)$  is an open set of  $\text{Spec}^s(M)$ .

**Lemma 2.15.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$ . Then the set  $B = \{X^s(a) | a \in L\}$  forms an open base for the Zariski topology on  $\text{Spec}^s(M)$ .*

**Proof.** Suppose that  $\text{Spec}^s(M)$  is non-empty and  $U$  is an open subset of  $\text{Spec}^s(M)$ . Then for  $N \in M$ ,  $U = \text{Spec}^s(M) - D^s(N) = \text{Spec}^s(M) - D^s((0_M : (0_M : N)))$  by Lemma 2.10(3). Therefore  $U = \text{Spec}^s(M) - D^s(N) = \text{Spec}^s(M) - D^s((0_M : (0_M : N))) = \text{Spec}^s(M) - D^s((0_M : \bigvee \{x \in L | xN = 0_M\})) = \text{Spec}^s(M) - D^s(\bigwedge_{\{x \in L | xN = 0_M\}}(0_M : x))$  by Lemma 2.6(5). By Theorem 2.9(2), we have  $D^s(\bigwedge_{\{x \in L | xN = 0_M\}}(0_M : x)) = \bigcap_{\{x \in L | xN = 0_M\}} D^s((0_M : x))$ , therefore  $U = \text{Spec}^s(M) - D^s(\bigwedge_{\{x \in L | xN = 0_M\}}(0_M : x)) = \text{Spec}^s(M) - \bigcap_{\{x \in L | xN = 0_M\}} D^s((0_M : x)) = \bigcup_{\{x \in L | xN = 0_M\}} (\text{Spec}^s(M) - D^s((0_M : x))) = \bigcup_{\{x \in L | xN = 0_M\}} X^s(x)$ .  $\blacksquare$

**Theorem 2.16.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$ . Then for  $a \in L$  and the natural map  $\psi^s : \text{Spec}^s(M) \rightarrow \text{Spec}(L/(0_M : 1_M))$ , the following statements hold.*

1.  $(\psi^s)^{-1}(X(\bar{a})) = X^s(a)$ , where  $X(\bar{a}) = \text{Spec}(\bar{L}) - D(\bar{a})$ .
2.  $\psi^s(X^s(a)) \subseteq X(\bar{a})$  and if  $\psi^s$  is surjective, then  $\psi^s(X^s(a)) = X(\bar{a})$ .

**Proof.** (1) Consider  $(\psi^s)^{-1}(X(\bar{a})) = (\psi^s)^{-1}(\text{Spec}(\bar{L}) - D(\bar{a})) = \text{Spec}^s(M) - (\psi^s)^{-1}(D(\bar{a})) = \text{Spec}^s(M) - D^s((0_M : a)) = X^s(a)$ , where  $(\psi^s)^{-1}D(\bar{a}) = D^s((0_M : a))$  for  $a \in L$  with  $(0_M : 1_M) \leq a$  by Lemma 2.11.

(2) Follows from (1). ■

**Theorem 2.17.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$ . Then  $X^s(ab) = X^s(a) \cap X^s(b)$  for  $a, b \in L$ .*

**Proof.** By Theorem 2.16(1), we have  $X^s(ab) = (\psi^s)^{-1}(X(\overline{ab}))$ . Therefore  $X^s(ab) = (\psi^s)^{-1}(X(\overline{ab})) = (\psi^s)^{-1}(\text{Spec}^s(\bar{L}) - D(\overline{ab})) = \text{Spec}^s(M) - (\psi^s)^{-1}(D(\overline{ab})) = \text{Spec}^s(M) - D^s((0_M : ab)) = \text{Spec}^s(M) - D^{s*}((0_M : ab))$  by Lemma 2.11 and Lemma 2.10(3). But by Proposition 2.8, we have  $D^{s*}((0_M : a)) \cup D^{s*}((0_M : b)) = D^{s*}((0_M : ab))$ , therefore  $X^s(ab) = \text{Spec}^s(M) - D^{s*}((0_M : ab)) = \text{Spec}^s(M) - (D^{s*}((0_M : a)) \cup D^{s*}((0_M : b))) = (\text{Spec}^s(M) - D^{s*}((0_M : a))) \cap (\text{Spec}^s(M) - D^{s*}((0_M : b))) = (\text{Spec}^s(M) - D^s((0_M : a))) \cap (\text{Spec}^s(M) - D^s((0_M : b))) = X^s(a) \cap X^s(b)$ . ■

A topological space  $Z$  is called *quasi-compact* if each of its open covers has a finite subcover (see [16]). We recall that  $\text{Spec}(L)$  is quasi-compact if  $L$  is compactly generated multiplicative lattice with  $1_L$  compact (see[18]).

**Theorem 2.18.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and the natural map  $\psi^s$  is surjective. Then for  $r \in L$ , the open set  $X^s(r)$  is quasi-compact. In particular, the space  $\text{Spec}^s(M)$  is quasi-compact.*

**Proof.** Suppose that the natural map  $\psi^s$  is surjective. By Lemma 2.15, the set  $B = \{X^s(a) | a \in L\}$  is an open base for the Zariski topology on  $\text{Spec}^s(M)$ . Let  $\{a_\lambda \in L | \lambda \in \Lambda\}$  be such that  $\text{Spec}^s(M) = \cup_{\lambda \in \Lambda} X^s(a_\lambda)$ . Then by Theorem 2.16(2),  $\text{Spec}(\bar{L}) = X(\bar{1}_L) = \psi^s(X^s(1_L)) = \psi^s(\text{Spec}^s(M) - D^s((0_M : 1_L))) = \psi^s(\text{Spec}^s(M)) = \psi^s(\cup_{\lambda \in \Lambda} X^s(a_\lambda)) = \cup_{\lambda \in \Lambda} \psi^s(X^s(a_\lambda)) = \cup_{\lambda \in \Lambda} X(\overline{a_\lambda})$ . Since  $\text{Spec}(\bar{L})$  is quasi-compact, there exists a finite subset  $\Lambda'$  of  $\Lambda$  such that  $\text{Spec}(\bar{L}) \subseteq \cup_{\lambda \in \Lambda'} X(\overline{a_\lambda})$  therefore by Theorem 2.16(1),  $\text{Spec}^s(M) = X^s(1_L) = (\psi^s)^{-1}(X(\bar{1}_L)) = (\psi^s)^{-1}(\text{Spec}(\bar{L})) \subseteq (\psi^s)^{-1}(\cup_{\lambda \in \Lambda'} X(\overline{a_\lambda})) \subseteq \cup_{\lambda \in \Lambda'} (\psi^s)^{-1}(X(\overline{a_\lambda})) = \cup_{\lambda \in \Lambda'} X^s(a_\lambda)$ . Consequently,  $\text{Spec}^s(M)$  is a quasi-compact space. ■

The following Theorem follows from Lemma 2.15, Theorem 2.17 and Theorem 2.18.

**Theorem 2.19.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and the natural map  $\psi^s$  is surjective. Then the family of quasi-compact open sets of  $\text{Spec}^s(M)$  is closed under finite intersection and forms an open base.*

Note that, by Theorem 2.9(3), the collection  $\{D^s(N) | N \in M\}$  is closed under finite union. Therefore each closed set is of the form of  $D^s(N)$  for  $N \in M$ .

A topological space  $Z$  is  $T_0$  if and only if the closures of distinct points are distinct and a topological space  $Z$  is  $T_1$  if and only if every singleton subset is closed (see [16]). Denote the closure of  $Y \subseteq \text{Spec}^s(M)$  by  $Cl(Y)$ , and the join of all elements in  $Y$  by  $T(Y)$ .

**Lemma 2.20.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and  $Y \subseteq \text{Spec}^s(M)$ . Then  $D^s(T(Y)) = Cl(Y)$ . Hence,  $Y$  is closed if and only if  $D^s(T(Y)) = Y$ .*

**Proof.** Suppose that  $Y \subseteq \text{Spec}^s(M)$  is closed. Clearly,  $Y \subseteq D^s(T(Y))$ . Now, suppose that  $D^s(N)$  is a closed subset of  $\text{Spec}^s(M)$  with  $Y \subseteq D^s(N)$ . Then  $(0_M : N) \leq (0_M : K)$  for each  $K \in Y$  and so  $(0_M : N) \leq \wedge_{K \in Y} (0_M : K)$ . But by Lemma 2.31(5), we have  $\wedge_{K \in Y} (0_M : K) = (0_M : \vee_{K \in Y} K)$ , therefore  $(0_M : N) \leq (0_M : \vee_{K \in Y} K) = (0_M : T(Y))$ . Thus  $(0_M : N) \leq (0_M : T(Y)) \leq (0_M : Q)$  for  $Q \in D^s(T(Y))$ . This implies  $D^s(T(Y)) \subseteq D^s(N)$  and hence  $D^s(T(Y))$  is the smallest closed subset of  $\text{Spec}^s(M)$  containing  $Y$ . Consequently,  $D^s(T(Y)) = Cl(Y)$ . ■

**Lemma 2.21** [10]. *Let  $M$  be a lattice  $L$ -module. Then  $M$  is a comultiplication lattice  $L$ -module if and only if  $N = (0_M : (0_M : N))$  for every  $N \in M$ .*

**Theorem 2.22.** *Let  $M$  be a comultiplication lattice module over a  $C$ -lattice  $L$ . Then  $\text{Spec}^s(M)$  is a  $T_0$ -space.*

**Proof.** Suppose that  $N, K \in \text{Spec}^s(M)$ . Then by Lemma 2.20 and Lemma 2.10(1), we have  $Cl(\{N\}) = Cl(\{K\})$  if and only if  $D^s(N) = D^s(K)$  if and only if  $(0_M : N) = (0_M : K)$  and by Lemma 2.6 and Lemma 2.21 we have  $(0_M : N) = (0_M : K)$  if and only if  $N = K$ . This implies closures of distinct points are distinct, and so  $\text{Spec}^s(M)$  is a  $T_0$ -space. ■

**Lemma 2.23.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and  $S \in \text{Spec}^s(M)$ . Then the following statements hold.*

1.  $Cl(\{S\}) = D^s(S)$ .
2.  $K \in Cl(\{S\})$  implies  $D^s(K) \subseteq D^s(S)$ . Also, the converse is true if  $K \in \text{Spec}^s(M)$ .

**Proof.** (1) Suppose that  $Y = \{S\}$ . Then  $T(Y) = \vee_{S \in Y} S = S$  and therefore by Lemma 2.20,  $Cl(\{S\}) = D^s(T(Y)) = D^s(S)$ .

(2) Suppose that  $K \in Cl(\{S\})$ . Then by (1), we have  $K \in D^s(S)$  and so by definition  $(0_M : S) \leq (0_M : K)$ . If  $P \in D^s(K)$ , then  $(0_M : K) \leq (0_M : P)$  and so, we have  $(0_M : S) \leq (0_M : K) \leq (0_M : P)$  which implies  $P \in D^s(S)$  and therefore  $D^s(K) \subseteq D^s(S)$ . Conversely, suppose that  $K \in \text{Spec}^s(M)$  and  $D^s(K) \subseteq D^s(S)$ . Then  $K \in D^s(K) \subseteq D^s(S)$ . Therefore  $(0_M : S) \leq (0_M : K)$  and hence  $K \in Cl(\{S\})$  by (1). ■

**Lemma 2.24.** *Let  $M$  be a principally generated lattice module over a  $C$ -lattice  $L$ . If  $K \in M$  is minimal then  $(0_M : K)$  is a maximal element of  $L$ .*

**Proof.** Suppose that  $K \in M$  is minimal and  $c \in L$  with  $(0_M : K) \leq c$ . Since  $K$  is minimal and  $cK \leq K$ , we have either  $cK = K$  or  $cK = 0_M$ . If  $cK = K$ , then  $1_L = (cK : K)$ . Since  $M$  is principally generated, we have  $(cK : K) = c \vee (0_M : K)$ , therefore  $1_L = (cK : K) = c \vee (0_M : K) = c$ . Now, if  $cK = 0_M$ , then  $c \leq (0_M : K)$  and hence  $c = (0_M : K)$ . This implies, for  $c \in L$  with  $(0_M : K) \leq c$ , either  $1_L = c$  or  $c = (0_M : K)$ . Consequently,  $(0_M : K)$  is a maximal element of  $L$ . ■

**Lemma 2.25** [10]. *Let  $M$  be a principally generated comultiplication lattice module over a multiplicative lattice  $L$ . Then  $M$  has a minimal element. In particular, every nonzero element of  $M$  has a minimal element.*

**Lemma 2.26** [10]. *Let  $M$  be a principally generated comultiplication lattice module over a multiplicative lattice  $L$ . Then  $K \in M$  is minimal if and only if  $K = (0_M : p) \neq 0_M$  for some maximal element  $p \in L$ .*

**Theorem 2.27.** *Let  $M$  be a principally generated comultiplication lattice module over a  $C$ -lattice  $L$  and  $S \in \text{Spec}^s(M)$ . Then  $\{S\}$  is closed in  $\text{Spec}^s(M)$  if and only if  $S$  is minimal element of  $M$  and  $\text{Spec}_p^s(M) = \{S\}$ .*

**Proof.** Suppose that  $S$  is a minimal element of  $M$  and  $\text{Spec}_p^s(M) = \{S\}$ . Then by Lemma 2.24,  $(0_M : S)$  is a maximal element of  $L$ . Now, suppose that  $K \in \text{Cl}(\{S\})$ . Then by Lemma 2.23 (1),  $K \in D^s(S)$ , and so  $(0_M : S) \leq (0_M : K)$ . Since  $(0_M : S)$  is a maximal element of  $L$ , we have  $p = (0_M : S) = (0_M : K)$ . Therefore  $S, K \in \text{Spec}_p^s(M) = \{S\}$ . This implies  $S = K$  and hence  $\text{Cl}(\{S\}) = \{S\}$ .

Conversely, suppose that  $\{S\}$  is closed in  $\text{Spec}^s(M)$  and  $S$  is not minimal. Then by Lemma 2.25, there exists a minimal element  $N \leq S$  and so  $(0_M : N)$  is a maximal element of  $L$  by Lemma 2.24. Since every maximal element is prime, we have  $(0_M : N)$  is a prime element of  $L$  and therefore  $N \in \text{Spec}^s(M)$  by Lemma 2.3. Now, we have  $N, S \in \text{Spec}^s(M)$  with  $N \leq S$ , therefore  $(0_M : S) \leq (0_M : N)$  by Lemma 2.6(3) and so  $N \in D^s(S) = \text{Cl}(\{S\}) = \{S\}$  by Lemma 2.23. Hence  $N = S$ , and so by Lemma 2.6(3)  $(0_M : N) = (0_M : S)$ . Consequently,  $S$  is a minimal element of  $M$  and  $\text{Spec}_p^s(M) = \{S\}$ . ■

A topological space  $Z$  is irreducible if for any decomposition  $Z \subseteq A_1 \cup A_2$  with closed subsets  $A_i$  of  $Z$  with  $i = 1, 2$ , we have  $A_1 = Z$  or  $A_2 = Z$ . A subset  $Y$  of  $Z$  is irreducible if it is irreducible as a subspace of  $Z$ . An irreducible component of a topological space  $Z$  is a maximal irreducible subset of  $Z$ . A singleton subset and its closure in  $Z$  are irreducible (see [2]).

**Lemma 2.28.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and  $S \in \text{Spec}^s(M)$ . Then  $D^s(S)$  is an irreducible closed subset of  $\text{Spec}^s(M)$ .*

**Proof.** Note that, for  $S \in \text{Spec}^s(M)$ , the set  $\{S\}$  is irreducible and also that  $Cl(\{S\})$  irreducible. But by Lemma 2.23(1), we have  $Cl(\{S\}) = D^s(S)$ . Therefore  $D^s(S)$  is an irreducible closed subset of  $\text{Spec}^s(M)$ . ■

**Lemma 2.29** [11]. *Let  $L$  be a multiplicative lattice and  $S \subseteq \text{Spec}(L)$ . Then  $S$  is irreducible if and only if the meet of all elements of  $S$  is prime.*

**Theorem 2.30.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and  $Y \subseteq \text{Spec}^s(M)$ . If  $T(Y)$  is a second element of  $M$ , then  $Y$  is irreducible. Conversely, if  $Y$  is irreducible, then  $K = \{(0_M : S) | S \in Y\}$  is an irreducible subset of  $\text{Spec}(L)$  such that  $T'(K) = (0_M : T(Y))$  is a prime element of  $L$ , where  $T'(K)$  is the meet of all elements of  $K$ .*

**Proof.** Suppose that  $Y \subseteq Y_1 \cup Y_2$ , where  $Y_1$  and  $Y_2$  are two closed subsets of  $\text{Spec}^s(M)$ . Then by Lemma 2.23(1), and Lemma 2.28, there exist  $N, K \in \text{Spec}^s(M)$  such that  $Y_1 = D^s(N)$  and  $Y_2 = D^s(K)$ . Therefore  $Y \subseteq D^s(N) \cup D^s(K)$ . By Theorem 2.9(3), we have  $D^s(N) \cup D^s(K) = D^s(N \vee K)$ , so  $Y \subseteq D^s(N \vee K)$ . This implies  $(0_M : (N \vee K)) \leq (0_M : P)$  for  $P \in Y$  and hence  $(0_M : (N \vee K)) \leq \bigwedge_{P \in Y} (0_M : P)$ . But by Lemma 2.2(3), we have  $\bigwedge_{P \in Y} (0_M : P) = (0_M : \bigvee_{P \in Y} P) = (0_M : T(Y))$ , therefore  $(0_M : (N \vee K)) = (0_M : N) \wedge (0_M : K) \leq (0_M : T(Y))$ . Since  $T(Y)$  is second,  $(0_M : T(Y))$  is prime by Lemma 2.2 and hence quasi-prime, therefore  $(0_M : N) \wedge (0_M : K) \leq (0_M : T(Y))$  implies either  $(0_M : N) \leq (0_M : T(Y))$  or  $(0_M : K) \leq (0_M : T(Y))$ . Hence for  $P \in Y$ ,  $(0_M : N) \leq (0_M : T(Y)) \leq (0_M : P)$  or  $(0_M : K) \leq (0_M : T(Y)) \leq (0_M : P)$ . This implies  $P \in D^s(N)$  or  $P \in D^s(K)$  and hence  $Y \subseteq D^s(N) = Y_1$  or  $Y \subseteq D^s(K) = Y_2$ . Consequently,  $Y$  is irreducible.

Conversely, suppose that  $Y$  is irreducible. Then  $\psi^s(Y) = K' = \overline{\{(0_M : S) | S \in Y\}}$  is an irreducible subset of  $\text{Spec}(L/(0_M : 1_M))$ , since  $\psi^s$  is continuous. Therefore  $K = \{(0_M : S) | S \in Y\}$  is an irreducible subset of  $\text{Spec}(L)$  and so  $T'(K) = \bigwedge_{S \in Y} (0_M : S)$  is a prime element of  $L$  by Lemma 2.29. But by Lemma 2.31(5),  $\bigwedge_{S \in Y} (0_M : S) = (0_M : \bigvee_{S \in Y} S) = (0_M : T(Y))$ , therefore  $T'(K) = (0_M : T(Y))$  is a prime element of  $L$  and so  $K = \{(0_M : S) | S \in Y\}$  is an irreducible subset of  $\text{Spec}(L)$  by Lemma 2.29. ■

**Corollary 2.31.** *Let  $M$  be a comultiplication lattice module over a  $C$ -lattice  $L$  and  $\text{Spec}_p^s(M)$  is non-empty, for  $p \in \text{Spec}(L)$ . Then the following statements hold.*

1.  $\text{Spec}_p^s(M)$  is irreducible.
2.  $\text{Spec}_p^s(M)$  is an irreducible closed subset of  $\text{Spec}^s(M)$ , if  $p$  is a maximal element of  $L$ .

**Proof.** (1) Suppose that  $\text{Spec}_p^s(M)$  is non-empty. Then  $(0_M : T(\text{Spec}_p^s(M))) = (0_M : \bigvee_{S \in \text{Spec}_p^s(M)} S) = \bigwedge_{S \in \text{Spec}_p^s(M)} (0_M : S)$  by Lemma 2.31(5). But  $(0_M : S) = p$  for  $S \in \text{Spec}_p^s(M)$ , therefore  $(0_M : T(\text{Spec}_p^s(M))) = \bigwedge_{S \in \text{Spec}_p^s(M)} (0_M : S) = \bigwedge_{S \in \text{Spec}_p^s(M)} p = p$  and hence  $(0_M : T(\text{Spec}_p^s(M)))$  is a prime element of  $L$ . Therefore  $T(\text{Spec}_p^s(M))$  is a second element of  $M$  by Lemma 2.3. Consequently,  $\text{Spec}_p^s(M)$  is irreducible by Theorem 2.30.

(2) Note that,  $\text{Spec}_p^s(M)$  is irreducible by (1).

Now, suppose that  $\text{Spec}_p^s(M)$  is non-empty with maximal element  $p \in L$ . Then  $\text{Spec}_p^s(M) = \{S \in \text{Spec}^s(M) \mid (0_M : S) = p\}$ . By Lemma 2.6(1), we have  $p \leq (0_M : (0_M : p))$ , therefore  $\text{Spec}_p^s(M) = \{S \in \text{Spec}^s(M) \mid p = (0_M : (0_M : p)) = (0_M : S)\}$  by maximality of  $p$  and so  $\text{Spec}_p^s(M) = D^s((0_M : p))$  is closed by Theorem 2.9. Consequently,  $\text{Spec}_p^s(M)$  is an irreducible closed subset of  $\text{Spec}^s(M)$ . ■

**Theorem 2.32.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and  $Y \subseteq \text{Spec}^s(M)$  with  $(0_M : T(Y)) = p$  is a prime element of  $L$ . Then  $Y$  is irreducible if  $\text{Spec}_p^s(M)$  is non-empty.*

**Proof.** Suppose that  $\text{Spec}_p^s(M)$  is non-empty and  $Y \subseteq \text{Spec}^s(M)$  with  $(0_M : T(Y)) = p$  is a prime element of  $L$ . Then  $(0_M : T(Y)) = p = (0_M : S)$  for each  $S \in \text{Spec}_p^s(M)$ . Therefore  $D^s(S) = D^s(T(Y))$  by Lemma 2.10(1) and so  $D^s(S) = D^s(T(Y)) = Cl(\{Y\})$  by Lemma 2.20. Hence  $Cl(\{Y\})$  is irreducible by Lemma 2.28. Consequently,  $Y$  is irreducible. ■

Let  $Y$  be a closed subset of a topological space. An element  $y \in Y$  is called a *generic point* of  $Y$ , if  $Y = Cl(\{y\})$  (see [2]). By Proposition 2.23(1), we observe that,  $S \in \text{Spec}^s(M)$  is a generic point of the irreducible closed subset  $D^s(S)$ .

**Theorem 2.33.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  with the surjective natural map  $\psi^s$  and  $Y \subseteq \text{Spec}^s(M)$ . Then  $Y$  is an irreducible closed subset of  $\text{Spec}^s(M)$  if and only if  $Y = D^s(S)$  for some  $S \in \text{Spec}^s(M)$ . Hence, every irreducible closed subset of  $\text{Spec}^s(M)$  has a generic point.*

**Proof.** By Lemma 2.28,  $Y = D^s(S)$  is an irreducible closed subset of  $\text{Spec}^s(M)$ .

Conversely, suppose that  $Y$  is an irreducible closed subset of  $\text{Spec}^s(M)$ . Then by Theorem 2.30,  $(0_M : T(Y)) = p$  is a prime element of  $L$ . Since  $\psi^s$  is surjective, there exists  $S \in \text{Spec}^s(M)$  with  $(0_M : S) = (0_M : T(Y)) = p$ , therefore  $D^s(S) = D^s(T(Y))$  by Lemma 2.10(1) and hence  $D^s(T(Y)) = Cl(Y)$  by Lemma 2.20. Thus  $D^s(S) = Cl(Y)$ . Since  $Y$  is closed,  $Cl(Y) = Y$ . Consequently,  $D^s(S) = Y$  for some  $S \in \text{Spec}^s(M)$ . ■

**Theorem 2.34.** *Let  $M$  be a principally generated comultiplication lattice module over a  $C$ -lattice  $L$ . Then  $\text{Spec}^s(M)$  is a  $T_1$ -space if and only if  $\text{Spec}^s(M) = \text{Min}(M)$ .*

**Proof.** Note that,  $Min(M) \subseteq Spec^s(M)$ . Suppose that  $Spec^s(M)$  is a  $T_1$ -space. Then for  $S \in Spec^s(M)$ ,  $\{S\}$  is closed in  $Spec^s(M)$ . Therefore  $S \in Min(M)$  by Theorem 2.27 and so  $Spec^s(M) \subseteq Min(M)$ . Consequently,  $Spec^s(M) = Min(M)$ .

Conversely, suppose that  $Spec^s(M) = Min(M)$  and  $S \in Spec^s(M)$ . Then  $(0_M : S) = p$  is a prime element of  $L$  by Lemma 2.2, therefore  $S \in Spec_p^s(M)$ . Now, suppose that  $N \in Spec_p^s(M)$ . Then  $N$  is second element with  $(0_M : N) = p$ . Since  $Spec^s(M) = Min(M)$ ,  $N$  is a minimal element of  $M$ . By Lemma 2.6(5),  $(0_M : S \vee N) = (0_M : S) \wedge (0_M : N) = p \wedge p = p$ , therefore  $S \vee N \in Spec^s(M)$  by Lemma 2.3 and hence  $S \vee N \in Min(M)$  since  $Spec^s(M) = Min(M)$ . Thus  $N = S \vee N$  and so  $S \leq N$ . Since  $N$  is a minimal element of  $M$ ,  $N = S$  and so  $\{S\}$  is closed in  $Spec^s(M)$  by Theorem 2.27. Thus every singleton subset is closed and consequently,  $Spec^s(M)$  is  $T_1$ -space. ■

**Definition 2.35** [12]. Topological space  $Z$  is spectral space if  $Z$  satisfy the conditions: (1)  $Z$  is a  $T_0$ -space, (2)  $Z$  is quasi-compact, (3) The quasi-compact open subsets of  $Z$  are closed under finite intersection and form an open base and (4) Each irreducible closed subset of  $Z$  has a generic point.

The following Theorem follows immediately from Theorem 2.17, Theorem 2.18 and Theorem 2.33.

**Theorem 2.36.** *Let  $M$  be a lattice module over a  $C$ -lattice  $L$  and  $\psi^s$  be the surjective natural map. Then  $Spec^s(M)$  is spectral if and only if it is  $T_0$ -space.*

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