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ALL MAXIMAL COMPLETELY REGULAR SUBMONOIDS OF $Hyp_G(2)$

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Abstract

In this paper we consider mappings σ which map the binary operation symbol f to the term $\sigma(f)$ which do not necessarily preserve the arity. These mapping are called generalized hypersubstitutions of type $\tau = (2)$ and we denote the set of all these generalized hypersubstitutions of type $\tau = (2)$ by $Hyp_G(2)$. The set $Hyp_G(2)$ together with a binary operation defined on this set and the identity generalized hypersubstitution which maps f to the term $f(x_1, x_2)$ forms a monoid. In this paper, we determine all maximal completely regular submonoids of this monoid.

Keywords: generalized hypersubstitution, regular element, completely regular.

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1. Introduction

Varieties are collections of algebras which are classified by identities and hypervarieties are collections of algebras which are classified by hyperidentities. The main tool used to study hyperidentities and hypervarieties is the concept of a hypersubstitution. The notion of a hypersubstitution originated by Denecke, Lau,

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Pöschel and Schweigert [2]. In 2000, Leeratanavalee and Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively [7]. The set of all generalized hypersubstitutions together with a binary operation defined on this set and the identity hypersubstitution forms a monoid. There are several published papers on algebraic properties of this monoid and its submonoids.

The concept of a regular subsemigroup plays an important role in the theory of semigroup. In 2011, Puninagool and Leeratanavalee determined all regular elements in the monoid of all generalized hypersubstitutions of type $\tau = (2)$, see [3]. In 2013, Boonmee and Leeratanavalee determined all completely regular elements in the monoid of all generalized hypersubstitutions of type $\tau = (n)$, see [1]. Since the set of all completely regular elements in the monoid of all generalized hypersubstitutions is not its submonoid, so in this paper we determine all maximal completely regular submonoids of $Hyp_G(2)$.

2. Monoid of all generalized hypersustitutions

In this section, we give briefly the concept of the monoid of all generalized hypersubstitutions.

By $W_\tau(X_n)$ is denoted the set of all n-ary terms of type τ. It is clear that every n-ary term is also an m-ary term for all $m \geq n$. Let $W_\tau(X) := \cup_{n=1}^{\infty} W_\tau(X_n)$ and is called the set of all terms of type τ. A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. To define a binary operation on the set of all generalized hypersubstitutions of type τ, firstly, we define the concept of a generalized superposition of terms $S^n : W_\tau(X)^{n+1} \longrightarrow W_\tau(X)$ by the following:

for any term $t \in W_\tau(X)$,

(i) $S^n(x_j, t_1, \ldots, t_n) = t_j$, if $t = x_j$ and $1 \leq j \leq n$;

(ii) $S^n(x_j, t_1, \ldots, t_n) = x_j$, if $t = x_j$ and $n < j$;

(iii) $S^n(t, t_1, \ldots, t_n) = f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_{n_i}, t_1, \ldots, t_n))$, if $t = f_i(s_1, \ldots, s_{n_i})$.

Every generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ by the following:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i and supposed that $\hat{\sigma}[t_i], 1 \leq j \leq n_i$ are already defined.

We define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma_1} \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, x_2, \ldots, x_{n_i})$. In [7], Leeratanavalee and Denecke proved that:

For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$.

By using the previous results, $\underline{Hyp_G(\tau)} := (Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid [7].

3. All maximal completely regular submonoids of $Hyp_G(2)$

To determine the set of all completely regular elements of $\underline{Hyp_G(n)}$, we first recall the definitions of regular and completely regular elements of a semigroup and introduce some notations.

An element a of a semigroup S is called regular (completely regular) if there exists $b \in S$ such that $aba = a$ ($a = aba$ and $ab = ba$).

For a type $\tau = (n)$ with an n-ary operation symbol f and $t \in W_{(n)}(X)$, we denote

$\sigma_t :=$ the generalized hypersubstitution σ of type $\tau = (n)$ which maps f to the term t;

$leftmost(t) :=$ the first variable (from the left) occurring in t;

$rightmost(t) :=$ the last variable occurring in t;

$var(t) :=$ the set of all variables occurring in the term t.

Let $\sigma_t \in Hyp_G(n)$, we denote

$R_1 := \{\sigma_{x_i} | x_i \in X\}$;

$R_2 := \{\sigma_t | t \in W_\tau(X) \setminus X \text{ and } var(t) \cap X_n = \varnothing\}$;

$R_3 := \{\sigma_t | t = f(t_1, \ldots, t_n)$ where $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m$ and for distinct $j_1, \ldots, j_m \in \{1, \ldots, n\}$ and $var(t) \cap X_n = \{x_{j_1}, \ldots, x_{j_m}\}\}$;

$CR(R_3) := \{\sigma_t | t = f(t_1, \ldots, t_n)$ where $t_{i_1} = x_{\pi(i_1)}, \ldots, t_{i_m} = x_{\pi(i_m)}$ and π is a bijective map on $\{i_1, \ldots, i_m\}$ for some $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and $var(t) \cap X_n = \{x_{\pi(i_1)}, \ldots, x_{\pi(i_m)}\}\}$.

It is clear that $CR(R_3) \subset R_3$. In 2010, Puninagool and Leeratanavalee [4] showed that $\bigcup_{i=1}^{3} R_i$ is the set of all regular elements in $Hyp_G(n)$.

Theorem 1 [1]**.** *For each $\sigma_t \in CR(R_3)$, σ_t is a completely regular element in $Hyp_G(n)$.*

Theorem 2 [1]. *Let $CR(Hyp_G(n)) := CR(R_3) \cup R_1 \cup R_2$. Then $CR(Hyp_G(n))$ is the set of all completely regular elements in $Hyp_G(n)$.*

Next, we will consider in case of $\tau = (2)$ that means we have only one binary operation, say f, and then

$R_1 := \{\sigma_{x_i} | x_i \in X\}$;

$R_2 := \{\sigma_t | t \in W_{(2)}(X) \setminus X$ and $var(t) \cap X_2 = \varnothing\}$;

$CR(R_3) := \{\sigma_t |\ t = f(t_1, t_2)$ where $t_i = x_i$ for some $i \in \{1, 2\}$ and $var(t) \cap X_2 = \{x_i\}\} \cup \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$.

It is easily to see that $R_1, R_2, CR(R_3)$ are pairwise disjoint and R_1, R_2 are subsemigroups of $Hyp_G(2)$ but $CR(R_3)$ need not be a submonoid of $Hyp_G(2)$ as the following example.

Example 3. Let $\sigma_s \in CR(R_3)$ and $\sigma_t \in CR(R_3)$ such that $s = f(x_1, f(x_4, x_1))$ and $t = f(x_2, x_1)$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(x_1, f(x_4, x_1))] \\
&= S^2(\sigma_t(f), \hat{\sigma_t}[x_1], \hat{\sigma_t}[f(x_4, x_1)]) \\
&= S^2(\sigma_t(f), x_1, S^2(\sigma_t(f), \hat{\sigma_t}[x_4], \hat{\sigma_t}[x_1])) \\
&= S^2(\sigma_t(f), x_1, S^2(f(x_2, x_1), x_4, x_1)) \\
&= S^2(\sigma_t(f), x_1, f(x_1, x_4)) \\
&= S^2(f(x_2, x_1), x_1, f(x_1, x_4)) \\
&= f(f(x_1, x_4), x_1).
\end{aligned}$$

So $\sigma_t \circ_G \sigma_s = \sigma_{f(f(x_1,x_4),x_1)} \notin CR(R_3)$.

Next, let $\sigma_t \in Hyp_G(2)$, we denote

$CR_1(R_3) := \{\sigma_t | t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ and $var(t) \cap X_2 = \{x_1\}\}$,

$CR_2(R_3) := \{\sigma_t | t = f(t', x_2)$ where $t' \in W_{(2)}(X)$ and $var(t) \cap X_2 = \{x_2\}\}$.

We see that $CR_1(R_3) \cup CR_2(R_3) \subset CR(R_3) \subset R_3$. We denote

$(MCR)_{Hyp_G(2)} = R_1 \cup R_2 \cup CR_1(R_3) \cup \{\sigma_{f(x_1,x_2)}\}, (MCR_1)_{Hyp_G(2)} = R_1 \cup R_2 \cup CR_2(R_3) \cup \{\sigma_{f(x_1,x_2)}\}$ and $(MCR_2)_{Hyp_G(2)} = R_1 \cup R_2 \cup \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$.

Proposition 4. *$CR_1(R_3) \cup \{\sigma_{f(x_1,x_2)}\}$ and $CR_2(R_3) \cup \{\sigma_{f(x_1,x_2)}\}$ are submonoids of $Hyp_G(2)$.*

Proof. It is clear that $CR_1(R_3) \subset Hyp_G(2)$. Next we show that $CR_1(R_3)$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in CR_1(R_3)$. Then $t = f(x_1, t'), s = f(x_1, s')$ where $t', s' \in W_{(2)}(X)$ and $var(t) \cap X_2 = \{x_1\}, var(s) \cap X_2 = \{x_1\}$.

Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_1, s')] \\ &= S^2(\sigma_t(f), \hat{\sigma}_t[x_1], \hat{\sigma}_t[s']) \\ &= S^2(f(x_1, t'), x_1, \hat{\sigma}_t[s']) \\ &= f(x_1, t') \qquad \text{since } \ var(t) \cap X_2 = \{x_1\} \\ &= \sigma_{f(x_1,t')}(f)\end{aligned}$$

and

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \hat{\sigma}_s[f(x_1, t')] \\ &= S^2(\sigma_s(f), \hat{\sigma}_s[x_1], \hat{\sigma}_s[t']) \\ &= S^2(f(x_1, s'), x_1, \hat{\sigma}_s[t']) \\ &= f(x_1, s') \qquad \text{since } \ var(s) \cap X_2 = \{x_1\} \\ &= \sigma_{f(x_1,s')}(f).\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in CR_1(R_3)$ and therefore $\underline{CR_1(R_3) \cup \{\sigma_{f(x_1,x_2)}\}}$ is a submonoid of $\underline{Hyp_G(2)}$. For $\underline{CR_2(R_3) \cup \{\sigma_{f(x_1,x_2)}\}}$ is a submonoid of $\underline{Hyp_G(2)}$, the proof is similar to the previous proof. ■

Theorem 5. *$\underline{(MCR)_{Hyp_G(2)}}$ and $\underline{(MCR_1)_{Hyp_G(2)}}$ are completely regular submonoids of $\underline{Hyp_G(2)}$.*

Proof. It is clear that $(MCR)_{Hyp_G(2)} \subset Hyp_G(2)$ and every element in $(MCR)_{Hyp_G(2)}$ is completely regular. Next we show that $(MCR)_{Hyp_G(2)}$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in (MCR)_{Hyp_G(2)} = R_1 \cup R_2 \cup CR_1(R_3) \cup \{\sigma_{f(x_1,x_2)}\}$.

Case 1. $\sigma_t \in R_1$. Then $t = x_i \in X$.

Case 1.1. $\sigma_s \in R_1$. It is clearly that $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in R_1 \subset (MCR)_{Hyp_G(2)}$.

Case 1.2. $\sigma_s \in R_2$. Then $s = f(s_1, s_2)$ where $s_1, s_2 \in W_{(2)}(X)$ and $var(s) \cap X_2 = \varnothing$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, s_2)] \\ &= S^2(\sigma_t(f), \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2]) \\ &= S^2(x_i, \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2]) \\ &= \begin{cases} \hat{\sigma}_t[s_i] \ , & \text{if } \ i \in \{1, 2\}; \\ x_i \ , & \text{if } \ i > 2 \end{cases} \\ &= \begin{cases} leftmost(s_1) \ , & \text{if } \ i = 1; \\ rightmost(s_2) \ , & \text{if } \ i = 2; \\ x_i \ , & \text{if } \ i > 2. \end{cases}\end{aligned}$$

Then we have $\sigma_t \circ_G \sigma_s = \sigma_{x_j}$ for some $x_j \in X$. Hence $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{HypG(2)}$.

Case 1.3. $\sigma_s \in CR_1(R_3)$. Then $s = f(x_1, s')$ where $s' \in W_{(2)}(X)$ and $var(s) \cap X_2 = \{x_1\}$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_1, s')] \\
&= S^2(\sigma_t(f), \hat{\sigma}_t[x_1], \hat{\sigma}_t[s']) \\
&= S^2(x_i, x_1, \hat{\sigma}_t[s']) \\
&= \begin{cases} x_1\ , & \text{if } \ i = 1; \\ \hat{\sigma}_t[s'] = rightmost(s')\ , & \text{if } \ i = 2; \\ x_i\ , & \text{if } \ i > 2. \end{cases}
\end{aligned}$$

Then we have $\sigma_t \circ_G \sigma_s = \sigma_{x_j}$ for some $x_j \in X$. Hence $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR)_{HypG(2)}$.

Case 2. $\sigma_t \in R_2$. Then $t \in W_{(2)}(X) \setminus X$ and $var(t) \cap X_2 = \varnothing$.

Case 2.1. $\sigma_s \in R_1$. It is obvious that $\sigma_t \circ_G \sigma_s = \sigma_s \in R_1 \subset (MCR)_{HypG(2)}$.

Case 2.2. $\sigma_s \in R_2$. Then $s \in W_{(2)}X \setminus X$ and $var(s) \cap X_2 = \varnothing$. Since R_2 is subsemigroup of $Hyp_G(2)$, so $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in R_2 \subset (MCR)_{HypG(2)}$.

Case 2.3. $\sigma_s \in CR_1(R_3)$. Then $s = f(x_1, s')$ where $s' \in W_{(2)}(X)$ and $var(s) \cap X_2 = \{x_1\}$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_1, s')] \\
&= S^2(\sigma_t(f), \hat{\sigma}_t[x_1], \hat{\sigma}_t[s']) \\
&= S^2(f(t_1, t_2), x_1, \hat{\sigma}_t[s']) \\
&= f(t_1, t_2) \qquad \text{since } \ var(t) \cap X_2 = \varnothing.
\end{aligned}$$

Then $\sigma_t \circ_G \sigma_S \in R_2 \subset (MCR)_{HypG(2)}$.

Case 3. $\sigma_t \in CR_1(R_3)$. Then $t = f(x_1, t')$ where $t' \in W_{(2)}(X)$ and $var(t) \cap X_2 = \{x_1\}$.

Case 3.1. $\sigma_s \in R_1$. It is obvious that $\sigma_t \circ_G \sigma_s = \sigma_s \in R_1 \subset (MCR)_{HypG(2)}$.

Case 3.2. $\sigma_s \in R_2$. Then $s = f(s_1, s_2)$ where $s_1, s_2 \in W_{(2)}(X)$ and $var(s) \cap X_2 = \varnothing$. Consider

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(s_1, s_2)] \\
&= S^2(\sigma_t(f), \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2]) \\
&= S^2(f(x_1, t'), \hat{\sigma}_t[s_1], \hat{\sigma}_t[s_2]) \\
&= f(\hat{\sigma}_t[s_1], t'') \qquad\qquad (\text{since } \ var(t) \cap X_2 = \{x_1\})
\end{aligned}$$

where t'' is a new term derived by substituting x_1 which occurs in t' by $\hat{\sigma}_t[s_1]$. Then $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR)_{Hyp_G(2)}$.

Case 3.3. $\sigma_s \in CR_1(R_3)$. By Proposition 3.6., we have that $\underline{CR_1(R_3) \cup \{\sigma_{f(x_1,x_2)}\}}$ is a submonoid of $\underline{Hyp_G(2)}$. So $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in \overline{CR_1(R_3)} \subset \underline{(MCR)_{Hyp_G(2)}}$.

Therefore $\underline{(MCR)_{Hyp_G(2)}}$ is a completely regular submonoid of $\underline{Hyp_G(2)}$. For $\underline{(MCR_1)_{Hyp_G(2)}}$ is a completely regular submonoid of $\underline{Hyp_G(2)}$, the proof is similar to the previous proof. ■

Theorem 6. $\underline{(MCR_2)_{Hyp_G(2)}}$ *is a completely regular submonoid of* $\underline{Hyp_G(2)}$.

Proof. It is clear that$(MCR_2)_{Hyp_G(2)} \subset Hyp_G(2)$ and every element in $(MCR_2)_{Hyp_G(2)}$ is completely regular. Next we show that $(MCR_2)_{Hyp_G(2)}$ is closed under $\circ_G$. Let $\sigma_t, \sigma_s \in (MCR_2)_{Hyp_G(2)} = R_1 \cup R_2 \cup \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$

Case 1. $\sigma_t \in R_1$. Then $t = x_i \in X$.

Case 1.1. $\sigma_s \in R_1$. Since R_1 is subsemigroup of $Hyp_G(2)$, so $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in (MCR_2)_{Hyp_G(2)}$.

Case 1.2. $\sigma_s \in R_2$. Then $s = f(s_1, s_2)$ where $s_1, s_2 \in W_{(2)}(X)$ and $var(s) \cap X_2 = \varnothing$. We can prove in the same maner as in Case 1.2 of Theorem 3.8. and conclude that $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_2)_{Hyp_G(2)}$.

Case 1.3. $\sigma_s \in \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$. It is obviously that $\sigma_t \circ_G \sigma_s = \sigma_t$ if $\sigma_s = f(x_1, x_2)$. If $\sigma_s = f(x_2, x_1)$, then

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma}_t[f(x_2, x_1)] \\
&= S^2(\sigma_t(f), \hat{\sigma}_t[x_2], \hat{\sigma}_t[x_1]) \\
&= S^2(x_i, x_2, x_1) \\
&= \begin{cases} x_2 \ , & \text{if } \ i = 1; \\ x_1 \ , & \text{if } \ i = 2; \\ x_i \ , & \text{if } \ i > 2. \end{cases}
\end{aligned}$$

Then $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_2)_{Hyp_G(2)}$.

Case 2. $\sigma_t \in R_2$. Then $t \in W_{(2)}(X) \setminus X$ and $var(t) \cap X_2 = \varnothing$. We can prove similar to Case 1, then $\sigma_t \circ_G \sigma_s \in R_2 \subset (MCR_2)_{Hyp_G(2)}$.

Case 3. $\sigma_t \in \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$.

Case 3.1. $\sigma_s \in R_1$. It is obviously that $\sigma_t \circ_G \sigma_s \in R_1 \subset (MCR_2)_{Hyp_G(2)}$.

Case 3.2. $\sigma_s \in R_2$. Then $s = f(s_1, s_2)$ where $s_1, s_2 \in W_{(2)}(X)$ and $var(s) \cap X_2 = \varnothing$. It is clearly that $\sigma_t \circ_G \sigma_s = \sigma_s$, if $t = f(x_1, x_2)$. If $t = f(x_2, x_1)$, then

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(s_1, s_2)] \\ &= S^2(\sigma_t(f), \hat{\sigma_t}[s_1], \hat{\sigma_t}[s_2]) \\ &= S^2(f(x_2, x_1), \hat{\sigma_t}[s_1], \hat{\sigma_t}[s_2]) \\ &= f(\hat{\sigma_t}[s_2], \hat{\sigma_t}[s_1]).\end{aligned}$$

Since $var(f(\hat{\sigma_t}[s_2], \hat{\sigma_t}[s_1])) \cap X_2 = \varnothing$, then $\sigma_{f(x_2,x_1)} \circ_G \sigma_s \in R_2 \subset (MCR_2)_{Hyp_G(2)}$.

Case 3.3. $\sigma_s \in \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$. If $\sigma_t = \sigma_{f(x_1,x_2)}$, then $\sigma_t \circ_G \sigma_s = \sigma_s \in \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\}$. If $\sigma_t = \sigma_{f(x_2,x_1)}$, then

$$\sigma_t \circ_G \sigma_s = \begin{cases} \sigma_{f(x_2,x_1)}, & \text{if } s = f(x_1, x_2); \\ \sigma_{f(x_1,x_2)}, & \text{if } s = f(x_2, x_1). \end{cases}$$

Then $\sigma_t \circ_G \sigma_s, \sigma_s \circ_G \sigma_t \in \{\sigma_{f(x_1,x_2)}, \sigma_{f(x_2,x_1)}\} \subset (MCR_2)_{Hyp_G}(2)$.
Therefore $\underline{(MCR_2)_{Hyp_G(2)}}$ is a completely regular submonoid of $\underline{Hyp_G(2)}$. ■

Theorem 7. *$\underline{(MCR)_{Hyp_G(2)}}$ and $\underline{(MCR_1)_{Hyp_G(2)}}$ are maximal completely regular submonoids of $\underline{Hyp_G(2)}$.*

Proof. Let $\underline{K}$ be a proper completely regular submonoid of $Hyp_G(2)$ such that $(MCR)_{Hyp_G(2)} \subseteq K \subset Hyp_G(2)$. Let $\sigma_t \in K$, then σ_t is completely regular.

Case 1. $\sigma_t = \sigma_{f(x_2,x_1)}$, choose $\sigma_s \in CR_1(R_3)$, then $s = f(x_1, s')$ where $s' \in W_{(2)}(X)$ and $var(s) \cap X_2 = \{x_1\}$. Consider

$$\begin{aligned}(\sigma_t \circ_G \sigma_s)(f) &= \hat{\sigma_t}[f(x_1, s')] \\ &= S^2(\sigma_t(f), \hat{\sigma_t}[x_1], \hat{\sigma_t}[s']) \\ &= S^2(f(x_2, x_1), x_1, \hat{\sigma_t}[s']) \\ &= f(\hat{\sigma_t}[s'], x_1).\end{aligned}$$

Since $\hat{\sigma_t}[s'] \in W_{(2)}(X) \setminus X$ and the second input of the term $f(\hat{\sigma_t}[s'], x_1)$ is x_1, so $\sigma_s \circ_G \sigma_t$ is not completely regular.

Case 2. $\sigma_t \in CR_2(R_3)$. Then $t = f(t', x_2)$ where $t' \in W_{(2)}(X)$ and $var(t) \cap X_2 = \{x_2\}$. Choose $\sigma_s = \sigma_{f(x_1,x_1)} \in CR_1(R_3)$. Consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \hat{\sigma_s}[f(t', x_2)] \\ &= S^2(\sigma_s(f), \hat{\sigma_s}[t'], \hat{\sigma_s}[x_2]) \\ &= S^2(f(x_1, x_1), \hat{\sigma_s}[t'], x_2) \\ &= f(\hat{\sigma_s}[t'], \hat{\sigma_s}[t']).\end{aligned}$$

Since $\hat{\sigma_s}[t'] \in W_{(2)}(X) \setminus X$, so $\sigma_s \circ_G \sigma_t$ is not completely regular. Then $\sigma_t \in \underline{(MCR)_{Hyp_G(2)}}$. Therefore $K \subseteq \underline{(MCR)_{Hyp_G(2)}}$ and thus $\underline{K} = \underline{(MCR)_{Hyp_G(2)}}$. For $\underline{(MCR_1)_{Hyp_G(2)}}$ is a maximal completely regular submonoid of $\underline{Hyp_G(2)}$, the proof is similar to the previous proof. ∎

Theorem 8. *$\underline{(MCR_2)_{Hyp_G(2)}}$ is a maximal completely regular submonoid of $\underline{Hyp_G(2)}$.*

Proof. Let $\underline{K}$ be a proper completely regular submonoid of $Hyp_G(2)$ such that $\underline{(MCR_2)_{Hyp_G(2)}} \subseteq K \subset Hyp_G(2)$. Let $\sigma_t \in K$, then σ_t is a completely regular element. If $\sigma_t \in CR_2(R_3)$ then $t = f(t', x_2)$ where $t' \in W_{(2)}(X)$ and $var(t) \cap X_2 = \{x_2\}$. Choose $\sigma_s = \sigma_{f(x_2,x_1)}$, consider

$$\begin{aligned}(\sigma_s \circ_G \sigma_t)(f) &= \hat{\sigma_s}[f(t', x_2)]\\ &= S^2(\sigma_s(f), \hat{\sigma_s}[t'], \hat{\sigma_s}[x_2])\\ &= S^2(f(x_1, x_2), \hat{\sigma_s}[t'], x_2)\\ &= f(x_2, \hat{\sigma_s}[t']).\end{aligned}$$

Since $\hat{\sigma_t}[s'] \in W_{(2)}(X) \setminus X$ and the first input of the term $f(x_2, \hat{\sigma_s}[t'])$ is x_2, so $\sigma_s \circ_G \sigma_t$ is not completely regular. Then $\sigma_t \in \underline{(MCR_2)_{Hyp_G(2)}}$. Therefore $K \subseteq \underline{(MCR_2)_{Hyp_G(2)}}$ and thus $\underline{K} = \underline{(MCR_2)_{Hyp_G(2)}}$. ∎

Corollary 9. *$\{\underline{(MCR)_{Hyp_G(2)}}, \underline{(MCR_1)_{Hyp_G(2)}}, \underline{(MCR_2)_{Hyp_G(2)}}\}$ is the set of all maximal completely regular submonoids of $Hyp_G(2)$.*

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