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ON THE UNRECOGNIZABILITY BY PRIME GRAPH FOR THE ALMOST SIMPLE GROUP PGL(2, 9)

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Abstract

The prime graph of a finite group G is denoted by $\Gamma(G)$. Also G is called recognizable by prime graph if and only if each finite group H with $\Gamma(H) = \Gamma(G)$, is isomorphic to G. In this paper, we classify all finite groups with the same prime graph as $\mathrm{PGL}(2, 9)$. In particular, we present some solvable groups with the same prime graph as $\mathrm{PGL}(2, 9)$.

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1. Introduction

Let n be a natural number. We denote by $\pi(n)$, the set of all prime divisors of n. Also Let G be a finite group. The set $\pi(|G|)$ is denoted by $\pi(G)$. The set of element orders of G is denoted by $\pi_e(G)$. We denote by $\mu(S)$, the maximal numbers of $\pi_e(G)$ under the divisibility relation. The *prime graph* of G is a graph whose vertex set is $\pi(G)$ and two distinct primes p and q are joined by an edge (and we write $p \sim q$), whenever G contains an element of order pq. The prime graph of G is denoted by $\Gamma(G)$. A finite group G is called *recognizable by prime graph* if for every finite group H such that $\Gamma(G) = \Gamma(H)$, then we have $G \cong H$. So G is *recognizable by prime graph* whenever there exists a fin finite group K such that $\Gamma(K) = \Gamma(G)$ in while K is not isomorphic to G.

For the almost simple group $\mathrm{PGL}(2, q)$, there are a lot of results about the recognition by prime graph. In [8], it is proved that if p is a prime number which is not a Mersenne or Fermat prime and $p \neq 11, 19$ and $\Gamma(G) = \Gamma(\mathrm{PGL}(2, p))$, then G has a unique nonabelian composition factor which is isomorphic to $\mathrm{PSL}(2, p)$ and if $p = 13$, then G has a unique nonabelian composition factor which is isomorphic to PSL$(2,13)$ or PSL$(2,27)$. We know that PGL$(2,2^\alpha) \cong$ PSL$(2,2^\alpha)$. For the characterization of such simple groups we refer to [9, 10]. In [1], it is proved that if $q = p^\alpha$, where p is an odd prime and α is an odd natural number, then PGL$(2,q)$ is uniquely determined by its prime graph.

By the above description, we get that the characterization by prime graph of PGL$(2,p^k)$, where p is an odd prime number and k is even, is an open problem. In this paper as the main result we consider the recognition by prime graph of the almost simple groups PGL$(2,3^2)$. Moreover, we construct some solvable group with the same prime graph as PGL$(2,3^2)$.

2. Preliminary results

Lemma 2.1. *Let G be a finite group and $N \trianglelefteq G$ such that G/N is a Frobenius group with kernel F and cyclic complement C. If $(|F|,|N|) = 1$ and F is not contained in $NC_G(N)/N$, then $p|C| \in \pi_e(G)$ for some prime divisor p of $|N|$.*

Lemma 2.2. *Let G be a Frobenius group with kernel F and complement C. Then the following assertions hold:*

(a) *F is a nilpotent group.*

(b) $|F| \equiv 1 \pmod{|C|}$.

(c) *Every subgroup of C of order pq, with p, q (not necessarily distinct) primes, is cyclic.*

In particular, every Sylow subgroup of C of odd order is cyclic and a Sylow 2-subgroup of C is either cyclic or generalized quaternion group. If C is a non-solvable group, then C has a subgroup of index at most 2 isomorphic to $SL(2,5) \times M$, where M has cyclic Sylow p-subgroups and $(|M|,30) = 1$.

By using [13, Theorem A] we have the following result:

Lemma 2.3. *Let G be a finite group with $t(G) \geq 2$. Then one of the following holds:*

(a) *G is a Frobenius or 2-Frobenius group;*

(b) *there exists a nonabelian simple group S such that $S \leq \overline{G} := G/N \leq Aut(S)$ for some nilpotent normal π_1-subgroup N of G and $\overline{G}/S$ is a π_1-group.*

3. Main results

Lemma 3.1. *There exists a Frobenius group $G = K : C$, where K is an abelian 3-group and $\pi(C) = \{2,5\}$, such that $\Gamma(G) = \Gamma(\mathrm{PGL}(2,9))$.*

Proof. Let F be a finite field with 3^4 elements. Also let V be the additive group of F and H be the multiplicative group $F \setminus \{0\}$. We know that H acts on V by right product. So $G := V \rtimes H$ is a finite group such that $\pi(G) = \{2, 3, 5\}$, since $|V| = 3^4$ and $|H| = 80$. On the other hand H acts fixed point freely on V, so G is a Frobenius group with kernel V and complement H. Since the multiplicative group $F \setminus \{0\}$ is cyclic, H is cyclic too. Therefore, G has an element of order 10, and so the prime graph of G consists just one edge, which is the edge between 2 and 5. This implies that $\Gamma(G) = \Gamma(\mathrm{PGL}(2, 9))$, as desired. ∎

Lemma 3.2. *There exists a Frobenius group $G = K : C$, where $\pi(K) = \{2, 5\}$ and C is a cyclic 3-group such that $\Gamma(G) = \Gamma(\mathrm{PGL}(2, 9))$.*

Proof. Let F_1 and F_2 be two fields with 2^2 and 5^2 elements, respectively. Let $V := F_1 \times F_2$ be the direct product of the additive groups F_1 and F_2. Also let $H := H_1 \times H_2$, be the direct product of H_1 and H_2, which are the multiplicative groups $F_1 \setminus \{0\}$ and $F_2 \setminus \{0\}$, respectively. We know that H_i acts fixed point freely on F_i, where $1 \le i \le 2$. So we define an action of H on V as follows: for each $(h, h') \in H$ and $(g, g') \in V$, we define $(g, g')^{(h,h')} := (hg, h'g')$. It is clear that this definition is well defined. So we may construct a finite group $G = V \rtimes H$. On the other hand H acts fixed point freely on V. So G is a Frobenius group with kernel V and complement H such that $\pi(V) = \{2, 5\}$ and $\pi(H) = \{3\}$. Finally, since V is nilpotent, we get that $\Gamma(G)$ contains an edge between 2 and 5, so $\Gamma(G) = \Gamma(\mathrm{PGL}(2, 9))$. ∎

Lemma 3.3. *There exists a 2-Frobenius group G with normal series $1 \lhd H \lhd K \lhd G$, such that $\pi(H) = \{5\}$, $\pi(G/K) = \{2\}$ and $\pi(K/H) = \{3\}$, such that $\Gamma(G) = \Gamma(\mathrm{PGL}(2, 9))$.*

Proof. Let F be a field with 5^2 elements and V be its additive group. We know that $F = \{0, 1, \alpha, \alpha^2, \ldots, \alpha^{23}\}$, where α is a generator of the multiplicative group $F \setminus \{0\}$. Hence $|\alpha| = 24$, and so $\beta := \alpha^8$ has order 3. Also $\langle\beta\rangle \cong Z_3$ and $\mathrm{Aut}(Z_3) \cong Z_2$. This argument implies that we may construct a Frobenius group $T := \langle\beta\rangle : \langle\gamma\rangle$, where γ is an involution.

Now we define an action of T on V as follows: for each $\beta^x\gamma^y \in T$ and $v \in V$, $v^{\beta^x\gamma^y} := \beta^x v$, where $1 \le x \le 3$ and $1 \le y \le 2$. Therefore $G := V : T$ is a 2 Frobenius group with desired properties. ∎

Theorem 3.4. *Let G be a finite group. Then $\Gamma(G) = \Gamma(\mathrm{PGL}(2, 3^2))$ if and only if G is isomorphic to one of the following groups:*

(1) *A Frobenius group $K : C$, where K is an abelian 3-group and $\pi(C) = \{2, 5\}$,*

(2) *A Frobenius group $K : C$, where $\pi(K) = \{2, 5\}$ and C is a cyclic 3-group,*

(3) *A 2-Frobenius group with normal series* $1 \lhd H \lhd K \lhd G$*, such that* $\pi(H) \subseteq \{2,5\}$*,* $\pi(G/K) = \{2\}$ *and* $\pi(K/H) = \{3\}$*,*

(4) *Almost simple group* $\mathrm{PGL}(2,3^2)$*.*

Proof. Throughout the proof, we assume that G is a finite group with the same prime graph as the almost simple group $\mathrm{PGL}(2,3^2)$. First we note that by [16, Lemma 7], we have:

$$\mu(\mathrm{PGL}(2,9)) = \{3,8,10\}.$$

Hence in $\Gamma(\mathrm{PGL}(2,3^2))$ (and so in $\Gamma(G)$), there is only one edge which is the edge between 2 and 5 and so 3 is an isolated vertex. This implies that $\Gamma(G)$ has two connected components $\{3\}$ and $\{2,5\}$. Thus by Lemma 2.3, we get that G is a Frobenius group or 2-Frobenius group or there exists a nonabelian simple group S such that $S \leq G/Fit(G) \leq Aut(S)$. We consider each possibility for G.

Let G be a Frobenius group with kernel K and complement C. We know that K is nilpotent and C is a connected component of the prime graph of G. Also by the above description, 3 is not adjacent to 2 and 5 in $\Gamma(G)$. This shows that either $\pi(K) = \{3\}$ or $\pi(C) = \{3\}$. We consider these cases, separately.

Case 1. Let $\pi(K) = \{3\}$. Hence the order of complement C, is even and so K is an abelain subgroup of G. Also by the above description, $\pi(C) = \{2,5\}$. Since C is a connected component of $\Gamma(G)$, there is and edge between 2 and 5. So it follows that $\Gamma(G) = \Gamma(\mathrm{PGL}(2,9))$, which implies groups satisfying in (1).

Case 2. Let $\pi(C) = \{3\}$. Hence $\pi(K) = \{2,5\}$. Since K is nilpotent, we get that 2 and 5 are adjacent in $\Gamma(G)$. This means $\Gamma(G) = \Gamma(\mathrm{PGL}(2,9))$ and so we get (2).

Case 3. Let G be a 2-Frobenius group with normal series $1 \lhd H \lhd K \lhd G$. Since $\pi(K/H)$ and $\pi(H) \cup \pi(G/K)$ are the connected components of $\Gamma(G)$, we get that $\pi(K/H) = \{3\}$ and $\pi(H) \cup \pi(G/K) = \{2,5\}$. This implies (3).

Case 4. Let there exist a nonabelian simple group S, such that $S \leq \bar{G} := G/\mathrm{Fit}(G) \leq \mathrm{Aut}(S)$. Since $\pi(S) \subseteq \pi(G)$, $\pi(S) = \{2,3,5\}$. The finite simple groups with this property are classified in [12, Table 8]. So we get that S is isomorphic to one of the simple groups A_5, $\mathrm{PSU}(4,2)$ and $\mathrm{PSL}(2,9)(\cong A_6)$.

Subcase 4.1. Let $S \cong A_5$. We know that $\mathrm{Aut}(A_5) = S_5$. So $\bar{G}$ is isomorphic to the alternating group A_5 or the symmetric group S_5. Since in the prime graph of S_5, there is an edge between 2 and 3, hence we get that $\bar{G}$ is not isomorphic to S_5. Thus, $G/\mathrm{Fit(G)} = A_5$. In the prime graph of A_5, 2 and 5 are nonadjacent. Then at least one of the prime numbers 2 or 5, belongs to $\pi(\mathrm{Fit}(G))$.

Let $5 \in \pi(\mathrm{Fit}(G))$. Let F_5 be a Sylow 5-subgroup of $\mathrm{Fit}(G)$. Since F_5 is a characteristic subgroup of $\mathrm{Fit}(G)$ and $\mathrm{Fit}(G)$ is a normal subgroup of G, $F_5 \trianglelefteq G$.

On the other hand in alternating group A_5, the subgroup $\langle(12)(34),(13)(24)\rangle : \langle(123)\rangle$ is a Frobenius subgroup isomorphic to $2^2 : 3$. We recall that by the previous argument, $F_5 \trianglelefteq G$, and so G has a subgroup isomorphic to $5^\alpha : (2^2 : 3)$. So by Lemma 2.1, we get that 3 is adjacent to 5, a contradiction.

Subcase 4.2. Let $S \cong \mathrm{PSU}(4,2)$. By [4], there is an edge between 3 and 2 which is a contradiction.

Subcase 4.3. Let $S \cong \mathrm{PSL}(2,9)$. Then $\bar{G}$ is isomorphic to $\mathrm{PSL}(2,9)$ or $\mathrm{PSL}(2,9) : \langle\theta\rangle$, where θ is a diagonal, field or diagonal-field automorphism of $\mathrm{PSL}(2,9)$. In particular θ is an involution. If θ is a field or diagonal-field automorphism of $\mathrm{PSL}(2,9)$, then the semidirect product $\mathrm{PSL}(2,9) : \langle\theta\rangle$ contains an element of order 6. Hence θ is neither a field automorphism nor a diagonal-field automorphism. Therefore θ is a diagonal automorphism and so $\bar{G} \cong \mathrm{PGL}(2,9)$.

By the above discussion, $G/\mathrm{Fit}(G) \cong \mathrm{PGL}(2,9)$. It is enough to prove that $\mathrm{Fit}(G) = 1$. On the contrary, let $r \in \pi(\mathrm{Fit}(G))$. Also let F_r be the Sylow r-subgroup of $\mathrm{Fit}(G)$. Since $\mathrm{Fit}(G)$ is nilpotent, we can write $\mathrm{Fit}(G) = O_{r'}(\mathrm{Fit}(G)) \times F_r$. So if we put $\tilde{G} = G/O_{r'}(\mathrm{Fit}(G))$, then we get that:

$$\mathrm{PGL}(2,9) \cong \frac{G}{\mathrm{Fit}(G)} \cong \frac{\tilde{G}}{F_r} \cong \frac{\tilde{G}/\Phi(F_r)}{F_r/\Phi(F_r)}.$$

Since $F_r/\Phi(F_r)$ is an elementary abelian group, without loose of generality we may assume that $F := \mathrm{Fit}(G)$ is an elementary abelian r-group and $G/F \cong \mathrm{PGL}(2,9)$.

If $r = 3$, then by , we conclude that in $\Gamma(G)$, 2 and 3 are adjacent, which is a contradiction. So let $r \neq 3$. Also let S_3 be a Sylow 3-subgroup of $\mathrm{PGL}(2,9)$. We know that S_3 is not cyclic. On the other hand $F \rtimes S_3$ is a Frobenius group since 3 is an isolated vertex of $\Gamma(G)$. This follows that S_3 is cyclic which is impossible. Therefore $F = 1$ and so $G \cong \mathrm{PGL}(2,9)$, which completes the proof. ■

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