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ON THE GENUS OF THE CAYLEY GRAPH OF A COMMUTATIVE RING

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Abstract

Let R be a commutative ring with non-zero identity and let $Z(R)$ be the set of all zero-divisors. The Cayley graph $\mathbb{CAY}(R)$ of R is the simple undirected graph whose vertices are elements of R and two distinct vertices x and y are joined by an edge if and only if $x - y \in Z(R)$. In this paper, we determine all isomorphism classes of finite commutative rings with identity whose $\mathbb{CAY}(R)$ has genus one.

Keywords: Cayley graph, local ring, zero-divisor, planar graph, genus.

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1. Introduction

The idea to associate a graph to a ring first appeared in [10]. There are a number of papers in which the graphs associated to rings were introduced (see for example [5, 6, 15]). The question concerning planarity of the zero-divisor graph was first posed in [6]. The more general problem, the one about the genus of graphs attached to rings, has received considerable attention. There are many extensive studies of this topic (see [11, 19, 21, 14]). Toroidal zero-divisor graphs were classified independently by Wang [19, 20] and Wickham [21]. Genus two zero-divisor graphs of local rings are studied by Bloomfield and Wickham [12]. In [4], Anderson *et al.* introduced and studied the total graph of a commutative ring R. The total graph of R, denoted by $T_\Gamma(R)$, is the undirected graph whose vertices are the elements in R and two distinct vertices x and y are adjacent if $x + y \in Z(R)$. In [14], Maimani *et al.* determined a lower bound for the genus of the total graph of a direct product of two fields. Also in that article, the authors classified the finite commutative rings R for which $T_\Gamma(R)$ is a planar or toroidal graph and then Tamizh Chelvam *et al.* characterized all commutative rings whose total graph has genus two. The complement of the total graph is denoted by $\overline{T_\Gamma(R)}$. Note that two distinct vertices x and y in $\overline{T_\Gamma(R)}$ are adjacent if $x + y \in Reg(R)$. In [3], Akhtar *et al.* defined the unitary cayley graph of R, denoted by $Cay(R, U(R))$, as the graph whose vertex set is R and two distinct vertices x and y are adjacent if $x - y \in U(R)$. Later on, Ashrafi *et al.* [7] introduced the unit graph of R, denoted by $G(R) = Cay(R^+, U(R))$. Actually $G(R)$ is the graph obtained by setting all the elements of R to be the vertices and defining distinct vertices x and y to be adjacent if $x + y \in U(R)$. Akhtar *et al.* determined all finite commutative rings whose unitary cayley graph has genus zero ([3], Theorem 8.2). Also Ashrafi *et al.* determined all finite commutative rings whose unit graph has genus zero ([7], Theorem 5.14) and then Tamizh Chelvam *et al.* characterized all commutative rings whose unit and unitary cayley graphs have genus one ([18], Theorem 4.3). Further, in variation to the concept of zero-divisor graph, Akbari *et al.* [1] introduced and studied the Cayley graph of a commutative ring R with respect to its zero-divisors. The Cayley graph of R, denoted by $\mathbb{CAY}(R)$, is the undirected graph whose vertices are elements of R and two distinct vertices x and y are adjacent if and only if $x - y \in Z(R)$. The purpose of this article is to explore the question of embedding a $\mathbb{CAY}(R)$ on surfaces of higher genus, the torus in particular. In this paper, we characterize all finite commutative rings R whose $\mathbb{CAY}(R)$ has genus one.

A ring R is called *local* if it has a unique maximal ideal. Note that $R^\times$ be the set of all units in R and $J(R)$ be the Jacobson radical of R. For any set X, let X^* denote the set of non-zero elements of X. We denote the ring of integers modulo n by $\mathbb{Z}_n$, the field with q elements by $\mathbb{F}_q$ and the set of all nilpotent elements in R by $N(R)$. For basic definitions on rings, one may refer [8].

By a graph $G = (V, E)$, we mean an undirected simple graph with vertex set V and edge set E. A graph in which each pair of distinct vertices is joined by an edge is called a complete graph. We use K_n to denote the complete graph on n vertices. An r-partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. The union of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2$ whose vertex set is $V_1 \cup V_2$ and whose edge set is $E_1 \cup E_2$. The cartesian product of graphs G_1 and G_2 is the graph $G_1 \Box G_2$ whose vertex set is $V(G_1) \times V(G_2)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G_1)$ and $v_1 = v_2$ or $v_1v_2 \in E(G_2)$ and $u_1 = u_2$. Let G and H be two graphs with disjoint vertex sets V_1, V_2 and edge sets E_1, E_2 respectively. Their *join* is denoted by $G + H$ and it consists of $G \cup H$ and all edges joining every vertex of V_1 with every vertex of V_2. The girth of a graph is the length of the shortest cycle in the graph. A clique in a graph G is a subset of pairwise adjacent vertices and the supremum of the size of cliques in G, denoted by $\omega(G)$, is called the clique number of G. For general references on graph theory, we use Chartrand [13].

Let S_k denote the sphere with k handles, where k is a nonnegative integer, that is, S_k is an oriented surface of genus k. The genus of a graph G, denoted $\gamma(G)$, is the minimal integer n such that the graph can be embedded in S_n. Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. A genus 0 graph is called a *planar graph* and a genus 1 graph is called a *toroidal graph*. We note here that if H is a subgraph of a graph G, then $\gamma(H) \leq \gamma(G)$. A minor of G is a graph obtained from G by contracting edges in G or deleting edges and isolated vertices in G. A classical theorem due to Wagner states that a graph G is planar if and only if G does not have K_5 or $K_{3,3}$ as a minor. It is well known that if G' is a minor of G, then $\gamma(G') \leq \gamma(G)$. For $xy \in E(G)$, we denote the contracted edge by the vertex $[x, y]$. Also if H is a subgraph of G and H' is a minor of H, then we call H' as a minor subgraph of G. Further note that if H is a subgraph of a graph G, then $g(H) \leq g(G)$. A result of Battle, Harary, Kodama, and Youngs states that the genus of a graph is the sum of the genera of its blocks [9]. For example, the graph $\mathbb{G}$ in Figure 1.1 has two blocks, both isomorphic to $K_{3,3}$, and so has genus 2 [21, C. Wickham]. For details on the notion of embedding a graph in a surface, see [22].

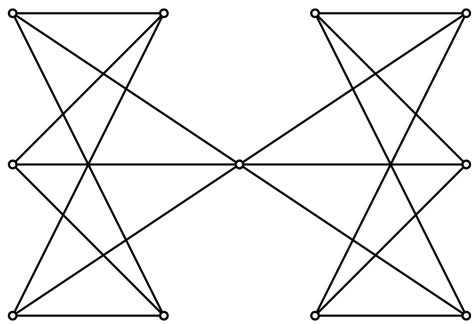

Figure 1.1. Graph $\mathbb{G}$ with $\gamma(\mathbb{G}) = 2$.

Now we summarize certain results and bounds on the genus of a graph.

Lemma 1.1 [17]**.** $\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ if $n \geq 3$. In particular, $\gamma(K_n) = 1$ if $n = 5, 6, 7$.

Lemma 1.2 [17]**.** $\gamma(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ *if* $m, n \geq 2$. *In particular,* $\gamma(K_{4,4}) = \gamma(K_{3,n}) = 1$ *if* $n = 3, 4, 5, 6$. *Also* $\gamma(K_{5,4}) = \gamma(K_{6,4}) = \gamma(K_{m,4}) = 2$ *if* $m = 7, 8, 9, 10$.

Lemma 1.3 [22, Euler formula]**.** *If* G *is a finite connected graph with* n *vertices,* m *edges, and genus* g*, then* $n - m + f = 2 - 2\gamma$*, where* f *is the number of faces created when* G *is minimally embedded on a surface of genus* γ*.*

Lemma 1.4 [9]**.** *If* G *is a connected graph having a subgraph* G_1 *and a block* G_2 *such that* $G = G_1 \cup G_2$*, and* $G_1 \cap G_2 = v$ *(a vertex of* G*), then* $\gamma(G) \geq \gamma(G_1) + \gamma(G_2)$*.*

Lemma 1.5 [16]**.** *For all integers* $n \not\equiv 5$ *or* $9 \ (mod \ \ 12)$ *with* $n \geq 2$*,* $\gamma(K_n \Box K_2) = \left\lceil \frac{(n-3)(n-2)}{6} \right\rceil$*. If* $n = 5$*, then* $\gamma(K_n \Box K_2) = 2$*.*

Theorem 1.6 [2]**.** *Let* R *be a ring which is not an integral domain. Then* $\omega(\mathbb{CAY}(R))$*,* $\chi(\mathbb{CAY}(R))$*,* $\omega(Z(\mathbb{CAY}(R)))$*,* $\omega(Z(\mathbb{CAY}(R)))$ *and* $\sup\{|\mathfrak{m}| : \mathfrak{m} \in Max(R)\}$ *are all infinite or all finite and equal, where* $Z(\mathbb{CAY}(R))$ *is the induced subgraphs of* $\mathbb{CAY}(R)$ *on* $Z(R)$*.*

Theorem 1.7 [2]**.** *Let* R *be a ring which is not an integral domain. Then* $\mathbb{CAY}(R)$ *is planar if and only if* R *is one of the following rings:*

$$\mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_4,\ \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle},\ \mathbb{Z}_9,\ \frac{\mathbb{Z}_3[x]}{\langle x^2 \rangle},\ \mathbb{Z}_8,\ \frac{\mathbb{Z}_2[x]}{\langle x^3 \rangle},\ \frac{\mathbb{Z}_2[x,y]}{\langle x,y \rangle^2},$$

$$\frac{\mathbb{Z}_4[x]}{\langle 2x, x^2-2 \rangle},\ \frac{\mathbb{Z}_4[x]}{\langle 2x, x^2 \rangle},\ \frac{\mathbb{Z}_4[x]}{\langle x^2+x+1 \rangle},\ \frac{\mathbb{F}_4[x]}{\langle x^2 \rangle}.$$

2. GENUS OF $\mathbb{CAY}(R)$

In this paper, we determine all isomorphism classes of finite commutative non-local rings with identity whose $\mathbb{CAY}(R)$ has genus one.

Theorem 2.1. *Let* $R = F_1 \times \cdots \times F_m$ *be a finite commutative ring with identity, where each* F_j *is a field and* $m \geq 2$*. Then* $\gamma(\mathbb{CAY}(R)) = 1$ *if and only if* R *is isomorphic to one of the following rings:* $\mathbb{Z}_3 \times \mathbb{Z}_3$*,* $\mathbb{Z}_2 \times \mathbb{F}_4$*.*

Proof. Assume that $\gamma(\mathbb{CAY}(R)) = 1$. Suppose $n \geq 4$, Then $|\mathfrak{m}| \geq 8$ for all $\mathfrak{m} \in Max(R)$ and by Theorem 1.6, $\omega(\mathbb{CAY}(R)) \geq 8$. Hence K_8 is a subgraph of $\mathbb{CAY}(R)$ and by Lemma 1.1, $\gamma(K_8) = 2$, a contradiction. Hence $n \leq 3$.

Suppose $n = 3$. Then $R = F_1 \times F_2 \times F_3$. Suppose that $|F_i| \geq 3$ for some i. Without loss of generality, we assume that $|F_1| \geq 3$. Let $\Omega = \{x_1, x_2, \ldots, x_{11}\}$ where $x_1 = (0,0,0), x_2 = (0,1,0), x_3 = (1,1,0), x_4 = (u,1,0), x_5 = (u,0,0), x_6 = (1,0,0), x_7 = (u,1,1), x_8 = (u,0,1), x_9 = (0,0,1), x_{10} = (1,0,1), x_{11} = (1,1,1)$. Then the subgraph induced by Ω in $\mathbb{CAY}(R)$ contains $\mathbb{G}$ (see Figure 1.1) as a subgraph. Since $\gamma(\mathbb{G}) = 2$, we get $\gamma(\mathbb{CAY}(R)) > 1$, a contradiction. Hence $F_1 = F_2 = F_3 \cong \mathbb{Z}_2$ and so $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Note that $\mathbb{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ is a subgraph of $\mathbb{CAY}(R)$. Since $\gamma(\mathbb{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)) > 1$, $\gamma(\mathbb{CAY}(R)) > 1$, a contradiction. Hence $n = 2$ and so $R = F_1 \times F_2$. By definition of Cayley graphs, $\mathbb{CAY}(R) = K_{|F_1|} \Box K_{|F_2|}$. If $|F_1| \geq 5$, then $K_5 \Box K_2$ is a subgraph of $\mathbb{CAY}(R)$. By Lemma 1.5, $\gamma(K_5 \Box K_2) = 2$ and so $\gamma(\mathbb{CAY}(R)) > 1$, a contradiction. Hence $|F_i| < 5$ for $i = 1, 2$. Since $\mathbb{CAY}(R)$ is non-planar and by Theorem 1.7, $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$. Hence R is isomorphic to one of the following rings:

$$\mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{F}_4 \text{ or } \mathbb{F}_4 \times \mathbb{F}_4.$$

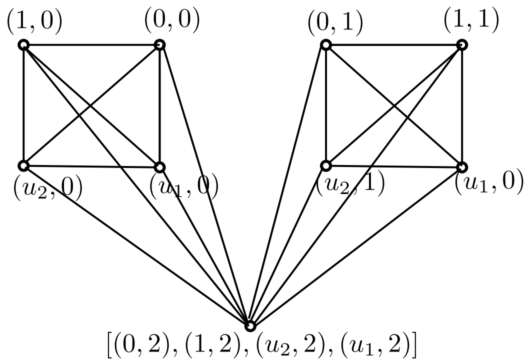

Figure 1.2. $H = (K_4 \cup K_4) + K_1$.

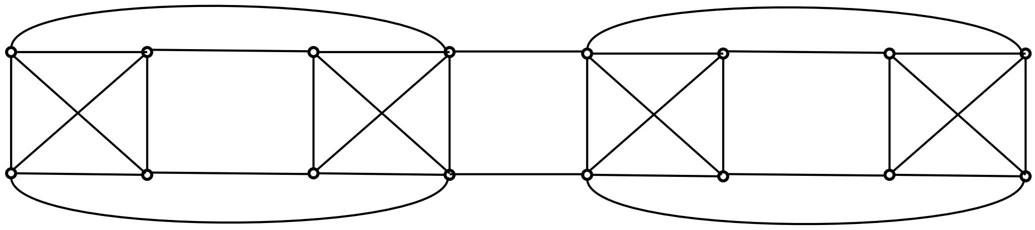

Figure 1.3. H_1

Suppose $R \cong \mathbb{Z}_3 \times \mathbb{F}_4$. Then $\mathbb{CAY}(R) \cong K_4 \Box K_3$ and so $H = (K_4 \cup K_4) + K_1$ (see Figure 1.2) is a minor subgraph of $\mathbb{CAY}(\mathbb{Z}_3 \times \mathbb{F}_4)$. By Lemma 1.4, $\gamma(H) \geq 2$.

This implies that $\gamma(\mathbb{CAY}(R)) > 1$, a contradiction. Hence $R \not\cong \mathbb{Z}_3 \times \mathbb{F}_4$.

Suppose $R \cong \mathbb{F}_4 \times \mathbb{F}_4$. Then $\mathbb{CAY}(R) \cong K_4 \square K_4$ and so H_1 (see Figure 1.3) is a subgraph of $\mathbb{CAY}(\mathbb{F}_4 \times \mathbb{F}_4)$ and $H_2 = (K_4 \square K_2) \cup (K_4 \square K_2)$ is a subgraph of H_1. By Lemma 1.5, $\gamma(H_2) = 2$ which gives $\gamma(H_1) \geq 1$ and hence $\gamma(\mathbb{CAY}(\mathbb{F}_4 \times \mathbb{F}_4)) > 1$, a contradiction. Hence $R \not\cong \mathbb{F}_4 \times \mathbb{F}_4$. Hence R is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{F}_4$.

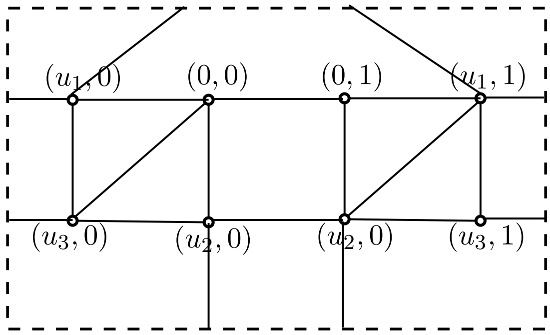

Figure 1.4. Embedding of $\mathbb{CAY}(\mathbb{Z}_2 \times \mathbb{F}_4) \cong \mathbb{CAY}(\mathbb{F}_4 \times \mathbb{Z}_2)$.

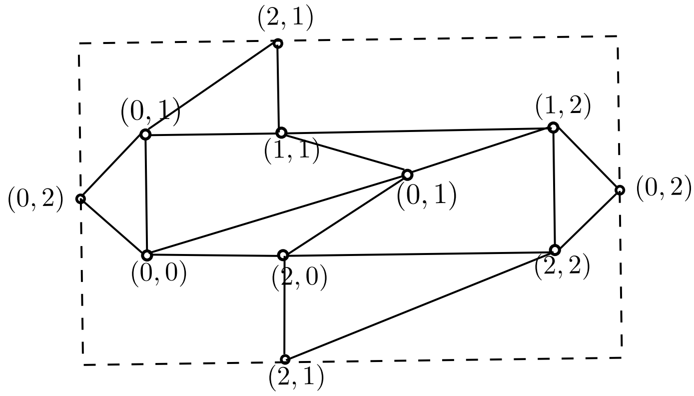

Figure 1.5. Embedding of $\mathbb{CAY}(\mathbb{Z}_3 \times \mathbb{Z}_3)$.

Converse follows from Figures 1.4 and 1.5. ■

Lemma 2.2. *Let* $R = \mathbb{Z}_4 \times \mathbb{Z}_4$. *Then* $\gamma(\mathbb{CAY}(\mathbb{Z}_4 \times \mathbb{Z}_4)) > 1$.

Proof. Clearly $(K_4 \square K_2) \cup (K_4 \square K_2)$ is a subgraph of $\mathbb{CAY}(\mathbb{Z}_4 \times \mathbb{Z}_4)$. By Lemma 1.5, $\gamma((K_4 \square K_2) \cup (K_4 \square K_2)) = 2$ and hence $\gamma(\mathbb{CAY}(\mathbb{Z}_4 \times \mathbb{Z}_4)) > 1$. ■

Theorem 2.3. *Let* $R = R_1 \times \cdots \times R_n$ *be a finite commutative ring with identity, where each* $(R_i, \mathfrak{m}_i)$ *is a local ring with* $\mathfrak{m}_i \neq \{0\}$ *and* $n \geq 2$. *Then* $\gamma(\mathbb{CAY}(R)) \geq 2$.

Proof. Since $\mathfrak{m}_i \neq \{0\}$ for all i, $|\mathfrak{m}_i| \geq 2$, $|R_i^\times| \geq 2$ for all i. This implies that $\mathbb{CAY}(\mathbb{Z}_4 \times \mathbb{Z}_4)$ is a subgraph of $\mathbb{CAY}(R)$ and by Lemma 2.2 $\gamma(\mathbb{CAY}(R)) \geq 2$. ■

Theorem 2.4. *Let $R = R_1 \times \cdots \times R_n \times F_1 \times \cdots \times F_m$ be a finite commutative ring with identity, where each $(R_i, \mathfrak{m}_i)$ is a local ring and F_j is a field, $m, n \geq 1$ and $m + n \geq 2$. Then $\gamma(\mathbb{CAY}(R)) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.*

Proof. Assume that $\gamma(\mathbb{CAY}(R)) = 1$. Then by Theorem 2.3, $n = 1$ and so $|R_1| \geq 4$. Suppose $m \geq 2$. By assumption that R_1 has at least four distinct elements, say 0, 1, r_1, r_2. Let $\Omega = \{x_1, x_2, \ldots, x_8\}$ where $x_1 = (0, 0, 0, \ldots, 0), x_2 = (1, 0, 0, \ldots, 0), x_3 = (r_1, 0, \ldots, 0), x_4 = (r_2, 0, \ldots, 0), x_5 = (0, 1, 0, 0, \ldots, 0), x_6 = (1, 1, 0, 0, \ldots, 0), x_7 = (r_1, 1, 0, \ldots, 0), x_8 = (r_2, 1, 0, \ldots, 0)$. Then the graph induced by Ω in $\mathbb{CAY}(R)$ contains K_8 as a subgraph and so by Lemma 1.1, $\gamma(\mathbb{CAY}(R)) > 1$, a contradiction. Hence $m = 1$ and so $R = R_1 \times F_1$.

If $|\mathfrak{m}_1| \geq 3$, then $|R_1| \geq 8$. Let $a_1, \ldots, a_8$ be distinct elements of R_1. Then $(a_1, 0), \ldots, (a_8, 0)$ induce a copy of K_9 and by Lemma 1.1, $\gamma(\mathbb{CAY}(R)) > 1$, a contradiction. Hence $|\mathfrak{m}_1| = 2$ and so $R_1 \cong \mathbb{Z}_4$, or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$.

Now $R = \mathbb{Z}_4 \times F_1$, or $\frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle} \times F_1$. Suppose $|F_1| \geq 4$. Then $\mathfrak{M}_1 = \langle 2 \rangle \times F_1$ and $\mathfrak{M}_2 = \mathbb{Z}_4 \times \langle 0 \rangle$ are maximal ideals in R and by Theorem 1.6, $\omega(\mathbb{CAY}(R)) = |\mathfrak{M}_1| \geq 8$ which gives K_8 is subgraph of $\mathbb{CAY}(R)$. By Lemma 1.1, $\gamma(K_8) > 1$ and so $\gamma(\mathbb{CAY}(R)) > 1$, a contradiction. Therefore $|F_1| \leq 3$ and so $F_1 \cong \mathbb{Z}_2$ or $\mathbb{Z}_3$. Hence R is isomorphic to one of the following rings:

$$\mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle},\ \mathbb{Z}_3 \times \mathbb{Z}_4,\ \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}.$$

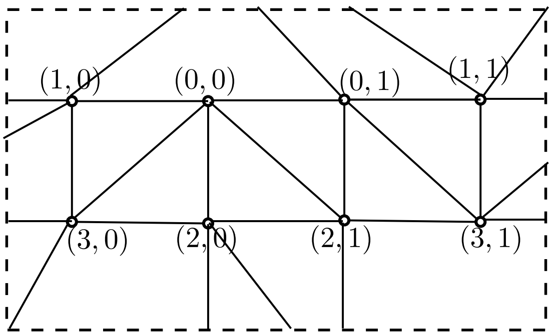

Figure 1.6. Embedding of $\mathbb{CAY}(\mathbb{Z}_2 \times \mathbb{Z}_4)$.

Suppose $R \cong \mathbb{Z}_3 \times \mathbb{Z}_4$ or $\mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$. Then $\mathbb{CAY}(R) \cong K_4 \Box K_3$ and so $(K_4 \cup K_4) + K_1$ is a minor subgraph of $\mathbb{CAY}(\mathbb{F}_4 \times \mathbb{Z}_3)$ and by Lemma 1.4, $\gamma((K_4 \cup K_4) + K_1) \geq 2$, $\gamma(\mathbb{CAY}(R)) > 1$, a contradiction. R is isomorphic to one of the following rings:

$$\mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}.$$

Converse follows from Figure 1.6 ■

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