

## ON CENTRALIZER OF SEMIPRIME INVERSE SEMIRING

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### Abstract

Let  $S$  be 2-torsion free semiprime inverse semiring satisfying  $A_2$  condition of Bandlet and Petrich [1]. We investigate, when an additive mapping  $T$  on  $S$  becomes centralizer.

**Keywords:** inverse semiring, semiprime inverse semiring, commutators, left(right) centralizer.

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### 1. INTRODUCTION AND PRELIMINARIES

Throughout this paper,  $S$  we will represent inverse semiring which satisfies  $A_2$  condition of Bandlet and Petrich [1].  $S$  is prime if  $aSb = (0)$  implies either  $a = 0$  or  $b = 0$  and  $S$  is semiprime if  $aSa = (0)$  implies  $a = 0$ .  $S$  is  $n$ -torsion free if  $nx = 0$ ,  $x \in S$  implies  $x = 0$ . Following Zalar [12], we canonically define left(right) centralizer of  $S$  as an additive mapping  $T : S \rightarrow S$  such that  $T(xy) = T(x)y$  ( $xT(y)$ ),  $\forall x, y \in S$  and  $T$  is called centralizer if it is both right and left centralizer.

Bresar and Zalar [2] have proved that an additive mapping  $T$  on 2-torsion free prime ring  $R$  which satisfies weaker condition  $T(x^2) = T(x)x$  is a left centralizer. Later, Zalar [12] generalized this result for semiprime rings. Motivated by the work of Zalar [12], Vukman [10] proved that an additive mapping on 2-torsion free semiprime ring satisfying  $T(xyz) = xT(y)x$  is a centralizer. In this paper, our objective is to explore the result of Vukman [10] in the setting of inverse semirings as follows: Let  $S$  be 2-torsion free semiprime inverse semiring and let

$T : S \rightarrow S$  be additive mapping such that  $T(xy) + xT(y) = 0$  holds  $\forall x, y \in S$  then  $T$  is a centralizer.

To prove this result we will first generalize Proposition 1.4 of [12] in the framework of inverse semirings.

By semiring we mean a nonempty set  $S$  with two binary operations '+' and '.' such that  $(S, +)$  and  $(S, \cdot)$  are semigroups where + is commutative with absorbing zero 0, i.e.,  $a + 0 = 0 + a = a$ ,  $a \cdot 0 = 0 \cdot a = a \forall a \in S$  and  $a \cdot (b + c) = a \cdot b + a \cdot c$ ,  $(b + c) \cdot a = b \cdot a + c \cdot a$  holds  $\forall a, b, c \in S$ . Introduced by Karvellas [6], a semiring  $S$  is an inverse semiring if for every  $a \in S$  there exist a unique element  $\acute{a} \in S$  such that  $a + \acute{a} + a = a$  and  $\acute{a} + a + \acute{a} = \acute{a}$ , where  $\acute{a}$  is called pseudo inverse of  $a$ . Karvellas [6] proved that for all  $a, b \in S$ ,  $(a \cdot b)^\acute{a} = \acute{a} \cdot b = a \cdot \acute{b}$  and  $\acute{a} \acute{b} = ab$ .

In this paper, inverse semirings satisfying the condition that for all  $a \in S$ ,  $a + \acute{a}$  is in center  $Z(S)$  of  $S$  are considered (see [4] for more details). Commutative inverse semirings and distributive lattices are natural examples of inverse semirings satisfying  $A_2$ . In a distributive lattice pseudo inverse of every element is itself. Also if  $R$  is commutative ring and  $I(R)$  is semiring of all two sided ideals of  $R$  with respect to ordinary addition and product of ideals and  $T$  is subsemiring of  $I(R)$  then set  $S_1 = \{(a, I) : a \in R, I \in T\}$ . Define on  $S_1$  addition  $\oplus$  and multiplication  $\odot$  by  $(a, I) \oplus (b, J) = (a + b, I + J)$  and  $(a, I) \odot (b, J) = (ab, IJ)$ . It is easy to see  $S_1$  is an inverse semiring with  $A_2$  condition where  $(a, I)^\acute{a} = (\acute{a}, I)$ .

By [4], commutator  $[.,.]$  in inverse semirings defines as  $[x, y] = xy + \acute{y}x$ . We will make use of commutator identities  $[x, y + z] = [x, y] + [x, z]$ ,  $[xy, z] = [x, z]y + x[y, z]$  and  $[x, yz] = [x, y]z + y[x, z]$  (see [4] for their proofs).

The following Lemmas are useful in establishing main result.

**Lemma 1.1.** For  $a, b \in S$ ,  $a + b = 0$  implies  $a = \acute{b}$ .

**Proof.** Let  $a + b = 0$  which implies  $a + b + \acute{a} + \acute{b} = 0$  or  $a + b + \acute{a} + \acute{b} + a = a$  or  $a + b + \acute{b} = a$  and by hypothesis, we get  $a = \acute{b}$ .

However, converse of Lemma 1.1. is not true for instance, in distributive lattice  $D$ , for  $a \in D$  we have  $a = \acute{a}$  but  $a + a = a$ .

**Lemma 1.2.** If  $x, y, z \in S$  then following identities are valid:

- (1)  $[xy, x] = x[y, x]$ ,  $[x, yx] = [x, y]x$ ,  $[x, xy] = x[x, y]$ ,  $[yx, x] = [y, x]x$
- (2)  $y[x, z] = [x, yz] + [x, y]z$ ,  $[x, y]z = \acute{y}[x, z] + [x, yz]$
- (3)  $x[y, z] = [xy, z] + [x, z]\acute{y}$ ,  $[x, z]y = [xy, z] + \acute{x}[y, z]$ .

**Proof.** (1)  $[xy, x] = xyx + \acute{x}xy = x(yx + \acute{x}y) = x[y, x]$ .  
(2)  $y[x, z] = (y + \acute{y} + y)(xz + \acute{z}x) = (y + \acute{y})xz + (y + \acute{y})\acute{z}x + yxz + y\acute{z}x = x(y + \acute{y})z + (y + \acute{y})\acute{z}x + yxz + y\acute{z}x = xyz + x\acute{y}z + y\acute{z}x + yxz = xyz + y\acute{z}x + x\acute{y}z + yxz = [x, yz] + [x, y]z$ .

Proof of the other identities can be obtained using similar techniques.

In the following, we extend Lemma 1.1 of Zalar [12] in a canonical fashion.

**Lemma 1.3.** *Let  $S$  be a semiprime inverse semiring such that for  $a, b \in S$ ,  $axb = 0, \forall x \in S$  then  $ab = ba = 0$ .*

**Definition 1.4.** A mapping  $f : S \times S \rightarrow S$  is biadditive if  $f(x_1 + x_2, y) = f(x_1, y) + f(x_2, y)$  and  $f(x, y_1 + y_2) = f(x, y_1) + f(x, y_2)$ ,  $\forall x, y, x_1, x_2, y_1, y_2 \in S$ .

**Example.** Define mappings  $f, g : S_1 \times S_1 \rightarrow S_1$  by  $f((a, I), (b, J)) = (ab, IJ)$  and  $g((a, I), (b, J)) = ([a, b], IJ)$ . Then  $f$  and  $g$  are biadditive.

Also, if  $(D, \wedge, \vee)$  is a distributive lattice then  $h : D \times D \rightarrow D$  defined by  $h(a, b) = a, \forall a, b \in D$  is a biadditive mapping.

**Lemma 1.5.** *Let  $S$  be semiprime inverse semiring and  $f, g : S \times S \rightarrow S$  are biadditive mappings such that  $f(x, y)wg(x, y) = 0, \forall x, y, w \in S$ , then  $f(x, y)wg(s, t) = 0, \forall x, y, s, t, w \in S$ .*

**Proof.** Replace  $x$  with  $x + s$  in  $f(x, y)wg(x, y) = 0$ , we get  $f(s, y)wg(x, y) + f(x, y)wg(s, y) = 0$ . By Lemma 1.1, we have  $f(x, y)wg(s, y) = f(s, y)wg(x, y)$ . This implies

$$(f(x, y)wg(s, y))z(f(x, y)wg(s, y)) = (f(s, y)wg(x, y))z(f(x, y)wg(s, y)) = 0$$

and semiprimeness of  $S$  implies that  $f(x, y)wg(s, y) = 0$ . Now replacing  $y$  with  $y + t$  in last equation and using similar approach we get the required result.

**Lemma 1.6.** *Let  $S$  be a semiprime inverse semiring and  $a \in S$  some fixed element. If  $a[x, y] = 0$  for all  $x, y \in S$ , then there exists an ideal  $I$  of  $S$  such that  $a \in I \subset Z(S)$  holds.*

**Proof.** By Lemma 1.2, we have  $[z, a]x[z, a] = zax[z, a] + \acute{a}zx[z, a] = za([z, xa] + [z, x]\acute{a}) + \acute{a}([z, zxa] + [z, zx]\acute{a}) = za[z, xa] + za[z, x]\acute{a} + \acute{a}[z, zxa] + \acute{a}[z, zx]a = 0$ .

Using semiprimeness of  $S$  and then Lemma 1.1, we get  $a \in Z(S)$ . By Lemma 1.2, we have  $zaw[x, y] = za([x, wy] + [x, w]\acute{y}) = 0, \forall x, y, z, w \in S$ . By similar argument, we can show that  $zaw \in Z(S)$  and hence  $SaS \subset Z(S)$ . Now it is easy to see that ideal generated by  $a$  is central.

**Lemma 1.7.** *Let  $S$  be semiprime inverse semiring and  $a, b, c \in S$  such that*

$$(1) \quad axb + bxc = 0$$

*holds for all  $x \in S$  then  $(a + c)xb = 0$  for all  $x \in S$ .*

**Proof.** Replace  $x$  with  $xy$  in (1), we get

$$(2) \quad axbyb + bxybc = 0, \quad x, y \in S.$$

Post multiplying (1) by  $yb$  gives

$$(3) \quad axbyb + bxcyb = 0, \quad x, y \in S.$$

Applying Lemma 1.1 on (2) and using it in (3), we have

$$(4) \quad bx(\acute{b}yc + cyb) = 0, \quad x, y \in S.$$

Replace  $x$  with  $ycx$  in (4), we get

$$(5) \quad bycx(\acute{b}yc + cyb) = 0, \quad x, y \in S.$$

Pre multiplying (4) by  $cy$  gives

$$(6) \quad cybx(\acute{b}yc + cyb) = 0, \quad x, y \in S.$$

Adding pseudo inverse of (5) and (6) we get

$$(\acute{b}yc + cyb)x(\acute{b}yc + cyb) = 0, \quad x, y \in S.$$

Using semiprimeness of  $S$  and Lemma 1.1, we get  $\acute{b}yc = cyb, y \in S$ . By using last relation in (1) we get the required result.

## 2. MAIN RESULTS

**Theorem 2.1.** *Let  $S$  be a 2-torsion free semiprime inverse semiring and  $T : S \rightarrow S$  be an additive mapping which satisfies  $T(x^2) + T(x)\acute{x} = 0, \forall x \in S$ . Then  $T$  is a left centralizer.*

**Proof.** Take,

$$(7) \quad T(x^2) + T(x)\acute{x} = 0, \quad x \in S.$$

Linearization of (7) gives

$$(8) \quad T(xy + yx) + T(x)\acute{y} + T(y)\acute{x} = 0, \quad x, y \in S.$$

Replace  $y$  with  $xy + yx$  in (8), we get

$$(9) \quad T(x^2y + yx^2) + 2T(xyx) + T(xy)\acute{x} + T(yx)\acute{x} + T(x)y\acute{x} + T(x)x\acute{y} = 0.$$

Using Lemma 1.1 in (8) and using it in (9), we have

$$(10) \quad T(x^2y + yx^2) + 2T(xyx) + T(x)y\acute{x} + T(y)\acute{x}^2 + T(x)y\acute{x} + T(x)x\acute{y} = 0.$$

Using Lemma 1.1 in (7) and using it in (10) we get

$$(11) \quad T(x^2y + yx^2) + 2T(xy x) + T(x)y\acute{x} + T(y)\acute{x}^2 + T(x)y\acute{x} + T(x^2)\acute{y} = 0.$$

Replace  $x$  with  $x^2$  in (8) we get

$$(12) \quad T(x^2y + yx^2) + T(x^2)\acute{y} + T(y)\acute{x}^2 = 0.$$

Using (12) in (11), we get

$$2T(xy x) + 2T(x)y\acute{x} = 0.$$

As  $S$  is 2-torsion free, so we have

$$(13) \quad T(xy x) + T(x)y\acute{x} = 0.$$

Linearization (by  $x = x + z$ ) of (13) gives

$$(14) \quad T(xyz + zyx) + T(x)y\acute{z} + T(z)y\acute{x} = 0.$$

Replace  $x$  with  $xy$ ,  $z$  with  $yx$  and  $y$  with  $z$  in (14), we get

$$(15) \quad T(xyzyx + yxzxy) + T(xy)zy\acute{x} + T(yx)zx\acute{y} = 0.$$

Replace  $y$  with  $zy$  in (13), we get

$$(16) \quad T(xyzyx) + T(x)zy\acute{y} = 0.$$

Replace  $x$  with  $y$  and  $y$  with  $xzx$  in (13), we get

$$(17) \quad T(yxzxxy) + T(y)xzx\acute{y} = 0.$$

By adding (16) and (17), we get

$$(18) \quad T(xyzyx + yxzxy) + T(x)zy\acute{y} + T(y)xzx\acute{y} = 0.$$

Using Lemma 1.1 in (15) and using the result in (18), we get

$$(19) \quad T(xy)zyx + T(yx)zxy + T(x)zy\acute{y} + T(y)xzx\acute{y} = 0.$$

Now if we define biadditive function  $f : S \times S \rightarrow S$  by  $f(x, y) = T(xy) + T(x)\acute{y}$ , then (19) can be written as

$$(20) \quad f(x, y)zyx + f(y, x)zxy = 0.$$

From (8) and Lemma 1.1, we have

$$(f(x, y))\acute{ } = f(y, x).$$

Thus (20) can be rewritten as

$$f(x, y)zyx + f(x, y)zx'y = 0, \text{ or}$$

$$f(x, y)z[x, y] = 0, x, y, z \in S.$$

Using Lemma 1.5 and then Lemma 1.3, we have  $f(x, y)[s, t] = 0$ ,  $x, y, s, t \in S$ . Now fix  $x, y$  then by Lemma 1.6, there exist ideal  $I \subset Z(S)$  such that  $f = f(x, y) \in I \subset Z(S)$ . This implies that  $bf, fb \in Z(S), \forall b \in S$ , thus we have

$$(21) \quad xfy = xyf = fxy = yfx \text{ and}$$

$$(22) \quad xf^2y = f^2xy = yf^2x = f^2yx.$$

Replace  $y$  with  $f^2y$  in (8), we get

$$2T(xf^2y + f^2yx) + 2T(x)f^2y' + 2T(f^2y)x' = 0.$$

Using (22), we get

$$(23) \quad 2T(yf^2x + f^2xy) + 2T(x)f^2y' + 2T(f^2y)x' = 0.$$

By Lemma 1.1, (8), (7) and (23), we have

$$2T(y)f^2x + 2T(f^2x)y + 2T(x)f^2y' + 2T(f^2y)x' = 0, \text{ or}$$

$$2T(y)f^2x + T(f^2x + f^2x)y + 2T(x)f^2y' + T(f^2y + f^2y)x' = 0, \text{ or}$$

$$2T(y)f^2x + T(f^2x + xf^2)y + 2T(x)f^2y' + T(f^2y + yf^2)x' = 0, \text{ or}$$

$$2T(y)f^2x + T(f^2)xy + T(x)f^2y + 2T(x)f^2y' + T(f^2)yx' + T(y)f^2x' = 0, \text{ or}$$

$$2T(y)f^2x + T(y)f^2x' + T(f^2)xy + T(x)f^2y + 2T(x)f^2y' + T(f^2)yx' = 0, \text{ or}$$

$$T(y)f^2x + T(f)fxy + T(x)f^2y' + T(f)fyx' = 0, \text{ or}$$

$$(24) \quad T(y)f^2x + T(x)f^2y' + T(f)fy(x' + x) = 0.$$

Now replace  $x$  with  $xy$  and  $y$  with  $f^2$  in (8) and then using (21) and (22), we get

$$2T(fxfy + fyyfx) + 2T(xy)f^2' + 2T(f^2)x'y = 0.$$

By Lemma 1.1, (8) and (7), we have

$$2T(fx)fy + 2T(fy)fx + 2T(xy)f^2' + 2T(f^2)x'y = 0, \text{ or}$$

$$T(fx + fx)fy + T(fy + fy)fx + 2T(xy)f^2' + 2T(f^2)x'y = 0$$

$$\begin{aligned}
& T(fx + xf)fy + T(fy + yf)fx + 2T(xy)f^2 + 2T(f^2)xy = 0 \\
& T(f)xfy + T(x)ffy + T(f)yfx + T(y)ffx + 2T(xy)f^2 + 2T(f^2)xy = 0 \\
& T(f)xfy + T(x)f^2y + T(f)fx y + T(y)f^2x + 2T(xy)f^2 + 2T(f)fx y = 0 \\
& T(f)fx y + 2T(f)fx y + T(f)fx y + T(x)f^2y + T(y)f^2x + 2T(xy)f^2 = 0 \\
& T(f)fy(x + x) + T(x)f^2y + T(y)f^2x + 2T(xy)f^2 = 0.
\end{aligned}$$

Using Lemma 1.1 in (24) and using the result in last equation, we get

$$\begin{aligned}
& 2T(x)f^2y + 2T(xy)f^2 = 0, \text{ or} \\
(25) \quad & T(x)f^2y + T(xy)f^2 = 0, \text{ or}
\end{aligned}$$

$(T(x)y + T(xy))f^2 = 0$  or  $f^3 = 0$  which implies

$$f^2Sf^2 = f^4 = (0) \Rightarrow f^2 = 0.$$

Thus  $fSf = f^2S = (0) \Rightarrow f = 0$ . Therefore  $T(xy) + T(x)y = 0$  and then Lemma 1.1 implies that  $T$  is a left centralizer.

**Theorem 2.2.** *Let  $S$  be a 2-torsion free semiprime inverse semiring and let  $T : S \rightarrow S$  be an additive mapping such that*

$$(26) \quad T(xyx) + xT(y)x = 0, \forall x, y \in S.$$

*Then  $T$  is a centralizer.*

**Proof.** First we show that

$$[[T(x), x], x] = 0.$$

Linearization of (26) gives

$$(27) \quad T(xyz + zyx) + xT(y)z + zT(y)x = 0, \forall x, y, z \in S.$$

Replace  $y$  with  $x$  and  $z$  with  $y$  in last equation, we get

$$(28) \quad T(x^2y + yx^2) + xT(x)y + yT(x)x = 0.$$

Replace  $z$  with  $x^3$  in (27), we get

$$(29) \quad T(xyx^3 + x^3yx) + xT(y)x^3 + x^3T(y)x = 0.$$

Replace  $y$  with  $xyx$  in (28), we get

$$(30) \quad T(x^3yx + xyx^3) + xyxT(x)\acute{x} + xT(x)xy\acute{x} = 0.$$

Replace  $y$  with  $x^2y + yx^2$  in (26), we have

$$(31) \quad T(x^3yx + xyx^3) + xT(x^2y + yx^2)\acute{x} = 0.$$

Using Lemma 1.1 in (30) and using the result in (31), we get

$$(32) \quad \begin{aligned} xyxT(x)x + xT(x)xyx + xT(x^2y + yx^2)\acute{x} &= 0, \text{ or} \\ x[T(x), x]yx + x\acute{y}[T(x), x]x &= 0. \end{aligned}$$

Using Lemma 1.7 in (32), we have

$$(33) \quad \begin{aligned} (x[T(x), x] + [T(x), x]\acute{x})yx &= 0, \text{ or} \\ [[T(x), x], x]yx &= 0. \end{aligned}$$

Replace  $y$  with  $y[T(x), x]$  in (33), we have

$$(34) \quad [[T(x), x], x]y[T(x), x]x = 0.$$

Post multiplication (33) with  $[T(x), x]$  gives

$$(35) \quad [[T(x), x], x]yx[T(x), x] = 0.$$

Adding pseudo inverse of (35) and (34), we have  $[[T(x), x], x]y[[T(x), x], x] = 0$  and then semiprimeness of  $S$  implies that

$$(36) \quad [[T(x), x], x] = 0, \forall x \in S \text{ or}$$

$$[T(x), x]x + \acute{x}[T(x), x] = 0 \text{ or}$$

$$[T(x), x]x + (x + \acute{x})[T(x), x] = x[T(x), x], \text{ or}$$

$$[T(x), x]x + [T(x), x](x + \acute{x}) = x[T(x), x], \text{ or}$$

$$(37) \quad [T(x), x]x = x[T(x), x], \forall x \in S.$$



Linearization of (36) gives

$$(38) \quad \begin{aligned} & [[T(x), x], y] + [[T(x), y], x] + [[T(y), y], x] + [[T(y), x], y] \\ & + [[T(x), y], y] + [[T(y), x], x] = 0. \end{aligned}$$

Replace  $x$  with  $\acute{x}$  in (38) and using again (38) and the fact that  $(T(x))' = T(\acute{x})$  we have

$$(39) \quad \begin{aligned} & 2[[T(x), x], y] + 2[[T(x), y], x] + [[T(y), y], x + \acute{x}] + [[T(y), x], y + \acute{y}] \\ & + [[T(x), y], y + \acute{y}] + 2[[T(y), x], x] = 0. \end{aligned}$$

Adding (38) in (39) and then using (38) again, we get

$$(40) \quad \begin{aligned} & 2[[T(x), x], y] + 2[[T(x), y], x] + 2[[T(y), x], x] = 0, \quad \forall x, y \in S. \\ & [[T(x), x], y] + [[T(x), y], x] + [[T(y), x], x] = 0, \quad \forall x, y \in S. \end{aligned}$$

Replace  $y$  with  $xyx$  in (40), we have

$$[[T(x), x], xyx] + [[T(x), xyx], x] + [[T(xyx), x], x] = 0, \quad \text{or}$$

Using Lemma 1.1 in (26) and using it in last equation, we get

$$[[T(x), x], xyx] + [[T(x), xyx], x] + [[xT(y)x], x] = 0.$$

Using Lemma 1.2, we have

$$\begin{aligned} & [[T(x), x], x]yx + x[[T(x), x], yx] + [[T(x), xy]x, x] + [xy[T(x), x], x] \\ & + [xT(y), x]x, x] = 0. \end{aligned}$$

Using (36) and Lemma 1.2, we get

$$x[[T(x), x], y]x + [xT(y), x]x, x] + [[T(x), xy], x]x + x[y[T(x), x], x] = 0.$$

Again using Lemma 1.2, and (36) we have

$$\begin{aligned} & x[[T(x), x], y]x + x[[T(y), x], x]x + [T(x), x][y, x]x \\ & + x[[T(x), y], x]x + x[y, x][T(x), x] = 0. \end{aligned}$$

Using (40) in last equation, we get

$$\begin{aligned} & [T(x), x][y, x]x + x[y, x][T(x), x] = 0 \\ & [T(x), x](yx + \acute{xy})x + x(yx + \acute{xy})[T(x), x] = 0 \end{aligned}$$

$$[T(x), x]yx^2 + [T(x), x]xyx + xyx[T(x), x] + x^2y[T(x), x] = 0.$$

Using (37), we get

$$[T(x), x]yx^2 + x^2y[T(x), x] + x[T(x), x]yx + xy[T(x), x]x = 0.$$

Using (32), we have

$$(41) \quad [T(x), x]yx^2 + x^2y[T(x), x] = 0.$$

Pre multiply (41) by  $x$  gives

$$(42) \quad x[T(x), x]yx^2 + x^3y[T(x), x] = 0.$$

Using Lemma 1.1 in (32) and using it in (42), we get

$$(43) \quad xy[T(x), x]x^2 + x^3y[T(x), x] = 0.$$

Pre multiply last equation by  $T(x)$ , we get

$$(44) \quad T(x)xy[T(x), x]x^2 + T(x)x^3y[T(x), x] = 0.$$

Replace  $y$  with  $T(x)y$  in (43), we get

$$(45) \quad xT(x)y[T(x), x]x^2 + x^3T(x)y[T(x), x] = 0.$$

Adding pseudo inverse of (45) and (44), we get

$$(46) \quad [T(x), x]y[T(x), x]x^2 + [T(x), x^3]y[T(x), x] = 0.$$

By applying Lemma 1.7 in (46), we get

$$([T(x), x]x^2 + [T(x), x^3])y[T(x), x] = 0$$

$$([T(x), x]x^2 + [T(x), x]x^2 + x[T(x), x^2])y[T(x), x] = 0$$

$$([T(x), x]x^2 + [T(x), x]x^2 + x[T(x), x]x + x^2[T(x), x])y[T(x), x] = 0.$$

Using (37) and the fact that  $S$  is inverse semiring, we have

$$x[T(x), x]xy[T(x), x] = 0.$$

And then semiprimeness of  $S$  implies that

$$(47) \quad x[T(x), x]x = 0, \forall x \in S.$$

Replace  $y$  with  $yx$  in (32) and using (47) we have

$$(48) \quad x[T(x), x]yx^2 = 0.$$

Replace  $y$  with  $yT(x)$  in (48), we get

$$(49) \quad x[T(x), x]yT(x)x^2 = 0.$$

Post multiplying (48) by  $T(x)$ , we get

$$(50) \quad x[T(x), x]yx^2T(x) = 0.$$

Adding pseudo inverse of (50) in (49), we get

$$\begin{aligned} x[T(x), x]y[T(x), x^2] &= 0 \\ x[T(x), x]y([T(x), x]x + x[T(x), x]) &= 0. \end{aligned}$$

Using (37) and the fact that  $S$  is 2-torsion free, we have

$$(51) \quad x[T(x), x] = 0 = [T(x), x]x, \quad x \in S.$$

As (40) obtained from (36), we can get following from (51)

$$(52) \quad [T(x), x]y + [T(x), y]x + [T(y), x]x = 0.$$

Post multiplying (52) by  $[T(x), x]$  and using (51), we get  $[T(x), x]y[T(x), x] = 0$ ,  $\forall y \in S$  which implies that

$$(53) \quad [T(x), x] = 0.$$

Replace  $y$  with  $xy + yx$  in (26), we have

$$(54) \quad T(x^2yx + xyx^2) + xT(xy + yx)\acute{x} = 0.$$

Replace  $z$  with  $x^2$  in (27), we get

$$(55) \quad T(xy x^2 + x^2 yx) + xT(y)x^2 + x^2 T(y)\acute{x} = 0.$$

Using Lemma 1.1 in (54) and using the result in (55) we get

$$x(T(xy + yx) + \acute{x}T(y) + T(y)\acute{x})x = 0.$$

Now if we define biadditive function  $g : S \times S \rightarrow S$  by  $g(x, y) = T(xy + yx) + T(y)\acute{x} + \acute{x}T(y)$  then last equation can be written as

$$(56) \quad xg(x, y)x = 0.$$

As (40) obtained from (36), we can obtain following from (56)

$$(57) \quad xg(x, y)z + xg(z, y)x + zg(x, y)x = 0, \quad \forall x, y, z \in S.$$

Post multiplication (57) by  $g(x, y)x$  and using (56) we get

$$(58) \quad xg(x, y)zg(x, y)x = 0.$$

Linearization of (53) gives

$$(59) \quad [T(x), y] + [T(y), x] = 0.$$

Replace  $y$  with  $xy + yx$  in above equation and using (53) we get

$$[T(xy + yx), x] + x[T(x), y] + [T(x), y]x = 0.$$

Using Lemma 1.1 in (59) and using the result in last equation, we get

$$\acute{x}[T(y), x] + [T(y), x]\acute{x} + [T(xy + yx), x] = 0.$$

Using Lemma 1.2 in last equation, we get

$$[\acute{x}T(y), x] + [T(y)\acute{x}, x] + [T(xy + yx), x] = 0, \quad \text{or}$$

$$[\acute{x}T(y) + T(y)\acute{x} + T(xy + yx), x] = 0, \quad \text{or}$$

$$(60) \quad [g(x, y), x] = 0.$$

which gives

$$(61) \quad g(x, y)x = xg(x, y), \quad x, y \in S.$$

By (58) and (61),  $g(x, y)xzg(x, y)x = 0$  this and (61) implies

$$(62) \quad xg(x, y) = 0 = g(x, y)x.$$

Linearization of (62) gives  $g(x, y)z + g(z, y)x = 0$ .

Post multiplying last equation by  $g(x, y)$  and using (62), we get  $g(x, y)zg(x, y) = 0$  and this implies  $g(x, y) = 0, x, y \in S$ . Put  $x = y$ , we get

$$(63) \quad 2T(x^2) + xT(x) + T(x)x = 0.$$

From (53) we can get  $T(x)x = xT(x)$ , using this and the fact that  $S$  is 2-torsion free, in (63), we get

$$T(x^2) + xT(x) = 0 \text{ and } T(x^2) + T(x)x = 0.$$

And therefore by Theorem 2.1, it follows that  $T$  is right and left centralizer. This completes the proof.

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