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RELATION BETWEEN (FUZZY) GÖDEL IDEALS AND (FUZZY) BOOLEAN IDEALS IN BL-ALGEBRAS

Akbar Paad

Department of Mathematics
University of Bojnord, Bojnord, Iran

e-mail: akbar.paad@gmail.com

Abstract

In this paper, we study relationships between among (fuzzy) Boolean ideals, (fuzzy) Gödel ideals, (fuzzy) implicative filters and (fuzzy) Boolean filters in BL-algebras. In [9], there is an example which shows that a Gödel ideal may not be a Boolean ideal, we show this example is not true and in the following we prove that the notions of (fuzzy) Gödel ideals and (fuzzy) Boolean ideals in BL-algebras coincide.

Keywords: BL-algebra, (fuzzy) filter, (fuzzy) Boolean ideal, (fuzzy) Gödel ideal.

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1. Introduction

BL-algebras are the algebraic structure for Hájek basic logic [10] in order to investigate many valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general"many-valued logic with truth values in [0,1] and BL-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0, 1]. In 1958, Chang [2] introduced the concept of an MV-algebra which is one of the most classes of BL-algebras. The notion of (fuzzy) ideal has been introduced in many algebraic structures such as lattices, rings, MV-algebras. Ideal theory is very effective tool for studying various algebraic and logical systems. In the theory of MV-algebras, as various algebraic structures, the notion of ideal is at the center, while in BL-algebras, the focus has been on deductive systems also filters. The study of BL-algebras has experienced a tremendous growth over resent years and the main focus has been on filters. In the meantime, several authors have claimed in recent works that the notion of ideals is missing in BL-algebras. Zhang *et al.* [11] studied the notion of fuzzy ideals in BL-algebras and in 2013, Lele [6] introduced the notions of Boolean ideal and analyzed the relationship between ideals and filters by using the set of complement elements. Meng *et al.* [9] proved that every fuzzy Boolean ideal is a fuzzy Gödel ideal, but the converse is not true.

Now, in this paper, we study relationship among Boolean ideals, Gödel ideals, implicative filters and Boolean filters in BL-algebras and we prove every Gödel ideal is a Boolean ideal. Also, we show example [9] is not true and every fuzzy Gödel ideal is a fuzzy Boolean ideal in BL-algebras. Finally, we study relationship between fuzzy prime ideals and fuzzy prime filters in BL-algebras.

2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, refer to the references.

Definition [10]**.** A BL-algebra is an algebra $(L, \vee, \wedge, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

$(BL1)$ $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,
$(BL2)$ $(L, \odot, 1)$ is a commutative monoid,
$(BL3)$ $z \leq x \to y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,
$(BL4)$ $x \wedge y = x \odot (x \to y)$,
$(BL5)$ $(x \to y) \vee (y \to x) = 1$.

We denote $x^n = \overbrace{x \odot \cdots \odot x}^{n-times}$, if $n \in N$, when N is the set of natural numbers and $x^0 = 1$. Also, we denote $(\overbrace{x \to (\ldots (x \to (x}^{n-times} \to y))) \ldots)$ by $x^n \to y$, for all $x, y \in L$.

A BL-algebra L is called a Gödel algebra, if $x^2 = x \odot x = x$, for all $x \in L$ and BL-algebra L is called an MV-algebra, if $(x^-)^- = x$, for all $x \in L$, where $x^- = x \to 0$. A BL-algebra L is called a Boolean algebra, if $x \vee x^- = 1$, for all $x \in L$. Moreover, BL-algebra L is a Boolean algebra if and only if L is a Gödel algebra and MV-algebra.

Proposition 1 [3, 4, 10]**.** *In any BL-algebra the following hold:*

$(BL6)$ $x \leq y$ *if and only if* $x \to y = 1$,
$(BL7)$ $x^{n+1} \leq x^n$, $\forall n \in N$,
$(BL8)$ $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$,
$(BL9)$ $x \to (y \to z) = x \odot y \to z$,
$(BL10)$ $1 \to x = x$, $x \to x = 1$ *and* $x \to 1 = 1$,
$(BL11)$ $x^{---} = x^{-}$,
$(BL12)$ $(x \wedge y)^{-} = x^{-} \vee y^{-}$ *and* $(x \vee y)^{-} = x^{-} \wedge y^{-}$,
for all $x, y, z \in L$.

The following theorems and definitions are from [1, 3, 5, 6, 7, 10] and we refer the reader to them, for more details.

Definition. Let L be a BL-algebra and F be a nonempty subset of L. Then

(i) F is called a *filter* of L, if $x \odot y \in F$, for any $x, y \in F$ and if $x \in F$ and $x \leq y$ then $y \in F$, for all $x, y \in L$.

(ii) F is called an *implicative filter* of L, if $1 \in F$ and

$$x \to (y \to z) \in F \text{ and } x \to y \in F \text{ imply } x \to z \in F, \text{ for all } x, y, z \in L.$$

(iii) F is called a *fantastic filter*, if $1 \in F$ and

$z \to (y \to x) \in F$ and $z \in F$ imply $(((x \to y) \to y) \to x) \in F$, for all $x, y, z \in L$.

Definition. Let L be a BL-algebra and I be a nonempty subset of L. Then

(i) I is called a *ideal* of L, if $x \oslash y = x^{-} \to y \in I$, for any $x, y \in I$ and if $y \in I$ and $x \leq y$ then $x \in I$, for all $x, y \in L$.

(ii) A proper ideal I of L is called *prime ideal* of L, if $x \wedge y \in I$ implies $x \in I$ or $y \in I$, for all $x, y \in L$.

(iii) An ideal I of L is called a *Gödel ideal*, if $(x^{-} \to (x^{-})^2)^{-} \in I$, for all $x \in L$.

(iv) An ideal I of L is called a *Boolean ideal*, if $x \wedge x^{-} \in I$, for all $x \in L$.

Definition. Let L be a BL-algebra and X any subset of L. Then

(i) The set of double complement elements is denoted by $D(X)$ and is defined by
$$D(X) = \{x \in L \mid x^{--} \in X\}.$$

(ii) The set of complement elements (with respect to X) is denoted by $N(X)$ and is defined by
$$N(X) = \{x \in L \mid x^{-} \in X\}.$$

(iii) The MV-center of L, denoted by $MV(L)$, is defined

$$MV(L) = \{x \in L \mid x^{--} = x\} = \{x^{-} \mid x \in L\}.$$

Hence, a BL-algebra L is an MV-algebra if and only if $L = MV(L)$.

Theorem 2. *Let F be a filter of BL-algebra L. Then the binary relation $\equiv_F$ on L which is defined by*

$$x \equiv_F y \quad \textit{if and only if } x \to y \in F \quad \textit{and } y \to x \in F$$

is a congruence relation on L. Define $\cdot$, $\rightharpoonup$, $\sqcup$, $\sqcap$ on $\frac{L}{F}$, the set of all congruence classes of L, as follows:

$$[x] \cdot [y] = [x \odot y],\ [x] \rightharpoonup [y] = [x \to y],\ [x] \sqcup [y] = [x \vee y],\ [x] \sqcap [y] = [x \wedge y].$$

Then $(\frac{L}{F}, \cdot, \rightharpoonup, \sqcup, \sqcap, [0], [1])$ is a BL-algebra which is called quotient BL-algebra with respect to F.

Theorem 3. *Let I be an ideal of BL-algebra L. Then the binary relation $\equiv_I$ on L which is defined by*

$$x \equiv_I y \quad \textit{if and only if } x^- \odot y \in I \quad \textit{and } y^- \odot x \in I$$

is a congruence relation on L. Define $\cdot$, $\rightharpoonup$, $\sqcup$, $\sqcap$ on $\frac{L}{I}$, the set of all congruence classes of L, as follows:

$$[x] \cdot [y] = [x \odot y],\ [x] \rightharpoonup [y] = [x \to y],\ [x] \sqcup [y] = [x \vee y],\ [x] \sqcap [y] = [x \wedge y].$$

Then $(\frac{L}{I}, \cdot, \rightharpoonup, \sqcup, \sqcap, [0], [1])$ is a BL-algebra which is called quotient BL-algebra with respect to I. In addition, it is clear $[x]^{--} = [x]$, for all $x \in L$. Consequently, the quotient BL-algebra via any ideal is always an MV-algebra.

Theorem 4. *Let F be a filter and I be an ideal of BL-algebra L. Then*

(i) *The set of complement elements $N(I)$ is a filter.*
(ii) *The set of complement elements $N(F)$ is the ideal generated by $\overline{F}$.*
(iii) *I is a Boolean ideal if and only if $N(I)$ is a Boolean filter.*
(iv) *I is a Boolean ideal if and only if $\frac{L}{I}$ is a Boolean algebra.*
(v) *If F is a Boolean filter, then $N(F)$ is a Boolean ideal.*

Theorem 5. *Let F be a filter and I be an ideal of BL-algebra L. Then*

(i) *F is an implicative filter of L if and only if $\frac{L}{F}$ is a Gödel algebra.*
(ii) *F is a fantastic filter of L if and only if $\frac{L}{F}$ is an MV-algebra.*
(iii) *F is a fantastic filter of L if and only if $F = D(F)$.*
(iv) *The MV-center of the quotient BL-algebra $\frac{L}{N(I)}$ is isomorphic to the MV-algebra $\frac{L}{I}$.*

In the following, we give some fuzzy algebraic results on BL-algebras that come form references [6, 8, 11].

Definition. Let L be a BL-algebra and $\mu : L \to [0,1]$ be a fuzzy set on L. Then

(i) μ is called a *fuzzy filter* on L, if and only if $\mu(x) \leq \mu(1)$ and $\mu(x \to y) \wedge \mu(x) \leq \mu(y)$, for all $x, y \in L$.
It is easy to see that for any fuzzy filter μ, if $x \leq y$ then $\mu(x) \leq \mu(y)$.

(ii) μ is called a *fuzzy implicative filter* on L, if $\mu(x) \leq \mu(1)$ and

$$\mu(x \to (y \to z)) \wedge \mu(x \to y) \leq \mu(x \to z), \text{ for all } x, y, z \in L.$$

(iii) μ is called a *fuzzy fantastic filter* on L, if $\mu(x) \leq \mu(1)$ and

$$\mu(z \to (y \to x)) \wedge \mu(z) \leq \mu(((x \to y) \to y) \to x), \text{ for all } x, y, z \in L.$$

(iv) A fuzzy filter μ is called a *fuzzy Boolean filter* on L, if $\mu(x \vee x^-) = \mu(1)$, for all $x \in L$. A fuzzy filter μ is a fuzzy Boolean filter if and only if μ is a fuzzy implicative filter and fuzzy fantastic filter.

(v) A fuzzy filter μ is called a *fuzzy prime filter* on L, if $\mu(x \vee y) = \mu(x) \vee \mu(y)$, for all $x, y \in L$.

Definition. Let L be a BL-algebra and $\mu : L \to [0,1]$ be a fuzzy set on L. Then

(i) μ is called a *fuzzy ideal* on L, if and only if $\mu(x) \leq \mu(0)$ and $\mu((x^- \to y^-)^-) \wedge \mu(x) \leq \mu(y)$, for all $x, y \in L$. It is easy to see that for any fuzzy ideal μ, $\mu(x^{--}) = \mu(x)$ and if $x \leq y$ then $\mu(y) \leq \mu(x)$.

(ii) A fuzzy ideal μ is called a *fuzzy Gödel ideal* on L, if $\mu((x^- \to (x^-)^2)^-) = \mu(1)$, for all $x \in L$.

(iii) A fuzzy ideal μ is called a *fuzzy Boolean ideal* on L, if $\mu(x \wedge x^-) = \mu(0)$, for all $x \in L$.

(iv) A fuzzy ideal μ is called a *fuzzy prime ideal* on L, if $\mu(x \wedge y) = \mu(x) \vee \mu(y)$, for all $x, y \in L$.

Theorem 6. *Let L be a BL-algebra, μ be a fuzzy set on L and $\mu_t = \{x \in L \mid \mu(x) \geq t\}$, for each $t \in [0,1]$. Then*

(i) *μ is a(an, a) fuzzy (implicative, fantastic, Boolean) filter on L if and only if for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is a(an, a) (implicative, fantastic, Boolean) filter of L and μ is a fuzzy ideal on L if and only if for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is a ideal of L.*

(ii) *μ is a fuzzy Gödel ideal on L if and only if for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is a Gödel ideal of L.*

(iii) *If μ is a fuzzy Boolean ideal on L, then μ is a fuzzy Gödel ideal on L.*

Note. From now on, in this paper we let L be a BL-algebra, unless otherwise is stated.

3. Relation between Gödel ideals and Boolean ideals in BL-algebras

In this section we study relation between Gödel(Boolean) ideals and Boolean filters in BL-algebras.

Theorem 7. *Let L be an MV-algebra and I be a Gödel ideal of L. Then $N(I)$ is an implicative filter of L.*

Proof. Let I be a Gödel ideal of MV-algebra L. Then by Theorem 4(i), $N(I)$ is a filter of L. Now, we prove that $\frac{L}{N(I)}$ is a Gödel algebra. Since I is a Gödel ideal, then $(x^- \to (x^-)^2)^- \in I$, for all $x \in L$. Let $[x] \in \frac{L}{N(I)}$. Since L is an MV-algebra, then

$$(x \to x^2)^- = (x^{--} \to (x^{--})^2)^- = ((x^-)^- \to ((x^-)^-)^2)^-.$$

Now, since $x^- \in L$ and I is a Gödel ideal, then

$$((x^-)^- \to ((x^-)^-)^2)^- \in I.$$

And so $(x \to x^2)^- \in I$. Hence, $(x \to x^2) \in N(I)$ and since by $(BL7)$, $1 = (x^2 \to x) \in N(I)$, then $[x] = [x]^2$. Therefore, $\frac{L}{N(I)}$ is a Gödel algebra and by Theorem 5, $N(I)$ is an implicative filter of L. ■

Corollary 8. *Let I be a Gödel ideal of MV-algebra L. Then $N(I)$ is a Boolean filter of L and I is a Boolean ideal of L.*

Proof. By Theorem 7, $N(I)$ is an implicative filter of L and since L is an MV-algebra, then $\frac{L}{N(I)}$ is an MV-algebra, too. Now, by Theorem 5(ii), $N(I)$ is a fantastic filter of L and so $N(I)$ is a Boolean filter. Therefore, by Theorem 4(iii), I is Boolean ideal of L. ■

Theorem 9. *Let I be an ideal of L. Then*

(i) *$N(I)$ is a fantastic filter of L and $\frac{L}{N(I)}$ is an MV-algebra.*

(ii) *MV-algebra $\frac{L}{N(I)}$ is isomorphic to the MV-algebra $\frac{L}{I}$.*

Proof. (i) Let I be an ideal of L. Then

$$\begin{aligned} D(N(I)) &= \{x \in L \mid x^{--} \in N(I)\} \\ &= \{x \in L \mid x^{---} \in I\} \\ &= \{x \in L \mid x^- \in I\}, \text{ by } (BL11) \\ &= N(I). \end{aligned}$$

Now, by Theorem 4(i), $N(I)$ is a filter of L and by Theorem 5(iii), $N(I)$ is a fantastic filter of L. Hence, by Theorem 5(ii), $\frac{L}{N(I)}$ is an MV-algebra.

(ii) Since by (i), $\frac{L}{N(I)}$ is an MV-algebra, then $MV(\frac{L}{N(I)}) = \frac{L}{N(I)}$. Moreover, since by Theorem 5(iv), the MV-center of the quotient BL-algebra $\frac{L}{N(I)}$ is isomorphic to the MV-algebra $\frac{L}{I}$, then MV-algebra $\frac{L}{N(I)}$ is isomorphic to the MV-algebra $\frac{L}{I}$. ∎

Now, we replace condition MV-algebra by BL-algebra in Theorem 7 and we prove the following theorem.

Theorem 10. *Let I be an ideal of BL-algebra L. Then*

(i) *$N(I)$ is an implicative filter if and only if I is a Gödel ideal of L.*

(ii) *$N(I)$ is a(an) Boolean (implicative) filter if and only if I is a Boolean ideal of L.*

Proof. (i) Let $N(I)$ be an implicative filter of L and $x \in L$. Then by $(BL9)$,

$$(x^- \to (x^- \to (x^-)^2))^- = (x^- \odot x^- \to (x^-)^2)^- = 1^- = 0 \in I.$$

Hence, $x^- \to (x^- \to (x^-)^2) \in N(I)$ and since $x^- \to x^- = 1 \in N(I)$ and $N(I)$ is an implicative filter, then $(x^- \to (x^-)^2) \in N(I)$. Hence, $(x^- \to (x^-)^2)^- \in I$, for all $x \in L$. Therefore, I is a Gödel ideal of L. Conversely, let I be a Gödel ideal of L. Then $(x^- \to (x^-)^2)^- \in I$, for all $x \in L$ and so $(x^- \to (x^-)^2) \in N(I)$, for all $x \in L$. Hence, $[x^-] = [x^-]^2$, for all $x \in L$. Now, since by Theorem 9(i), $\frac{L}{N(I)}$ is an MV-algebra, then $MV(\frac{L}{N(I)}) = \frac{L}{N(I)}$ and so for any $[0] \neq [\alpha] \in \frac{L}{N(I)}$, there is a $[0] \neq [\beta] \in \frac{L}{N(I)}$ which $[\alpha] = [\beta]^-$. Hence,

$$[\alpha]^2 = ([\beta]^-)^2 = [\beta^-]^2 = [\beta^-] = [\alpha].$$

Moreover, if $[\alpha] = [0]$, then $[\alpha]^2 = [\alpha] = 0$. Therefore, $\frac{L}{N(I)}$ is a Gödel algebra and so by Theorem 5(i), $N(I)$ is an implicative filter of L.

(ii) Since by Theorem 9(i), $\frac{L}{N(I)}$ is an MV-algebra, then by theorem 5(ii), $N(I)$ is a fantastic filter of L. Now, since $N(I)$ is an implicative filter, then $N(I)$ is a Boolean filter of L and so by Theorem 4(iii), I is a Boolean ideal of L. Conversely, if I is a Boolean ideal of L, then by Theorem 4(iii), $N(I)$ is a Boolean filter and so $N(I)$ is a implicative filter of L. ∎

Corollary 11. *Let I be an ideal of BL-algebra L. Then I is a Gödel ideal if and only if I is a Boolean ideal.*

Proof. By Theorem 10, the proof is clear. ∎

Theorem 12. *Let I be an ideal of L. Then I is a Gödel ideal if and only if $\frac{L}{I}$ is a Gödel algebra.*

Proof. Let I be a Gödel ideal of L. Then by Corollary 11, I is a Boolean ideal of L. Hence, by Theorem 4(iv), $\frac{L}{I}$ is Boolean algebra and so $\frac{L}{I}$ is a Gödel algebra. Conversely, let $\frac{L}{I}$ be a Gödel algebra. Since $\frac{L}{I}$ is a MV-algebra, then $\frac{L}{I}$ is a Boolean algebra and so by Theorem 4(iv), I is a Boolean ideal of L. Hence, by Corollary 11, I is a Gödel ideal of L. ∎

4. Relation between fuzzy Gödel ideals and fuzzy Boolean ideals in BL-algebras

Zhang *et al.* [11] introduced the notion of fuzzy Boolean ideals in BL-algebras and Meng *et al.* [9] introduced the notion of fuzzy Gödel ideals in BL-algebra. Moreover, he proved each fuzzy Boolean ideal in a BL-algebra is a fuzzy Gödel ideal and by the following example showed the converse is not correct in general. In follow, we study relationship among fuzzy Boolean ideals, fuzzy Gödel ideals, fuzzy implicative filters and fuzzy Boolean filters.

Example 13 [9]**.** Let $L = \{0, a, b, c, d, 1\}$. Define $\odot$,$\rightarrow$, $\vee$ and $\wedge$ as follows:

Table 1. Product

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	d	c	0	d	a
b	0	c	b	c	0	b
c	0	0	c	0	0	c
d	0	d	0	0	d	d
1	0	a	b	c	d	1

Table 2. Implication

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	b	a	1
b	d	a	1	a	d	1
c	a	1	1	1	a	1
d	b	1	b	b	1	1
1	0	a	b	c	d	1

Table 3. Join

$\vee$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	1	a	a	1
b	b	1	b	b	1	1
c	c	a	b	c	a	1
d	d	a	1	a	d	1
1	1	1	1	1	1	1

Table 4. Meet

$\wedge$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	0	b
c	0	c	c	c	0	c
d	0	d	0	0	d	d
1	0	a	b	c	d	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Define fuzzy set μ on L by $\mu(a) = \mu(b) = \mu(c) = \mu(1) = 0.5$ and $\mu(0) = \mu(d) = 0.8$. One can easily check that μ is a fuzzy Gödel ideal on L, but it is not a fuzzy Boolean ideal, since $\mu(a \wedge a^-) = \mu(a \wedge c) = \mu(c) = 0.5 \neq 0.8 = \mu(0)$.

But this example is not true, since $\mu((c^- \to (c^-)^2)^-) = \mu((a \to a^2)^-) = \mu((a \to d)^-) = \mu(a^-) = \mu(c) = 0.5 \neq 0.8 = \mu(0)$. Therefore, μ is not a fuzzy Gödel ideal on L. In the following we prove that each fuzzy Gödel ideal in a BL-algebra is a fuzzy Boolean ideal.

Theorem 14 [11]**.** *Let μ be a fuzzy ideal on L. Then μ is a fuzzy Boolean ideal if and only if for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is a Boolean ideal of L.*

Theorem 15. *Let μ be a fuzzy Gödel ideal on L. Then μ is a fuzzy Boolean ideal on L.*

Proof. Let μ be a fuzzy Gödel ideal on L. Then by Theorem 6(ii), for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is a Gödel ideal of L. Now, by Corollary 11, for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is a Boolean ideal of L. Hence, by Theorem 14, μ is a fuzzy Boolean ideal on L. ■

Corollary 16. *Let μ be a fuzzy set on L. Then μ is a fuzzy Boolean ideal if and only if μ is a fuzzy Gödel ideal on L.*

Proof. By Theorem 6(iii) and Theorem 15 the proof is clear. ■

Definition [7]**.** Let μ be a fuzzy subset on L. The fuzzy subsets $N(\mu)$ and $D(\mu)$ from $L \to [0,1]$ are defined by $N(\mu)(x) = \mu(x^-)$ and $D(\mu)(x) = \mu(x^{--})$, for all $x \in L$, are called the set of complement and the set of double complement of the fuzzy subset μ.

Theorem 17 [7]**.** *Let μ be a fuzzy on L. Then*

(i) *If μ is a fuzzy filter, then $N(\mu)$ is fuzzy ideal on L.*

(ii) *If μ is a fuzzy ideal, then $N(\mu)$ is fuzzy filter on L.*

Theorem 18. *Let μ be a fuzzy ideal on L. Then μ is a fuzzy Boolean ideal on L if and only if $N(\mu)$ is a fuzzy Boolean filter on L.*

Proof. Let μ be a fuzzy Boolean ideal on L. Then $\mu(z \wedge z^-) = \mu(0)$, for all $z \in L$ and by Theorem 17(ii), $N(\mu)$ is fuzzy filter on L. Now, for all $x \in L$,

$$\begin{aligned} N(\mu)(x \vee x^-) &= \mu((x \vee x^-)^-) \\ &= \mu(((x^-) \wedge (x^-)^-), \text{ by } (BL12) \\ &= \mu(0) \\ &= N(\mu)(1). \end{aligned}$$

Therefore, $N(\mu)$ is a fuzzy Boolean filter on L.

Conversely, let $N(\mu)$ be a fuzzy Boolean filter on L. Then $N(\mu)(z \vee z^-) = N(\mu)(1)$, for all $z \in L$. Now, for all $x \in L$,

$$\begin{aligned}\mu(x \wedge x^-) &= \mu((x \wedge x^-)^{--})\\ &= N(\mu)((x \wedge x^-)^-)\\ &= N(\mu)((x^-) \vee (x^-)^-), \text{ by } (BL12)\\ &= N(\mu)(1)\\ &= \mu(0).\end{aligned}$$

Therefore, μ is a fuzzy Boolean ideal on L. ■

Theorem 19. *Let μ be a fuzzy set on L. Then μ is a fuzzy Boolean ideal if and only if $N(\mu)$ is a fuzzy implicative filter on L.*

Proof. Let μ be a fuzzy Boolean ideal. Then by Theorem 18, $N(\mu)$ is a fuzzy Boolean filter on L and so $N(\mu)$ is a fuzzy implicative filter on L. Conversely, Let $N(\mu)$ be a fuzzy implicative filter. Then by Theorem 6(ii), for each $t \in [0,1]$, $\emptyset \neq (N(\mu))_t$ is an implicative filter of L. Now, for each $t \in [0,1]$,

$$\begin{aligned}(N(\mu))_t &= \{x \in L \mid (N(\mu))(x) \geq t\}\\ &= \{x \in L \mid \mu(x^-) \geq t\}\\ &= \{x \in L \mid x^- \in \mu_t\}\\ &= N(\mu_t).\end{aligned}$$

Hence, for each $t \in [0,1]$, $\emptyset \neq N(\mu_t)$ is an implicative filter of L and so by Theorem 10(ii), for each $t \in [0,1]$, $\emptyset \neq \mu_t$ is Boolean ideal of L. Therefore, by Theorem 14, μ is a fuzzy Boolean ideal on L. ■

Theorem 20. *Let μ be a fuzzy ideal on L. Then $N(\mu)$ is a fuzzy fantastic filter on L.*

Proof. By Definition 4 and $(BL11)$, for all $x \in L$,

$$D(N(\mu))(x) = N(\mu)(x^{--}) = \mu(x^{---}) = \mu(x^-) = N(\mu)(x).$$

Hence, $D(N(\mu)) = N(\mu)$. Moreover, for each $t \in [0,1]$,

$$\begin{aligned}(D(N(\mu)))_t &= \{x \in L \mid D(N(\mu))(x) \geq t\}\\ &= \{x \in L \mid N(\mu)(x^{--}) \geq t\}\\ &= \{x \in L \mid x^{--} \in (N(\mu))_t\}\\ &= D((N(\mu))_t).\end{aligned}$$

Now, since $D(N(\mu)) = N(\mu)$, then for each $t \in [0,1]$, $(D(N(\mu)))_t = (N(\mu))_t$. Hence, for each $t \in [0,1]$, $D((N(\mu))_t) = (N(\mu))_t$ and so by Theorem 5(iii), $(N(\mu))_t$, for each $t \in [0,1]$, is a fantastic filter of L. Therefore, by Theorem 6(i), $N(\mu)$ is a fuzzy fantastic filter on L. ■

Corollary 21. *A fuzzy subset μ on L is a fuzzy Boolean ideal if and only if $N(\mu)$ is a fuzzy Boolean filter.*

Proof. By Theorem 19 and Theorem 20, the proof is clear. ■

Theorem 22. *Let μ be a fuzzy ideal on L. Then μ is a fuzzy prime ideal if and only if $N(\mu)$ is a fuzzy prime filter on L.*

Proof. Let μ be a fuzzy prime ideal on L. Then by Theorem 17(ii), $N(\mu)$ is a fuzzy filter on L. Now, for all $x, y \in L$,

$$\begin{aligned} N(\mu)(x \vee y) &= \mu((x \vee y)^-) \\ &= \mu(x^- \wedge y^-), \text{by } (BL12), \\ &= \mu(x^-) \vee \mu(y^-) \\ &= N(\mu)(x) \vee N(\mu)(y). \end{aligned}$$

Therefore, $N(\mu)$ is a fuzzy prime filter on L.
Conversely, let $N(\mu)$ be a fuzzy prime filter on L. Then for all $x, y \in L$,

$$\begin{aligned} \mu(x \wedge y) &= \mu((x \wedge y)^{--}) \\ &= \mu((x^- \vee y^-)^-), \text{by } (BL12), \\ &= N(\mu)(x^- \vee y^-) \\ &= N(\mu)(x^-) \vee N(\mu)(y^-) \\ &= \mu(x^{--}) \vee \mu(y^{--}) \\ &= \mu(x) \vee \mu(y). \end{aligned}$$

Therefore, μ is a fuzzy prime ideal on L. ■

Theorem 23. *Let μ be a fuzzy prime filter on L. Then $N(\mu)$ is a fuzzy prime ideal on L.*

Proof. Let μ be a fuzzy prime filter on L. Then by Theorem 17(i), $N(\mu)$ is a fuzzy ideal on L. Now, for all $x, y \in L$,

$$\begin{aligned} N(\mu)(x \wedge y) &= \mu((x \wedge y)^-) \\ &= \mu(x^- \vee y^-), \text{by } (BL12), \\ &= \mu(x^-) \vee \mu(y^-) \\ &= N(\mu)(x) \vee N(\mu)(y). \end{aligned}$$

Therefore, $N(\mu)$ is a fuzzy prime ideal on L. ■

The following example shows that the converse of Theorem 23, is not correct in general.

Example 24 [1]**.** Let $L = \{0, a, b, c, 1\}$. Define $\wedge$,$\vee$,$\odot$ and $\to$ on L as follows:

Table 5. Meet

$\wedge$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Table 6. Join

$\vee$	0	c	a	b	1
0	0	c	a	b	1
c	c	c	a	b	1
a	a	a	a	1	1
b	b	b	1	b	1
1	1	1	1	1	1

Table 7. Product

$\odot$	0	c	a	b	1
0	0	0	0	0	0
c	0	c	c	c	c
a	0	c	a	c	a
b	0	c	c	b	b
1	0	c	a	b	1

Table 8. Implication

$\to$	0	c	a	b	1
0	1	1	1	1	1
c	0	1	1	1	1
a	0	b	1	b	1
b	0	a	a	1	1
1	0	c	a	b	1

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a BL-algebra. Now, let the fuzzy set μ on L is defined by

$$\mu(0) = 0.5,\ \mu(a) = \mu(b) = \mu(a) = 0.6,\ \mu(1) = 0.8.$$

It is easy to check that μ is a fuzzy filter, but it is not a fuzzy prime filter. Because,

$$\mu(a \vee b) = \mu(1) = \mu(a) = 0.8 \neq 0.6 = \mu(a) \vee \mu(b).$$

Now, fuzzy set $N(\mu)$ which is equal to

$$N(\mu)(1) = N(\mu)(a) = N(\mu)(b) = N(\mu)(c) = \mu(0) = 0.5,\ N(\mu)(0) = \mu(1) = 0.8$$

is a fuzzy prime ideal.

Theorem 25. *Let μ be a fuzzy Boolean filter on L. Then $N(\mu)$ is a fuzzy Boolean ideal on L.*

Proof. Let μ be a fuzzy Boolean filter on L. Then by Theorem 6(i), for each $t \in [0, 1]$, $\emptyset \neq \mu_t$ is a Boolean filter of L. Moreover by Theorem 17(i), $N(\mu)$ is a fuzzy ideal on L. Now, since for each $t \in [0, 1]$, $(N(\mu))_t = N(\mu)_t$ and $\emptyset \neq \mu_t$ is a Boolean filter of L, then by Theorem 4(v), for each $t \in [0, 1]$, $(N(\mu))_t = N(\mu)_t$ is a Boolean ideal and so by Theorem 14, $N(\mu)$ is a fuzzy Boolean ideal on L. ■

The following example shows that the converse of Theorem 25, is not correct in general.

Example 26 [8]**.** Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\rightarrow$ are defined as the following tables:

Table 9. Product

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

Table 10. Implication

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Now, let the fuzzy set μ on L is defined by

$$\mu(0) = 0.4, \ \mu(a) = \mu(b) = 0.6, \ \mu(1) = 0.8.$$

It is easy to check that μ is a fuzzy filter, but it is not a fuzzy Boolean filter. Because,

$$\mu(a \vee a^-) = \mu(a \vee 0) = \mu(a) = 0.6 \neq 0.8 = \mu(1).$$

Now, fuzzy set $N(\mu)$ which is equal to

$$N(\mu)(1) = N(\mu)(a) = N(\mu)(b) = \mu(0) = 0.4, \ N(\mu)(0) = \mu(1) = 0.8$$

is a fuzzy ideal on L and since

$$N(\mu)(a \wedge a^-) = N(\mu)(b \wedge b^-) = N(\mu)(1 \wedge 1^-) = N(\mu)(0) = 0.8.$$

Therefore, $N(\mu)$ is a fuzzy Boolean ideal on L.

5. CONCLUSION

The results of this paper will be devoted to study relationship among (fuzzy) Boolean ideals, (fuzzy) Gödel ideals, (fuzzy) implicative filters and (fuzzy) Boolean filters on BL-algebras. Moreover, in this paper we proved the notions of (fuzzy) Gödel ideals and (fuzzy) Boolean ideals on BL-algebras are coincide.

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