

ON THE ASSOCIATED PRIME IDEALS OF LOCAL  
COHOMOLOGY MODULES DEFINED BY A PAIR  
OF IDEALS

MARYAM JAHANGIRI<sup>1</sup>

*Faculty of Mathematical Sciences and Computer*  
*Kharazmi University, Tehran, Iran*  
*and*  
*School of Mathematics*  
*Institute for Research in Fundamental Sciences (IPM)*  
*P.O. Box 19395-5746, Tehran, Iran*

**e-mail:** mjahangiri@ipm.ir  
jahangiri.maryam@gmail.com

ZOHREH HABIBI AND KHADIJEH AHMADI AMOLI

*Payame Noor University*  
*P.O. Box 19395-3697, Tehran, Iran*

**e-mail:** z.habibi@pnu.ac.ir  
khahmadi@pnu.ac.ir

**Abstract**

Let  $I$  and  $J$  be two ideals of a commutative Noetherian ring  $R$  and  $M$  be an  $R$ -module. For a non-negative integer  $n$  it is shown that, if the sets  $\text{Ass}_R(\text{Ext}_R^n(R/I, M))$  and  $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$  are finite for all  $i \leq n + 1$  and all  $j < n$ , then so is  $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^n(M)))$ . We also study the finiteness of  $\text{Ass}_R(\text{Ext}_R^i(R/I, H_{I,J}^n(M)))$  for  $i = 1, 2$ .

**Keywords:** local cohomology modules defined by a pair of ideals, spectral sequences, associated prime ideals.

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<sup>1</sup>Corresponding author.

## 1. INTRODUCTION

Let  $R$  be a commutative Noetherian ring,  $I$  and  $J$  be two ideals of  $R$  and  $M$  be an  $R$ -module. For all  $i \in \mathbb{N}_0$  the  $i$ -th local cohomology functor with respect to  $(I, J)$ , denoted by  $H_{I,J}^i(-)$ , defined by Takahashi *et al.* in [14] as the  $i$ -th right derived functor of the  $(I, J)$ -torsion functor  $\Gamma_{I,J}(-)$ , where

$$\Gamma_{I,J}(M) := \{x \in M : I^n x \subseteq Jx \text{ for } n \gg 1\}.$$

This notion coincides with the ordinary local cohomology functor  $H_I^i(-)$  when  $J = 0$ , see [5].

The main motivation for this generalization comes from the study of a dual of ordinary local cohomology modules  $H_I^i(M)$  ([12]). Basic facts and more information about local cohomology defined by a pair of ideals can be obtained from [7, 8] and [14].

Hartshorne in [9] proposed the following conjecture: “Let  $M$  be a finitely generated  $R$ -module and  $\mathfrak{a}$  be an ideal of  $R$ . Then  $\text{Ext}_R^i(R/\mathfrak{a}, H_{\mathfrak{a}}^j(M))$  is finitely generated for all  $i \geq 0$  and  $j \geq 0$ .”

Also, Huneke in [10] raised some crucial problems on local cohomology modules. One of them was about the finiteness of the set of associated prime ideals of the local cohomology modules  $H_I^i(M)$ .

Although there are some counterexamples to these conjectures, see [13], but there are some partial positive answers in some special cases too, see for example [3] or [4].

In this paper, we consider these two problems for local cohomology modules defined by a pair of ideals over not necessary finitely generated modules. In particular, we investigate certain conditions on  $H_{I,J}^j(M)$  such that the set of associated prime ideals of  $\text{Ext}_R^i(R/I, H_{I,J}^j(M))$  is finite.

More precisely, let  $n \in \mathbb{N}_0$  and assume that the sets  $\text{Ass}_R(\text{Ext}_R^n(R/I, M))$  and  $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$  are finite for all  $i \leq n+1$  and all  $j < n$  then, we use a spectral sequence argument to show that  $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^n(M)))$  is finite, too (Theorem 2.3). Moreover, it is shown that if the sets  $\text{Ass}_R(\text{Ext}_R^{n+1}(R/I, M))$  and  $\text{Supp}(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$  are finite for all  $i \leq n+2$  and all  $j < n$  then, so is  $\text{Ass}_R(\text{Ext}_R^1(R/I, H_{I,J}^n(M)))$  (Theorem 2.7).

We also present a necessary and sufficient condition for the finiteness of the set  $\text{Ass}_R(\text{Ext}_R^2(R/I, H_{I,J}^n(M)))$  (Theorem 2.8). These, also, generalize some known results concerning ordinary local cohomology modules.

Moreover, we study the grade  $\mathfrak{p} := \inf\{i \in \mathbb{N}_0 : H_{\mathfrak{p}}^i(M) \neq 0\}$  for  $\mathfrak{p} \in \text{Ass}_R(H_{I,J}^t(M))$ , where  $t = \inf\{i \in \mathbb{N}_0 : H_{I,J}^i(M) \neq 0\}$  (Theorem 2.11).

## 2. ASSOCIATED PRIME IDEALS

In this section, first, we are going to study the set of associated prime ideals of some Ext-modules of local cohomology modules defined by a pair of ideals.

The following relation between associated prime ideals of modules in an exact sequence, which can be proved easily, is frequently used in our results.

**Lemma 2.1.** *Let  $M \rightarrow N \rightarrow K \rightarrow 0$  be an exact sequence of  $R$ -modules. Then  $\text{Ass}(K) \subseteq \text{Supp}(M) \cup \text{Ass}(N)$ .*

Next lemma describes a convergence of Grothendieck spectral sequences.

**Lemma 2.2.** *Let  $M$  be an  $R$ -module. Then the following convergence of spectral sequences exists*

$$\text{Ext}_R^i(R/I, H_{I,J}^j(M)) \xrightarrow{i} \text{Ext}_R^{i+j}(R/I, M).$$

**Proof.** It is easy to see that  $\text{Hom}_R(R/I, \Gamma_{I,J}(M)) = \text{Hom}_R(R/I, M)$ . Also, for any injective  $R$ -module  $E$ ,  $\Gamma_{I,J}(E)$  is an injective  $R$ -module, by [14, 3.2] and [5, 2.1.4]. Now, in view of [11, 10.47], the assertion follows. ■

The following theorem, which concerns with Hartshorne's problem mentioned in the introduction, is one of the main results in this paper.

**Theorem 2.3.** *Let  $n$  be a non-negative integer and  $M$  be an  $R$ -module such that  $\text{Ass}_R(\text{Ext}_R^n(R/I, M))$  and  $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$  are finite for all  $i \leq n+1$  and all  $j < n$ . Then so is  $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^n(M)))$ .*

**Proof.** Consider the convergence of spectral sequences in Lemma 2.2 and note that  $E_2^{i,j} = 0$  for all  $i < 0$ . Therefore, for all  $2 \leq r \leq n+1$  there exists an exact sequence

$$(2.1) \quad 0 \rightarrow E_{r+1}^{0,n} \rightarrow E_r^{0,n} \xrightarrow{d_r^{0,n}} E_r^{r,n+1-r}.$$

Since,  $E_r^{r,n+1-r}$  is a subquotient of  $E_2^{r,n+1-r} = \text{Ext}_R^r(R/I, H_{I,J}^{n+1-r}(M))$ ,  $\text{Supp}_R(E_r^{r,n+1-r})$  is a finite set. So, the above exact sequence implies that  $\sharp \text{Ass}_R(E_r^{0,n}) < \infty$  if  $\sharp \text{Ass}_R(E_{r+1}^{0,n}) < \infty$ . Also, from the fact that  $E_2^{i,j} = 0$  for all  $j < 0$ , we have  $E_\infty^{0,n} \cong E_{n+2}^{0,n}$ . Therefore, to prove the assertion it is enough to show that  $\text{Ass}_R(E_\infty^{0,n})$  is a finite set.

Using the concept of the convergence of spectral sequences, there exists a bounded filtration

$$0 = \varphi^{n+1}H^n \subseteq \varphi^n H^n \subseteq \cdots \subseteq \varphi^1 H^n \subseteq \varphi^0 H^n = \text{Ext}_R^n(R/I, M)$$

of submodules of  $\text{Ext}_R^n(R/I, M)$  such that

$$E_\infty^{i,n-i} \cong \varphi^i H^n / \varphi^{i+1} H^n \text{ for all } i = 0, \dots, n.$$

Therefore,  $E_{n+1}^{n,0} \cong E_\infty^{n,0} \cong \varphi^n H^n$  is a subquotient of  $E_2^{n,0} = \text{Ext}_R^n(R/I, \Gamma_{I,J}(M))$ . So, by the assumption,  $\text{Supp}_R(\varphi^n H^n)$  is a finite set. Now, assume inductively that  $\#\text{Supp}_R(\varphi^i H^n) < \infty$  for all  $1 < i \leq n$ . Then, since

$$E_{n+1}^{1,n-1} \cong E_\infty^{1,n-1} \cong \varphi^1 H^n / \varphi^2 H^n$$

is a subquotient of  $E_2^{1,n-1} = \text{Ext}_R^1(R/I, H_{I,J}^{n-1}(M))$ , we deduce that  $\text{Supp}_R(\varphi^1 H^n)$  is finite. But,

$$E_\infty^{0,n} \cong \text{Ext}_R^n(R/I, M) / \varphi^1 H^n$$

and Lemma 2.1 implies that  $\#\text{Ass}_R(E_\infty^{0,n}) < \infty$ , as desired.  $\blacksquare$

As some immediate consequences of Theorem 2.3, we obtain the following results.

**Corollary 2.4.** *Let  $M$  be a finite  $R$ -module. Suppose that there is an integer  $n$  such that for all  $i < n$  the set  $\text{Supp}_R(H_{I,J}^i(M))$  is finite. Then  $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^n(M)))$  is finite.*

**Proof.** Using the fact that  $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M))) \subseteq V(I) \cap \text{Supp}_R(H_{I,J}^i(M))$  for all  $i$  and  $j$ , the result follows from Theorem 2.3.  $\blacksquare$

**Corollary 2.5.** *Let  $M$  be a finite  $R$ -module. Suppose that  $q = \inf\{i : H_{I,J}^i(M) \text{ is not Artinian}\}$  is an integer, then  $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^q(M)))$  is finite.*

**Proof.** By [2, IV, p. 275, Proposition 7],  $\text{Supp}_R(H_{I,J}^i(M))$  is finite for all  $i < q$ . Now, the result follows from Corollary 2.4.  $\blacksquare$

For an  $R$ -module  $M$  and an ideal  $\mathfrak{a}$  of  $R$ , the grade of  $\mathfrak{a}$  on  $M$  is defined by

$$\text{grade } \mathfrak{a} := \inf_M \{i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M) \neq 0\},$$

if this infimum exists, and  $\infty$  otherwise. If  $M$  is a finite  $R$ -module and  $\mathfrak{a}M \neq M$ , this definition coincides with the length of a maximal  $M$ -sequence in  $\mathfrak{a}$  (cf. [5, 6.2.7]).

**Corollary 2.6.** *Let  $M$  be a finite  $R$ -module and  $t = \inf\{i | H_{I,J}^i(M) \neq 0\}$  be an integer. Then  $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^t(M)))$  is finite. If in addition,  $\text{grade } I = t$ , then for a maximal  $M$ -sequence  $x_1, \dots, x_t$  in  $I$ , we have*

$$\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^t(M))) = \{\mathfrak{p} \in \text{Ass}_R(M/(x_1, \dots, x_t)M) \cap V(I); \text{grade } \mathfrak{p} = t\}.$$

**Proof.** In view of Theorem 2.3,  $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^t(M)))$  is finite. In the case where grade  $I = t$ , using [1, 2.4(i)] and [6, 1.2.27], we have

$$\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^t(M))) = \text{Ass}_R(\text{Hom}_R(R/I, H_I^t(M))) = \text{Ass}_R(H_I^t(M)).$$

Now, the assertion follows by [15, 3.10]. ■

In the rest of this paper we consider the set of associated prime ideals of some Ext modules of local cohomology modules defined by a pair of ideals.

**Theorem 2.7.** *Let  $n$  be a non-negative integer and  $M$  be an  $R$ -module such that  $\text{Ass}_R(\text{Ext}_R^{n+1}(R/I, M))$  and  $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$  are finite for all  $i \leq n+2$  and all  $j < n$ . Then so is  $\text{Ass}_R(\text{Ext}_R^1(R/I, H_{I,J}^n(M)))$ .*

**Proof.** Considering the convergence of the spectral sequences of Lemma 2.2, we have to show that  $\text{Ass}_R(E_2^{1,n})$  is a finite set. Using similar arguments as used in Theorem 2.3, one can see that it is enough to show that  $\text{Ass}_R(E_\infty^{1,n}) = \text{Ass}_R(E_{n+2}^{1,n})$  is a finite set.

By the concept of convergence of spectral sequences, there exists a filtration

$$0 = \varphi^{n+2}H^{n+1} \subseteq \varphi^{n+1}H^{n+1} \subseteq \dots \subseteq \varphi^1H^{n+1} \subseteq \varphi^0H^{n+1} = \text{Ext}_R^{n+1}(R/I, M)$$

of submodules of  $\text{Ext}_R^{n+1}(R/I, M)$  such that  $E_\infty^{i,n+1-i} \cong \varphi^iH^{n+1}/\varphi^{i+1}H^{n+1}$  for all  $i = 0, \dots, n+1$ . Using the fact that  $\sharp \text{Supp}_R(E_2^{i,j}) < \infty$  for all  $i \leq n+2$  and all  $j < n$  one can see that  $\text{Supp}_R(\varphi^iH^{n+1})$  is a finite set for all  $i = 2, \dots, n+2$ . Also,  $\sharp \text{Ass}_R(\varphi^1H^{n+1}) < \infty$ . Now, since

$$E_{n+2}^{1,n} \cong E_\infty^{1,n} \cong \varphi^1H^{n+1}/\varphi^2H^{n+1},$$

using Lemma 2.1, we have  $\sharp \text{Ass}_R(E_\infty^{1,n}) < \infty$ , and the result follows. ■

The following theorem presents a necessary and sufficient condition for the finiteness of the set  $\text{Ass}_R(\text{Ext}_R^i(R/I, H_{I,J}^n(M)))$  when  $i = 0, 2$ .

**Theorem 2.8.** *Let  $n$  be a non-negative integer and  $M$  be an  $R$ -module such that the sets  $\text{Supp}_R(\text{Ext}_R^{n+1}(R/I, M))$  and  $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$  are finite for all  $i \leq n+2$  and all  $j < n$ . Then  $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^{n+1}(M)))$  is finite if and only if  $\text{Ass}_R(\text{Ext}_R^2(R/I, H_{I,J}^n(M)))$  is finite.*

**Proof.** ( $\Leftarrow$ ) Again, consider the convergence of spectral sequences of Lemma 2.2 and assume that  $\text{Ass}_R(E_2^{2,n})$  is finite. Since  $E_2^{i,j} = 0$  for all  $i < 0$  or  $j < 0$ , using similar arguments as used in Theorem 2.3, one can see that  $E_\infty^{0,n+1} \cong E_{n+3}^{0,n+1}$  and in order to prove that  $\sharp \text{Ass}_R(E_2^{0,n+1}) < \infty$  we have to show that  $\sharp \text{Ass}_R(E_\infty^{0,n+1}) < \infty$ .

There exists a filtration

$$0 = \varphi^{n+2}H^{n+1} \subseteq \varphi^{n+1}H^{n+1} \subseteq \dots \subseteq \varphi^1H^{n+1} \subseteq \varphi^0H^{n+1} = \text{Ext}_R^{n+1}(R/I, M)$$

of submodules of  $\text{Ext}_R^{n+1}(R/I, M)$  such that  $E_\infty^{0,n+1} \cong \text{Ext}_R^{n+1}(R/I, M)/\varphi^1H^{n+1}$ . Since  $\#\text{Supp}_R(\text{Ext}_R^{n+1}(R/I, M)) < \infty$  we have  $\#\text{Ass}_R(E_\infty^{0,n+1}) < \infty$ , as desired.

( $\Rightarrow$ ) Now, assume that  $\text{Ass}_R(\text{Hom}_R(R/I, H_{I,J}^{n+1}(M))) < \infty$  and consider the exact sequence

$$0 \rightarrow \text{Ker } d_2^{0,n+1} \rightarrow E_2^{0,n+1} \xrightarrow{d_2^{0,n+1}} \text{Im } d_2^{0,n+1} \rightarrow 0.$$

Since  $\text{Ker } d_2^{0,n+1} = E_3^{0,n+1}$  and  $\#\text{Supp}_R(E_3^{0,n+1}) < \infty$ , in view of Lemma 2.1, we have  $\#\text{Ass}_R(\text{Im } d_2^{0,n+1}) < \infty$ . Now, using the exact sequence

$$0 \rightarrow \text{Im } d_2^{0,n+1} \rightarrow E_2^{2,n} \xrightarrow{d_2^{2,n}} E_2^{4,n-1}$$

and the fact that  $E_2^{4,n-1} = \text{Ext}_R^4(R/I, H_{I,J}^{n-1}(M))$  has finite support, we have  $\#\text{Ass}_R(E_2^{2,n}) < \infty$ , as desired.  $\blacksquare$

**Theorem 2.9.** *Let  $n$  be a non-negative integer and  $M$  be an  $R$ -module of dimension  $d$ , such that  $\text{Ass}_R(\text{Ext}_R^{n+d}(R/I, M))$  and  $\text{Supp}_R(\text{Ext}_R^i(R/I, H_{I,J}^j(M)))$  are finite for all  $i \geq n+1$  and all  $j < d$ . Then  $\text{Ass}_R(\text{Ext}_R^n(R/I, H_{I,J}^d(M)))$  is finite.*

**Proof.** The method of the proof is similar to the Theorem 2.7, considering [14, 4.7].  $\blacksquare$

In the rest of this paper, we study the grade of prime ideals  $\mathfrak{p} \in \text{Ass}_R(H_{I,J}^t(M))$  on  $M$ . For that, we shall use the following notations introduced in [14].

$$W(I, J) := \{\mathfrak{p} \in \text{Spec}(R) : I^n \subseteq \mathfrak{p} + J \text{ for some integer } n \geq 1\},$$

and

$$\widetilde{W}(I, J) := \{\mathfrak{a} : \mathfrak{a} \text{ is an ideal of } R \text{ and } I^n \subseteq \mathfrak{a} + J \text{ for some integer } n \geq 1\}.$$

The following lemma can be proved using [14, 3.2].

**Lemma 2.10.** *For any non-negative integer  $i$  and any  $R$ -module  $M$ ,*

- (i)  $\text{Supp}_R(H_{I,J}^i(M)) \subseteq \bigcup_{\mathfrak{a} \in \widetilde{W}(I,J)} \text{Supp}(H_{\mathfrak{a}}^i(M))$ .
- (ii)  $\text{Supp}_R(H_{I,J}^i(M)) \subseteq \text{Supp}_R(M) \cap W(I, J)$ .

In [15, 3.6] the authors study the grade  $\mathfrak{p}$  for  $\mathfrak{p} \in \text{Ass}_R(H_{I,J}^t(M))$ , where

$$t = \inf \{i \in \mathbb{N}_0 : H_{I,J}^i(M) \neq 0\}$$

in the case where  $M$  is a finitely generated  $R$ -module. But their proof is not correct. Actually, they use the equality  $\text{Supp}_R(M_x) = \{\mathfrak{p} \in \text{Supp}_R(M) : x \notin \mathfrak{p}\}$  which is not true. Here, we also made a correction to this result for not necessary finite modules.

**Theorem 2.11.** *Let  $M$  be an  $R$ -module and  $t = \inf\{i \in \mathbb{N}_0 : H_{I,J}^i(M) \neq 0\}$  be a non-negative integer. Then for all  $\mathfrak{p} \in \text{Ass}_R(H_{I,J}^t(M))$ , grade  $\mathfrak{p} = t$ .*

**Proof.** We use induction on  $t$ . Let  $t = 0$  and  $\mathfrak{p} \in \text{Ass}_R(\Gamma_{I,J}(M))$ . Then  $\mathfrak{p} = (0 :_R x)$  for some  $x \in \Gamma_{I,J}(M)$ . Hence  $x \in \Gamma_{\mathfrak{p}}(M)$  and so  $\Gamma_{\mathfrak{p}}(M) \neq 0$ .

Now suppose that  $t > 0$  and the case  $t - 1$  is settled. Let  $\mathfrak{p} \in \text{Ass}_R(H_{I,J}^t(M))$  and consider the exact sequence  $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$ , where  $E = E_R(M)$  is the injective envelope of  $M$ . Therefore, using [15, 2.2],  $H_{I,J}^i(L) \cong H_{I,J}^{i+1}(M)$  for all  $i \geq 0$  and we get

$$\inf \{i \in \mathbb{N}_0 : H_{I,J}^i(L) \neq 0\} = \inf \{i \in \mathbb{N}_0 : H_{I,J}^i(M) \neq 0\} - 1 = t - 1$$

and that  $\mathfrak{p} \in \text{Ass}_R(H_{I,J}^{t-1}(L))$ . Thus, by inductive hypothesis, grade  $\mathfrak{p} = t - 1$ .

Now, consider the long exact sequence

$$H_{\mathfrak{p}}^{i-1}(M) \rightarrow H_{\mathfrak{p}}^{i-1}(E) \rightarrow H_{\mathfrak{p}}^{i-1}(L) \rightarrow H_{\mathfrak{p}}^i(M).$$

If  $t > 1$ , then  $H_{\mathfrak{p}}^i(M) \cong H_{\mathfrak{p}}^{i-1}(L) = 0$  for all  $i < t$  and  $H_{\mathfrak{p}}^t(M) \cong H_{\mathfrak{p}}^{t-1}(L) \neq 0$ . Thus grade  $\mathfrak{p} = t$ .

Let  $t = 1$ . Then  $\Gamma_{\mathfrak{p}}(L) \neq 0$ . By the above exact sequence, it is enough to show that  $\Gamma_{\mathfrak{p}}(E) = 0$ . On the contrary, assume that  $\Gamma_{\mathfrak{p}}(E) \neq 0$ . Then there exists a non-zero element  $x \in E$  and  $n \in \mathbb{N}$  such that  $\mathfrak{p}^n x = 0$ . We may assume that  $\mathfrak{p}^n x = 0$  and  $\mathfrak{p}^{n-1} x \neq 0$ . So, there exists  $r \in \mathfrak{p}^{n-1}$  such that  $rx \neq 0$ . Thus  $\mathfrak{p} \subseteq (0 :_R rx)$ . On the other hand, by Lemma 2.10,

$$\mathfrak{p} \in \text{Ass}_R(H_{I,J}^1(M)) \subseteq \text{Supp}_R(H_{I,J}^1(M)) \subseteq \bigcup_{\mathfrak{a} \in \widetilde{W}(I,J)} \text{Supp}_R(H_{\mathfrak{a}}^1(M)).$$

So that there exists  $\mathfrak{a} \in \widetilde{W}(I,J)$  such that  $\mathfrak{a} \subseteq \mathfrak{p}$ . Let  $m \in \mathbb{N}$  with  $I^m \subseteq \mathfrak{a} + J \subseteq \mathfrak{p} + J \subseteq (0 :_R rx) + J$ . Hence  $rx \in \Gamma_{I,J}(M)$  which contradicts with hypothesis and the choice of  $rx$ . Therefore  $\Gamma_{\mathfrak{p}}(E) = 0$  and so grade  $\mathfrak{p} = 1$ .  $\blacksquare$

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