

POINTED PRINCIPALLY ORDERED REGULAR SEMIGROUPS

T.S. BLYTH

Mathematical Institute
University of St Andrews, Scotland

e-mail: tsb@st-and.ac.uk

AND

G.A. PINTO

School of Mathematics, Physics and Technology
College of The Bahamas, Freeport, Commonwealth of The Bahamas

e-mail: gcapinto@gmail.com

Abstract

An ordered semigroup S is said to be *principally ordered* if, for every $x \in S$ there exists $x^* = \max \{y \in S \mid xyx \leq x\}$. Here we investigate those principally ordered regular semigroups that are *pointed* in the sense that the classes modulo Green's relations \mathcal{L} , \mathcal{R} , \mathcal{D} have biggest elements which are idempotent. Such a semigroup is necessarily a semiband. In particular we describe the subalgebra of $(S; *)$ generated by a pair of comparable idempotents that are \mathcal{D} -related. We also prove that those \mathcal{D} -classes which are subsemigroups are ordered rectangular bands.

Keywords: regular semigroup, principally ordered, naturally ordered, Green's relations.

2010 Mathematics Subject Classification: 06F05, 20M17.

An ordered regular semigroup S is said to be *principally ordered* [3] if, for every $x \in S$ there exists $x^* = \max \{y \in S \mid xyx \leq x\}$. The basic properties of the unary operation $x \mapsto x^*$ in such a semigroup were established in [3] and are listed in [1, Theorem 13.26]. In particular, we recall for the reader's convenience that in such a semigroup the following properties hold and will be used throughout what follows:

$$(P_1) (\forall x \in S) x = xx^*x;$$

(P_2) every \mathcal{L} -class $[x]_{\mathcal{L}}$ contains a biggest idempotent, namely x^*x ;

- (P₃) every \mathcal{R} -class $[x]_{\mathcal{R}}$ contains a biggest idempotent, namely xx^* ;
- (P₄) $(\forall x \in S)x^{***} = x^*$;
- (P₅) every $x \in S$ has a biggest inverse, namely $x^\circ = x^*xx^*$;
- (P₆) $(\forall x \in S)x^\circ \leq x^*$;
- (P₇) $(\forall x \in S)x \leq x^{**} = x^{\circ*} = x^{*\circ}$.

The point of departure for our investigation here is the following observation.

Theorem 1. *If S is a principally ordered regular semigroup then the following statements are equivalent:*

- (1) every \mathcal{L} -class has a biggest element which is idempotent;
- (2) $(\forall x \in S)x^*x = \max[x]_{\mathcal{L}}$;
- (3) every \mathcal{R} -class has a biggest element which is idempotent;
- (4) $(\forall x \in S)xx^* = \max[x]_{\mathcal{R}}$;
- (5) $(\forall x \in S)x^2 \leq x$;
- (6) $(\forall x \in S)x^* \in E(S)$.

Moreover, if S satisfies any of the above conditions then

- (7) $(\forall x \in S) \max[x^*]_{\mathcal{R}} = x^* = x^{**} = \max[x^*]_{\mathcal{L}}$;
- (8) S is a semiband and Green's relation \mathcal{H} is equality;
- (9) $x \in S$ is completely regular if and only if $x \in E(S)$.

Proof. (1) \Leftrightarrow (2): If (1) holds and $e = e^2 = \max[x]_{\mathcal{L}}$ then, by (P₂), $e = e^*e = x^*x$ whence (2) holds. The converse is clear.

(3) \Leftrightarrow (4): This is dual to (1) \Leftrightarrow (2).

(2) \Rightarrow (5): If (2) holds then $x \leq x^*x$ gives, by (P₁), $x^2 \leq xx^*x = x$.

(5) \Rightarrow (2): If (5) holds then $x^3 \leq x^2 \leq x$, so $x \leq x^*$. Then $xx^* \leq x^{*2} \leq x^*$ and consequently $x = xx^*x \leq x^*x$ for every $x \in S$. Every $y \in [x]_{\mathcal{L}}$ is then such that $y \leq y^*y = x^*x$, whence it follows that $x^*x = \max[x]_{\mathcal{L}}$.

(4) \Leftrightarrow (5): This is dual to (2) \Leftrightarrow (5).

(5) \Rightarrow (6): Suppose that (5) holds. Then by the above so do (2) and (4). Now if $y \mathcal{R} x$ then, by (4), we have $y \leq xx^*$ whence $yx \leq xx^*x = x$. It follows by (5) that $xyx \leq x^2 \leq x$ and so $y \leq x^*$. In particular, on taking $y = xx^*$ we obtain $xx^* \leq x^*$ for every $x \in S$. Replacing x by x^{**} in this, we obtain $x^{**}x^* \leq x^*$ and it follows by (2) that $x^* = x^{**}x^* \in E(S)$.

(6) \Rightarrow (2): Clearly, every $e \in E(S)$ is such that $e \leq e^*$. Thus, if (6) holds then $x^* \leq x^{**}$ and $x^{**} \leq x^{***} = x^*$. Consequently $x^* = x^{**}$ for every $x \in S$.

We now observe that

$$y \equiv x(\mathcal{R}) \implies y^* = x^*.$$

Indeed, if $y \equiv x(\mathcal{R})$ then, by (6), $xx^*y^*x = yy^*y^*x = yy^*x = xx^*x = x$ whence $x^*y^* \leq x^*$. Then $x^*y^*x^* \leq x^{*2} = x^*$ and consequently $y^* \leq x^{**} = x^*$. Interchanging x and y produces the reverse inequality and therefore $y^* = x^*$.

Taking in particular $y = xx^*$ we then have $xx^* \leq (xx^*)^* = x^*$ whence $x = xx^*x \leq x^*x$ for every $x \in S$. If now $z \in [x]_{\mathcal{L}}$ then it follows that $z \leq z^*z = x^*x$ and therefore $x^*x = \max[x]_{\mathcal{L}}$, which is (2).

Suppose now that the above conditions are satisfied.

(7) As shown in (6) \Rightarrow (2), $x^* = x^{**} \in E(S)$, and therefore $x^* = x^{**}x^*$. Then, by (4) and (P_4) , $x^* = \max[x^{**}]_{\mathcal{R}} = \max[x^*]_{\mathcal{R}}$. Dually, we see that also $x^* = \max[x^*]_{\mathcal{L}}$.

(8) Since, by (6), each x^* is idempotent, we have $x = xx^*x = xx^* \cdot x^*x$ and so every $x \in S$ is a product of two idempotents, whence S is a semiband. Moreover, if $x \mathcal{H} y$ then $x = xx^*x = xx^* \cdot x^*x = yy^* \cdot y^*y = yy^*y = y$ whence \mathcal{H} reduces to equality.

(9) If $x \in S$ is completely regular then there exists $x' \in V(x)$ such that $xx' = x'x$. Then, by (5), $x' = x'xx' = x'^2x \leq x'x$ from which it follows that $x = xx'x \leq x'xx = x^2$ and consequently $x \in E(S)$. The converse is clear. ■

Definition. We shall say that a principally ordered regular semigroup is *pointed* whenever it satisfies any of the six equivalent properties of Theorem 1.

By way of providing a source of examples, we recall that the *natural order* \leq_n on the idempotents of a regular semigroup is defined by

$$e \leq_n f \iff e = ef = fe,$$

and that an ordered regular semigroup $(T; \leq)$ is said to be *naturally ordered* if the order \leq extends the natural order, in the sense that if $e \leq_n f$ then $e \leq f$. In this case, a fundamental property is that if $e \leq f$ then $e = efe$; see, for example, [1, Theorem 13.11].

Theorem 2. *If T is a naturally ordered regular semigroup with a biggest idempotent ξ then the semiband $\langle E(T) \rangle$ is a pointed principally ordered regular semigroup.*

Proof. If $\bar{e} = e_1 \cdots e_n \in \langle E(T) \rangle$ then, since ξ is the biggest element of $\langle E(T) \rangle$, we have that $e\xi e = e$ for every $e \in E(T)$, and consequently

$$\bar{e} \xi \bar{e} = e_1 \cdots e_n \xi e_1 \cdots e_n \leq e_1 \xi e_1 \cdots e_n = e_1 \cdots e_n = \bar{e}.$$

It follows that the regular subsemigroup $\langle E(T) \rangle$ is principally ordered with $\bar{e}^* = \xi$ for every $\bar{e} \in \langle E(T) \rangle$. Furthermore, $\bar{e}^2 = \bar{e}e_1\bar{e} \leq \bar{e} \xi \bar{e} \leq \bar{e}$ and it follows by Theorem 1(5) that $\langle E(T) \rangle$ is pointed. ■

To avoid unnecessary repetition throughout what follows, S will always denote a pointed principally ordered regular semigroup.

As we have seen above, a characteristic property of S is that the classes modulo Green's relations \mathcal{R} and \mathcal{L} have biggest elements which are idempotent. We now show that the same is true for Green's relation \mathcal{D} .

Theorem 3. *Green's relation \mathcal{D} on S is given by*

$$(x, y) \in \mathcal{D} \iff x^\circ = y^\circ.$$

Moreover, every \mathcal{D} -class has a biggest element which is idempotent. Specifically,

$$(\forall x \in S) \quad x^\circ = x^{\circ\circ} = \max [x^*x]_{\mathcal{R}} = \max [xx^*]_{\mathcal{L}} = \max [x]_{\mathcal{D}} \in E(S).$$

Proof. As observed in the proof of Theorem 1, we have $(xx^*)^* = x^*$ and therefore, by Theorem 1(4),

$$x^\circ = x^*xx^* = x^*x(x^*x)^* = \max [x^*x]_{\mathcal{R}} \in E(S),$$

and dually for \mathcal{L} . Moreover, by (P_7) and Theorem 1(6,7),

$$x^{\circ\circ} = x^{\circ*}x^\circ x^{\circ*} = x^{**}x^*xx^*x^{**} = x^*xx^* = x^\circ.$$

If now $x \mathcal{D} y$ then there exists $z \in S$ such that $x \mathcal{L} z \mathcal{R} y$. Then $x^*x = z^*z$ and $zz^* = yy^*$. It follows from the above that $x^\circ = z^\circ = y^\circ$. On the other hand, $x \mathcal{L} x^*x \mathcal{R} x^\circ$ gives $x \mathcal{D} x^\circ$. Consequently $x \mathcal{D} y \iff x^\circ = y^\circ$. Finally, by Theorem 1(2,4) we see that $x \leq xx^* \leq x^*xx^* = x^\circ$ whence it follows that $x^\circ = \max [x]_{\mathcal{D}} \in E(S)$. ■

Theorem 4. (1) $x \in S$ is a maximal idempotent if and only if it is a maximal element;

(2) S contains at most one maximal element.

Proof. (1) Suppose that e is a maximal idempotent of S . If $x \in S$ is such that $e \leq x$ then we have $e \leq x \leq x^* \in E(S)$, whence the hypothesis that e is maximal in $E(S)$ gives $e = x$. Thus e is a maximal element of S . Conversely, if $x \in S$ is a maximal element then $x \leq x^*$ gives $x = x^*$ whence, by Theorem 1(6), $x \in E(S)$.

(2) Let e and f be maximal elements of S . By (1), each is then idempotent. Now, by Theorem 1(5), $ef \cdot e \cdot ef = (ef)^2 \leq ef$ and gives $e \leq (ef)^*$. It follows that $e = (ef)^*$ and likewise $e = (fe)^*$. Similarly, $f = (fe)^* = (ef)^*$ and consequently $e = f$. ■

By [4, Theorem 3.3], a principally ordered regular semigroup is naturally ordered if and only if the assignment $x \mapsto x^*$ is antitone. In this case, as shown in [1, Theorem 13.29], each $(xx^*)^*$ is a maximal idempotent. Using this fact in the case where S is pointed, we obtain the following characterisation.

Theorem 5. *The following statements are equivalent:*

- (1) *S is naturally ordered;*
- (2) *S has a biggest element ξ and $x^* = \xi$ for every $x \in S$.*

Proof. (1) \Rightarrow (2): If (1) holds then each $(xx^*)^* = x^*$ is a maximal idempotent and $x \leq x^*$. Then property (2) follows immediately from Theorem 4.

(2) \Rightarrow (1): Suppose conversely that (2) holds and let $e, f \in E(S)$ be such that $e \leq_n f$. By (2), $e^* = \xi = f^*$ and consequently $e = ef = fef \leq fe^*f = ff^*f = f$. Thus S is naturally ordered. ■

Corollary. *If S is naturally ordered then Green's relations \mathcal{D} and \mathcal{J} coincide.*

Proof. By Theorem 5, $(x^2)^* = \xi = x^*$. Consequently, $x^2 = x^2(x^2)^*x^2 = x^2x^*x^2 = x^3$. Then $x^2 \in E(S)$ and so S is group bound. It follows by [6, Theorem 1.2.20] that \mathcal{D} and \mathcal{J} coincide. ■

Consider now the subset $S^* = \{x^* \mid x \in S\}$. This is related to the subset S° and to the set $C = \{x \in S \mid x^* = x^\circ\}$ of *compact elements* as follows.

Theorem 6. $S^* = C \cap S^\circ$.

Proof. The identity $x^{**} = x^{\circ\circ}$ shows that $S^* \subseteq C$. Similarly, $x^* = x^{***} = x^{**\circ} = x^{\circ\circ}$ shows that $S^* \subseteq S^\circ$. Thus $S^* \subseteq C \cap S^\circ$. Conversely, if $x \in C \cap S^\circ$ then $x^* = x^\circ$ and $x = x^{\circ\circ}$, whence $x = x^{\circ\circ} = x^{\circ\circ} = x^{**} \in S^*$. ■

As the following example shows, S^* is not in general a subsemigroup of S .

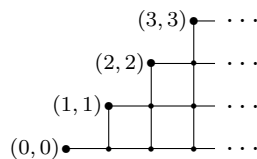
Example 1. Let L be a lattice and consider the cartesian ordered set

$$L^{[2]} = \{(x, y) \in L \times L \mid y \leq x\}.$$

With respect to the multiplication defined by

$$(x, y)(a, b) = (x \vee a, y \wedge b),$$

it is clear that $L^{[2]}$ is an ordered band. It is readily verified that $L^{[2]}$ is principally ordered with $(x, y)^* = (x, x)$. By Theorem 1(5), $L^{[2]}$ is pointed with $(L^{[2]})^* = \{(x, x) \mid x \in L\}$. Now $(L^{[2]})^*$ is not a subsemigroup, for clearly $(x, y)^*(a, b)^* = (x \vee a, x \wedge a)$ and this belongs to $(L^{[2]})^*$ if and only if $x = a$. The particular case of $\mathbb{N}^{[2]}$ is illustrated as follows:



However, in the presence of an identity element 1 the subset S^* has a particular description.

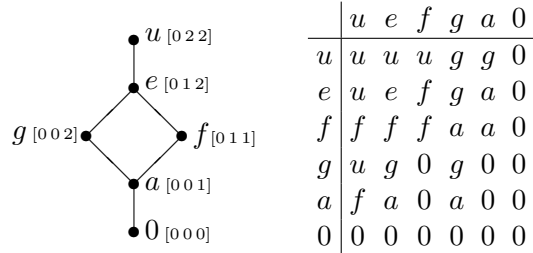
Theorem 7. *If S has an identity element 1, then $S^* = \{x \in S \mid 1 \leq x\}$ and is a join semilattice in which $x \vee y = xy$.*

Proof. If $x \in S$, then since $x1x = x^2 \leq x$ we have $1 \leq x^*$. Conversely, let $1 \leq x$. Then

$$x^* = \begin{cases} 1x^* \leq xx^* \leq x^*x^* = x^* & \text{whence } x^* = xx^*; \\ x^*1 \leq x^*x \leq x^*x^* = x^* & \text{whence } x^* = x^*x. \end{cases}$$

Hence $x^* \mathcal{H} x$ and so $x^* = x$ by Theorem 1(8). Thus we see that $S^* = \{x \in \langle E(S) \rangle \mid 1 \leq x\}$ and is a sub-band. Now if $x, y \in S^*$ then $x = x1 \leq xy$ and $y = 1y \leq xy$, so that xy is an upper bound for $\{x, y\}$. Furthermore, if $z \in S$ is any upper bound for $\{x, y\}$ then necessarily $z \in S^*$ whence $xy \leq z^2 = z$. Consequently, S^* is a join semilattice in which $x \vee y = xy$. ■

Example 2. Let $\mathbf{3}$ denote the 3-element chain $0 < 1 < 2$ and consider the ordered regular semigroup consisting of those isotone mappings f on $\mathbf{3}$ which are such that $f(0) = 0$. Equivalently, this is the semigroup $\text{Res } \mathbf{3}$ of residuated mappings on $\mathbf{3}$ [2]. It has the following Hasse diagram and Cayley table, in which $[0 a b]$ denotes the mapping f such that $f(0) = 0, f(1) = a, f(2) = b$.



This semiband is principally ordered and pointed, with identity element e . Here we have $x^* = u$ for $x \neq e$ and $e^* = e$, so that $S^* = \{e, u\}$.

Example 3. Consider, for $n \geq 2$, the ordered semigroup \mathbf{B}_n of $n \times n$ matrices with entries in a boolean algebra \mathbf{B} . For the basic operations in \mathbf{B} we use the notation $a + b$ (for $a \vee b$) and ab (for $a \wedge b$).

As shown in [1], this semigroup is regular if and only if $n = 2$. Moreover, as is established in [5], \mathbf{B}_2 is principally ordered with

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} b' + c' + d & a' + d' + b \\ a' + d' + c & b' + c' + a \end{bmatrix}.$$

The set of idempotents is

$$E(\mathbf{B}_2) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid b + c \leq a + d, bc \leq ad \right\},$$

and the regular subsemigroup they generate is

$$\langle E(\mathbf{B}_2) \rangle = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid bc \leq ad \right\}.$$

The semiband $\langle E(\mathbf{B}_2) \rangle$ is also principally ordered and pointed. This follows from Theorem 1 and the observation that $bc \leq ad$ gives $b' + c' \geq a' + d'$ whence

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} 1 & a' + d' + b \\ a' + d' + c & 1 \end{bmatrix} \in E(\mathbf{B}_2).$$

Since \mathbf{B}_2 has an identity element it follows from Theorem 7 that $\langle E(\mathbf{B}_2) \rangle^*$ is the join semilattice

$$\langle E(\mathbf{B}_2) \rangle^* = \{X \in \mathbf{B}_2 \mid I_2 \leq X\} = \left\{ \begin{bmatrix} 1 & x \\ y & 1 \end{bmatrix} \mid x, y \in \mathbf{B} \right\}.$$

We can also identify the compact elements of $\langle E(\mathbf{B}_2) \rangle$. For this we recall from (P_5) that every $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in \mathbf{B}_2 has a biggest inverse, namely

$$X^\circ = X^* X X^* = \begin{bmatrix} b'(a+c) + c'(a+b) + d & a'(c+d) + d'(a+c) + b \\ a'(b+d) + d'(a+b) + c & b'(c+d) + c'(b+d) + a \end{bmatrix}.$$

In particular, if $X \in \langle E(\mathbf{B}_2) \rangle$ then the inequality $bc \leq ad$ gives $d = d + bc$ and $a = a + bc$, so that we obtain

$$X^\circ = \begin{bmatrix} a + b + c + d & a'(c+d) + d'(a+c) + b \\ a'(b+d) + d'(a+b) + c & a + b + c + d \end{bmatrix}.$$

Thus, if $X \in \langle E(\mathbf{B}_2) \rangle$ is compact then necessarily $a + b + c + d = 1$. Conversely, if the property $a + b + c + d = 1$ holds then

$$\begin{aligned} a'(c+d) + d'(a+c) + b &\geq a'(a+b)' + d'(b+d)' + b \\ &= a'b' + b'd' + b \\ &= a' + d' + b. \end{aligned}$$

Clearly, the reverse inequality holds, so that $a'(c+d) + d'(a+c) + b = a' + d' + b$. Likewise, we see that $a'(b+d) + d'(a+b) + c = a' + d' + c$ and consequently $X^\circ = X^*$. Hence the set of compact elements of $\langle E(\mathbf{B}_2) \rangle$ is

$$C = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \langle E(\mathbf{B}_2) \rangle \mid a + b + c + d = 1 \right\}.$$

We now turn attention to the \mathcal{D} -classes of S . For idempotents e, f with $e \leq f$ and $(e, f) \in \mathcal{D}$ we first focus on the structure of the subalgebra of $(S; *)$ generated by $\{e, f\}$. In this connection the following observation is important.

Theorem 8. *Any two comparable \mathcal{D} -related idempotents of S are mutually inverse.*

Proof. Let $e, f \in E(S)$ be such that $e \leq f$ and $e \mathcal{D} f$. Then, by Theorem 3, $e^\circ = f^\circ$. Consequently, by Theorem 1(7) and (P_7) , $e^* = e^{**} = e^{\circ*} = f^{\circ*} = f^{**} = f^*$. Moreover, the idempotents e° and e^* are such that $e^\circ e^* = e^\circ = e^* e^\circ$, whence $e^\circ \leq_n e^*$.

We first observe that $e = eee \leq efe \leq ef^*e = ee^*e = e$ so that $e = efe$.

Consider now fee^* . That $fee^* \in E(S)$ follows from the inequalities

$$fee^* = fee^*eee^* \leq fee^* \cdot fee^* \leq fff^*fee^* = fee^*.$$

Now $fee^* \cdot e^\circ = fee^\circ = fee^*$ and

$$e^\circ \cdot fee^* \begin{cases} \leq f^\circ ffe^* = f^\circ ff^* = f^\circ ff^\circ = f^\circ = e^\circ; \\ \geq e^\circ ee^* = e^\circ ee^\circ = e^\circ, \end{cases}$$

so that $e^\circ \cdot fee^* = e^\circ$. Consequently $fee^* \mathcal{L} e^\circ = f^\circ \mathcal{L} ff^*$.

Furthermore, $fee^* \cdot ff^* = fef^*ff^* = fef^\circ = fee^\circ = fee^*$ and $ff^* \cdot fee^* = fee^*$ show that $fee^* \leq_n ff^*$. Since these idempotents are also \mathcal{L} -equivalent it follows that $fee^* = ff^*$.

Using the above observations, we see that $fef \cdot ff^* = fef^* = fefee^* = fee^* = ff^*$ whence $fef \mathcal{R} ff^* \mathcal{R} f$. Since $fef \in E(S)$ with $fef \leq_n f$ it follows that $f = fef$.

Thus e and f are mutually inverse. ■

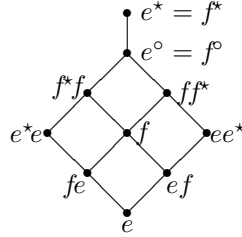
Corollary. *The following statements are equivalent:*

- (1) S is completely simple;
- (2) S is compact and naturally ordered.

Proof. (1) \Rightarrow (2): If S is completely simple then, since \leq_n reduces to equality, S is trivially naturally ordered. Since the idempotents x°, x^* are such that $x^\circ \leq_n x^*$, it follows that $x^\circ = x^*$ and therefore S is compact.

(2) \Rightarrow (1): Suppose that (2) holds and that $e, f \in E(S)$ are such that $e \leq_n f$. By Theorem 5, S has a biggest element ξ and $f^* = \xi = e^*$. Compactness now gives $f^\circ = e^\circ$ whence, by Theorem 3, $(e, f) \in \mathcal{D}$. Since also the natural order implies that $e \leq f$, it follows by Theorem 8 that the idempotents e and f are mutually inverse. Consequently, $f = fef = e$. Thus \leq_n reduces to equality and S is completely simple. ■

Theorem 9. *Let e, f be idempotents of S such that $e \leq f$ and $e \mathcal{D} f$. If T is the subalgebra of $(S; \star)$ generated by $\{e, f\}$ then T is a band having at most 10 elements. In the case where T has precisely 10 elements it is represented by the Hasse diagram*



in which elements joined by lines of positive gradient are \mathcal{R} -related, those joined by lines of negative gradient are \mathcal{L} -related, and the vertical line also indicates \leq_n .

Proof. Since $e \mathcal{D} f$ it follows from Theorem 3 that $e^\circ = f^\circ$ whence $e^\star = f^\star$. The elements of T are then finite products of the elements e, f and $e^\star [= f^\star]$. Moreover, since $e \leq f$, every $x \in T$ is such that $e \leq x \leq e^\star$. By Theorem 8, e and f are mutually inverse, so for every $x \in T$ we have

$$f = fef \leq fxf \leq fe^\star f = ff^\star f = f$$

whence $f = fTf$. In a similar way we see that $e = eTe$ and likewise

$$ee^\star = eTe^\star, \quad e^\star e = e^\star Te, \quad ff^\star = fTf^\star, \quad f^\star f = f^\star Tf, \quad ef = eTf, \quad fe = fTe.$$

For example, $ee^\star = eee^\star \leq exe^\star \leq ee^\star e^\star = ee^\star$ gives $ee^\star = eTe^\star$. It follows from this that $ee^\star = efe^\star = e f f^\star$ whence $ee^\star f = ef$ and then $ef = eef \leq e f f \leq ee^\star f = ef$ and consequently $ef = eTf$. Finally, it is readily seen from the above that $e^\star T e^\star = \{e^\circ, e^\star\}$. It now follows from these observations that T is a band which consists of at most 10 elements, has precisely two \mathcal{D} -classes, and is as described in the above Hasse diagram. ■

Example 4. In the semigroup $\langle E(\mathbf{B}_2) \rangle$ of Example 3, let $|\mathbf{B}| \geq 8$ and consider the idempotents

$$e = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} a & b \\ b & b \end{bmatrix} \quad \text{where } 0 < b < a < 1.$$

Simple calculations which use the expressions for X^\star and X° given in Example 3 reveal that $e^\star = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = f^\star$, and that $e^\circ = \begin{bmatrix} a & a \\ a & a \end{bmatrix} = f^\circ$ whence $e \mathcal{D} f$ with $e < f$.

Furthermore,

$$ee^* = \begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix}, ff^* = \begin{bmatrix} a & a \\ b & b \end{bmatrix}, e^*e = \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix}, f^*f = \begin{bmatrix} a & b \\ a & b \end{bmatrix}, ef = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}, fe = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}.$$

Consequently we have a copy of the band depicted in Theorem 9.

We now proceed to describe the structure of those \mathcal{D} -classes that are subsemigroups of S (which is the case for D_e in Theorem 9, but not so for D_u in Example 2 since $gf = 0$).

Theorem 10. *Given $e \in E(S)$, suppose that D_e is a subsemigroup of S . Then L_{e° is a left zero semigroup, R_{e° is a right zero semigroup, and D_e is isomorphic to the ordered rectangular band $L_{e^\circ} \times R_{e^\circ}$.*

Proof. We observe first that, since $x^\circ = e^\circ$ for every $x \in D_e$,

$$\begin{aligned} x \in L_{e^\circ} &\iff x^\circ x = e^\circ \iff x = xx^\circ \in D_e; \\ x \in R_{e^\circ} &\iff xx^\circ = e^\circ \iff x = x^\circ x \in D_e. \end{aligned}$$

If therefore $x, y \in L_{e^\circ}$ we have $xy = xx^\circ y = xe^\circ y = xy^\circ y = xe^\circ = xx^\circ = x$ and consequently L_{e° is a left zero semigroup. Likewise, R_{e° is a right zero semigroup. Then

$$L_{e^\circ} \times R_{e^\circ} = \{(xe^\circ, e^\circ y) \mid x, y \in D_e\}$$

is a rectangular band. Consider therefore the mapping $\vartheta : D_e \rightarrow L_{e^\circ} \times R_{e^\circ}$ given by the prescription $\vartheta(x) = (xe^\circ, e^\circ x)$, which is clearly isotone.

Now if $(a, b) \in L_{e^\circ} \times R_{e^\circ}$ then, since $ab \in D_e$ by the hypothesis with

$$\vartheta(ab) = (abe^\circ, e^\circ ab) = (abb^\circ, a^\circ ab) = (ae^\circ, e^\circ b) = (a, b),$$

we see that ϑ is surjective. Moreover,

$$\begin{aligned} \vartheta(x) \leq \vartheta(y) &\iff xe^\circ \leq ye^\circ, e^\circ x \leq e^\circ y \\ &\iff x = xe^\circ x \leq ye^\circ y = y. \end{aligned}$$

It follows from these observations that ϑ is an order isomorphism.

We now observe that if e, f are \mathcal{D} -equivalent idempotents such that $e \leq_n f$ then $e = ef = fe \leq fe^\circ = ff^\circ$ and consequently $e = ef \leq ff^\circ f = f$. Thus D_e is a naturally ordered regular semigroup with a biggest idempotent e° . Since $(xy)^\circ = e^\circ = y^\circ x^\circ$ for all $x, y \in D_e$, it follows by [1, Theorem 13.18] that e° is a middle unit of D_e . Using this fact, we see that

$$\vartheta(x)\vartheta(y) = (xe^\circ, e^\circ x)(ye^\circ, e^\circ y) = (xe^\circ ye^\circ, e^\circ xe^\circ y) = (xye^\circ, e^\circ xy) = \vartheta(xy),$$

whence we conclude that ϑ defines an ordered semigroup isomorphism $D_e \simeq L_{e^\circ} \times R_{e^\circ}$. ■

REFERENCES

- [1] T.S. Blyth, *Lattices and Ordered Algebraic Structures* (Springer, 2005). doi:10.1007/b139095
- [2] T.S. Blyth and M.F. Janowitz, *Residuation Theory* (Pergamon, 1972).
- [3] T.S. Blyth and G.A. Pinto, *Principally ordered regular semigroups*, Glasgow Math. J. **32** (1990) 349–364. doi:10.1017/S0017089500009435
- [4] T.S. Blyth and G.A. Pinto, *Idempotents in principally ordered regular semigroups*, Communications in Algebra **19** (1991) 1549–1563. doi:10.1080/00927879108824220
- [5] T.S. Blyth and M.H. Almeida Santos, *On weakly multiplicative inverse transversals*, Proc. Edinburgh Math. Soc. **37** (1993) 93–99. doi:10.1017/S001309150001871X
- [6] P.M. Higgins, *Techniques of Semigroup Theory* (Oxford Science Publications, 1992). doi:10.1007/BF02573500

Received 11 January 2016

Revised 20 May 2016