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IF-FILTERS OF PSEUDO-BL-ALGEBRAS

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Abstract

Characterizations of IF-filters of a pseudo-BL-algebra are established. Some related properties are investigated. The notation of prime IF- filters and a characterization of a pseudo-BL-chain are given. Homomorphisms of IF-filters and direct product of IF-filters are studied.

Keywords: pseudo-BL-algebra, filter, IF-filter, prime IF-filters, pseudo-BL-chain, homomorphism, direct product.

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1. Introduction

In 1958, Chang [2] gave a notation and a characterization of MV-algebras. In 1998, Hájek [8] introduced BL-algebras, which contain the class of MV-algebras. Georgescu and Iorgulescu [5] and independently Rachůnek [10] introduced pseudo MV-algebras as a noncommutative extension of MV-algebras. Finally, in 2000 there were given a notion of pseudo-BL-algebras, which are a noncommutative extension of BL-algebras. Some important properties of pseudo-BL-algebras were studied in [3, 4] and [7].

Zadeh [14] introduced fuzzy sets. Fuzzy sets and filters of pseudo-BL-algebras were studied in [11] and anti fuzzy filters were investigated in [13]. In 1983, Atanassov [1] gave a notion of intuitionistic fuzzy sets as a generalization of fuzzy sets. Takeuti and Titants [12] introduced a intuitionistic fuzzy logic.

In this paper, we introduce a notation of intuitionistic fuzzy filters of pseudo-BL-algebras and study their properties. We introduce prime intuitionistic fuzzy filters and using them we give a characterization of a pseudo-BL-chain.We investigate a homomorphism of intuitionistic fuzzy filters. Finally, we study a direct product of intuitionistic fuzzy filters. We will write shortly IF-filters instead of intuitionistic fuzzy filters.

2. Preliminaries

Definition 1. In [6], there were introduced a pseudo-BL-algebra A as an algebra $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following axioms for all $x, y, z \in A$:

(C1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;

(C2) $(A, \odot, 1)$ is a monoid;

(C3) $x \odot y \leq z \Leftrightarrow x \leq y \rightarrow z \Leftrightarrow y \leq x \rightsquigarrow z$;

(C4) $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$;

(C5) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

Lemma 1 ([7]). *Let* $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ *be a pseudo-BL-algebra. Then for all* $x, y, z \in A$:

(i) $y \leq x \rightarrow y$ *and* $y \leq x \rightsquigarrow y$;

(ii) $x \odot y \leq x \wedge y$;

(iii) $x \odot y \leq x$ *and* $x \odot y \leq y$;

(iv) $x \rightarrow 1 = x \rightsquigarrow 1 = 1$;

(v) $x \leq y \Leftrightarrow x \rightarrow y = x \rightsquigarrow y = 1$;

(vi) $x \rightarrow x = x \rightsquigarrow x = 1$;

(vii) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$ *and* $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$.

We will write shortly A instead of $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$.

Definition 2. A nonempty subset F of a pseudo-BL-algebra A is called a filter if it satisfies the following two conditions:

(F1) if $x, y \in F$, then $x \odot y \in F$;

(F2) if $x \in F$ and $x \leq y$, then $y \in F$.

A filter F of a pseudo-BL-algebra A is called *proper* if $F \neq A$. The proper filter F is prime if for all $x, y \in A$

$$x \vee y \in F \text{ implies } (x \in F \text{ or } y \in F).$$

Now, we give definitions of a fuzzy filter and an anti fuzzy filter of a pseudo-BL-algebra A and their some properties.

Recall that a *fuzzy set* of A is a function $\nu : A \to [0,1]$. For any fuzzy set ν and real number $\alpha \in [0,1]$ there are defined two sets:

$$\begin{aligned} U(\nu,\alpha) &= \{x \in A : \nu(x) \geq \alpha\}; \\ L(\nu,\alpha) &= \{x \in A : \nu(x) \leq \alpha\}; \end{aligned}$$

which are called an upper and a lower α-level set of ν.

Definition 3. Let ν be a fuzzy set of pseudo-BL-algebra A. A *complement* of ν is the fuzzy set ν^C defined as follows

$$\nu^C(x) = 1 - \nu(x)$$

for any $x \in A$.

A fuzzy set μ is called:

1. *a fuzzy filter*, if for all $x, y \in A$

   (ff1) $\mu(x \odot y) \geq \mu(x) \wedge \mu(y)$;
   (ff2) $x \leq y \Rightarrow \mu(x) \leq \mu(y)$.

2. *an anti fuzzy filter*, if for all $x, y \in A$

   (af1) $\mu(x \odot y) \leq \mu(x) \vee \mu(y)$;
   (af2) $x \leq y \Rightarrow \mu(y) \leq \mu(x)$.

Remark 1. Let μ and ν be a fuzzy sets of a pseudo-BL-algebra A. Then:

(i) μ is a fuzzy filter of A iff μ^C is an anti fuzzy filter of A;

(ii) ν is an anti fuzzy filter of A iff ν^C is a fuzzy filter of A.

Definition 4 ([11])**.** Let F be a filter of a pseudo-BL-algebra A and $\alpha, \beta \in [0,1]$ such that $\alpha > \beta$. Let us define a fuzzy filter $\mu_F(\alpha,\beta)$ as follows

$$\mu_F(\alpha,\beta)(x) = \begin{cases} \alpha \text{ if } x \in F, \\ \beta \text{ otherwise.} \end{cases}$$

Remark 2 ([13])**.** A fuzzy set $\mu_F^C(\alpha,\beta)$ is an anti fuzzy filter of A.

We denote by χ_F the characteristic function of F and by χ_F^C the complement of the characteristic function of F.

Definition 5. Let A be a pseudo-BL-algebra and ν be a fuzzy filter of A. Then ν is called a *fuzzy prime filter* if

$$\nu(x \vee y) = \nu(x) \vee \nu(y)$$

for all $x, y \in A$.

Definition 6. Let A be a pseudo-BL-algebra and μ be an anti fuzzy filter of A. Then μ is called an anti *fuzzy prime filter* if

$$\mu(x \vee y) = \mu(x) \wedge \mu(y)$$

for all $x, y \in A$.

For a fuzzy filter ν of pseudo-BL-algebra A we define a set

$$M_\nu = \{x \in A : \nu(x) = \nu(1)\}$$

and similarly, for an anti fuzzy filter μ we define a set

$$A_\mu = \{x \in A : \mu(x) = \mu(1)\}.$$

Remark 3. It is proved in [11] and [13] that a fuzzy filter ν of A is a fuzzy prime filter (an anti fuzzy filter μ of A is an anti fuzzy prime filter) iff M_ν (A_μ) is a prime filter of A.

3. IF-filters

Definition 7. A mapping $\mathcal{B} : A \to [0,1] \times [0,1]$ such that $\mathcal{B}(x) = (\nu_\mathcal{B}(x), \mu_\mathcal{B}(x))$, in which $\nu_\mathcal{B}(x) + \mu_\mathcal{B}(x) \leq 1$ for any $x \in A$, is called an IF-set of A.

In particular, we use $0_\sim$ and $1_\sim$ to denote the IF-empty set and the IF-whole set in a set A such that $0_\sim(x) = (0; 1)$ and $1_\sim(x) = (1; 0)$ for each $x \in A$, respectively.

For IF-sets $\mathcal{B} =(\nu_\mathcal{B}, \mu_\mathcal{B})$ and $\mathcal{C} =(\nu_\mathcal{C}, \mu_\mathcal{C})$ we define a relation $\leq$ as follows:

$$\mathcal{B} \leq \mathcal{C} \Leftrightarrow (\nu_\mathcal{B}(x) < \nu_\mathcal{C}(x) \text{ or } (\nu_\mathcal{B}(x) = \nu_\mathcal{C}(x) \text{ and } \mu_\mathcal{B}(x) < \mu_\mathcal{C}(x)) \text{ for any } x \in A).$$

Now, we give the definition of an IF-filter of a pseudo-BL-algebra. From this place an IF-set $\mathcal{B} = (\nu_\mathcal{B}, \mu_\mathcal{B})$ will be denoted by $\mathcal{B}$.

Definition 8. An IF-set $\mathcal{B}$ of pseudo-BL-algebra A is an IF-filter of A if it satisies the following conditions for all $x, y \in A$:

(IF1) $\nu_\mathcal{B}(x \odot y) \geq \nu_\mathcal{B}(x) \wedge \nu_\mathcal{B}(y)$;

(IF2) $\mu_\mathcal{B}(x \odot y) \leq \mu_\mathcal{B}(x) \vee \mu_\mathcal{B}(y)$;

(IF3) $x \leq y \Rightarrow (\nu_\mathcal{B}(x) \leq \nu_\mathcal{B}(y)$ and $\mu_\mathcal{B}(x) \geq \mu_\mathcal{B}(y))$.

Remark 4. An IF-set $\mathcal{B}$ of a pseudo-BL-algebra A is an IF-filter of A iff $\nu_\mathcal{B}$ is a fuzzy filter and $\mu_\mathcal{B}$ is an anti fuzzy filter of A.

It is easy to see, that (IF3) implies

(IF4) $\nu_{\mathcal{B}}(x) \leq \nu_{\mathcal{B}}(1)$ and $\mu_{\mathcal{B}}(x) \geq \mu_{\mathcal{B}}(1)$ for every $x \in A$;

(IF4') $\nu_{\mathcal{B}}(0) \leq \nu_{\mathcal{B}}(x)$ and $\mu_{\mathcal{B}}(0) \geq \mu_{\mathcal{B}}(x)$ for every $x \in A$.

Proposition 1. *Let $\mathcal{B}$ be an IF-set of a pseudo-BL-algebra A. Then $\mathcal{B}$ is an IF-filter of A iff $\mathcal{B}_C = \left(\nu_{\mathcal{B}}, \nu_{\mathcal{B}}^C\right)$ and ${}_C\mathcal{B} = \left(\mu_{\mathcal{B}}^C, \mu_{\mathcal{B}}\right)$ are IF-filters of A.*

Proof. $\Rightarrow$: Let $\mathcal{B}$ be an IF-set of a pseudo-BL-algebra A. By Remark 4 $\nu_{\mathcal{B}}$ is a fuzzy filter and $\mu_{\mathcal{B}}$ is an anti fuzzy filter of A. Then $\nu_{\mathcal{B}}^C$ is an anti fuzzy filter and $\mu_{\mathcal{B}}^C$ is a fuzzy filter of A. Using Remark 4 once again we obtain that $\mathcal{B}_C = \left(\nu_{\mathcal{B}}, \nu_{\mathcal{B}}^C\right)$ and ${}_C\mathcal{B} = \left(\mu_{\mathcal{B}}^C, \mu_{\mathcal{B}}\right)$ are IF-filters of A.
$\Leftarrow$: By Remark 4. ∎

Example 1. Let F be a filter of a pseudo-BL-algebra A and $\mathcal{B}(F) = \left(\nu_{\mathcal{B}(F)}, \mu_{\mathcal{B}(F)}\right)$ be an IF-set of A defined as follows

$$\nu_{\mathcal{B}(F)}(x) := \begin{cases} \alpha \text{ if } x \in F; \\ \beta \text{ otherwise} \end{cases} \text{ and } \mu_{\mathcal{B}(F)}(x) := \begin{cases} \alpha_1 \text{ if } x \in F; \\ \beta_1 \text{ otherwise.} \end{cases}$$

where $\alpha, \alpha_1, \beta, \beta_1 \in [0,1]$, $\alpha > \beta, \alpha_1 < \beta_1$ and $\alpha + \alpha_1, \beta + \beta_1 \leq 1$.

By Definition 4 and Remark 2, $\nu_{\mathcal{B}(F)}$ is a fuzzy filter of A and $\mu_{\mathcal{B}(F)}$ is an anti fuzzy filter of A. Hence, by Remark 4, $\mathcal{B}(F)$ is an IF-filter of A.

Proposition 2. *Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ be an IF-filter of a pseudo-BL-algebra A, then for all $x, y \in A$:*

(i) $\nu_{\mathcal{B}}(x \vee y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$;

(ii) $\nu_{\mathcal{B}}(x \wedge y) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$;

(iii) $\nu_{\mathcal{B}}(x \odot y) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$;

(iv) $\mu_{\mathcal{B}}(x \wedge y) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$;

(v) $\mu_{\mathcal{B}}(x \odot y) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$;

(vi) $\mu_{\mathcal{B}}(x \vee y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$.

Proof. By Lemma 1 (ii) $x \odot y \leq x \wedge y \leq x \vee y$. Then, by definition of an IF-filter, $\nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y) \leq \nu_{\mathcal{B}}(x \odot y) \leq \nu_{\mathcal{B}}(x \wedge y) \leq \nu_{\mathcal{B}}(x \vee y)$ and $\mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y) \geq \mu_{\mathcal{B}}(x \odot y) \geq \mu_{\mathcal{B}}(x \wedge y) \geq \mu_{\mathcal{B}}(x \vee y)$. (i) and (vi) are proved. Applying Lemma 1 (iii), we have $\nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y) \leq \nu_{\mathcal{B}}(x \odot y) \leq \nu_{\mathcal{B}}(x \wedge y) \leq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$ and $\mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y) \geq \mu_{\mathcal{B}}(x \odot y) \geq \mu_{\mathcal{B}}(x \wedge y) \geq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$. The proofs for (ii), (iii), (iv) and (v) are finished. ∎

Proposition 3. *An IF-set $\mathcal{B}$ of a pseudo-BL-algebra A is an IF-filter of A if and only if it satisfies* (IF1), (IF2) *and*

(IF5) $\nu_{\mathcal{B}}(x \vee y) \geq \nu_{\mathcal{B}}(x)$ *and* $\mu_{\mathcal{B}}(x \vee y) \leq \mu_{\mathcal{B}}(x)$ *for all* $x, y \in A$.

Proof. $\Rightarrow$: Let us suppose that $\mathcal{B}$ is an IF-filter of A. Then, by (IF3), $\nu_{\mathcal{B}}(x \vee y) \geq \nu_{\mathcal{B}}(x)$ and $\mu_{\mathcal{B}}(x \vee y) \leq \mu_{\mathcal{B}}(x)$ for all $x, y \in A$.
$\Leftarrow$: Conversely, let $\mathcal{B}$ satisfies (IF1), (IF2) and (IF5). We need to show that $\mathcal{B}$ satisfies (IF3). Let $x, y \in A$ be such that $x \leq y$. By (IF5) we have $\nu_{\mathcal{B}}(y) = \nu_{\mathcal{B}}(x \vee y) \geq \nu_{\mathcal{B}}(x)$ and $\mu_{\mathcal{B}}(y) = \mu_{\mathcal{B}}(x \vee y) \leq \mu_{\mathcal{B}}(x)$. Hence (IF3) is satisfied. ■

Theorem 1. *Let $\mathcal{B}$ be an IF-set of a pseudo-BL-algebra A. The following are equivalent:*

(i) *$\mathcal{B}$ is an IF-filter;*

(ii) *$\mathcal{B}$ satisfies* (IF3) *and for all $x, y \in A$*

$$\nu_{\mathcal{B}}(y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(x \to y), \tag{1}$$

$$\mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(x \to y), \tag{2}$$

(iii) *$\mathcal{B}$ satisfies* (IF3) *and for all $x, y \in A$*

$$\nu_{\mathcal{B}}(y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(x \rightsquigarrow y), \tag{3}$$

$$\mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(x \rightsquigarrow y). \tag{4}$$

Proof. Using Remark 4 of this paper, Proposition 3.3 and Corollary 3.4 of [13] and Theorem 3.3 of [11] we have the thesis. ■

Proposition 4. *Let $\mathcal{B}$ be an IF-set of a pseudo-BL-algebra A. The following are equivalent:*

(i) *$\mathcal{B}$ is an IF-filter;*

(ii) *for all $x, y, z \in A$*

$$x \to (y \to z) = 1 \Rightarrow \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y), \tag{5}$$

$$x \to (y \to z) = 1 \Rightarrow \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y). \tag{6}$$

(iii) *for all $x, y, z \in A$*

$$x \rightsquigarrow (y \rightsquigarrow z) = 1 \Rightarrow \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y), \tag{7}$$

$$x \rightsquigarrow (y \rightsquigarrow z) = 1 \Rightarrow \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y). \tag{8}$$

Proof. (i)⇒(ii) Suppose that $\mathcal{B}$ is an IF-filter of a pseudo-BL-algebra A. Let $x, y, z \in A$ be such that $x \to (y \to z) = 1$. By Theorem 1 (ii)

$$\nu_{\mathcal{B}}(y \to z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(x \to (y \to z)) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(1) = \nu_{\mathcal{B}}(x), \tag{9}$$

$$\mu_{\mathcal{B}}(y \to z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(x \to (y \to z)) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(1) = \mu_{\mathcal{B}}(x). \tag{10}$$

Aplying Theorem 1 (ii) the secound time we obtain

$$\nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(y) \wedge \nu_{\mathcal{B}}(y \to z), \tag{11}$$

$$\mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(y) \vee \mu_{\mathcal{B}}(y \to z). \tag{12}$$

(9), (10), (11) and (12) force $\nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$ and $\mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$.

(ii)⇒(i) Let $\mathcal{B}$ be an IF-set of a pseudo-BL-algebra A which satisfies (3). Let $x, y \in A$ be such that $x \leq y$. By Lemma 1 (iv) and (v),

$$x \to (x \to y) = 1,$$

hence applying (5) and (6) we have

$$\nu_{\mathcal{B}}(y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(x) = \nu_{\mathcal{B}}(x),$$
$$\mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(x) = \mu_{\mathcal{B}}(x),$$

that is, (IF3) holds.

Now we prove that (1) and (2) hold. By Lemma 1 (vi), $(x \to y) \to (x \to y) = 1$. Thus, applying (5) and (6) we get

$$\nu_{\mathcal{B}}(y) \geq \nu_{\mathcal{B}}(x \to y) \wedge \nu_{\mathcal{B}}(x) \text{ and}$$
$$\mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{B}}(x \to y) \vee \mu_{\mathcal{B}}(x).$$

Hence by Theorem 1, $\mathcal{B}$ is an IF-filter.

(iii)⇔(i) Analogously. ■

Proposition 5. *Let $\mathcal{B}$ be an IF-set of a pseudo-BL-algebra A. The following are equivalent:*

(i) *$\mathcal{B}$ is an IF-filter;*

(ii) *for all $x, y, z \in A$*

$$(x \odot y) \to z = 1 \Rightarrow \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y),$$
$$(x \odot y) \to z = 1 \Rightarrow \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y),$$

(iii) *for all* $x, y, z \in A$

$$(x \odot y) \rightsquigarrow z = 1 \Rightarrow \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y),$$
$$(x \odot y) \rightsquigarrow z = 1 \Rightarrow \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y).$$

Proof. By Proposition 4 and Lemma 1 (vii). ∎

Let $\mathcal{B}_i = (\nu_{\mathcal{B}_i}, \mu_{\mathcal{B}_i})$ be IF-filters of a pseudo-BL-algebra A for every $i \in I$. We define fuzzy sets $\bigwedge_{i\in I} \nu_{\mathcal{B}_i}$ and $\bigvee_{i\in I} \mu_{\mathcal{B}_i}$ as follows:

$$\left(\bigwedge_{i\in I} \nu_{\mathcal{B}_i}\right)(x) = \bigwedge\{\nu_{\mathcal{B}_i}(x) : i \in I\},$$
$$\left(\bigvee_{i\in I} \mu_{\mathcal{B}_i}\right)(x) = \bigvee\{\mu_{\mathcal{B}_i}(x) : i \in I\}.$$

For any IF-filters $\mathcal{B}_i = (\nu_{\mathcal{B}_i}, \mu_{\mathcal{B}_i})$ for $i \in I$, of a pseudo-BL-algebra A we define the IF-set $\bigcap_{i\in I} \mathcal{B}_i$ of A by

$$\bigcap_{i\in I} \mathcal{B}_i = \left(\bigwedge_{i\in I} \nu_{\mathcal{B}_i}, \bigvee_{i\in I} \mu_{\mathcal{B}_i}\right).$$

Theorem 2. *Let* $\mathcal{B}_i = (\nu_{\mathcal{B}_i}, \mu_{\mathcal{B}_i})$ *for* $i \in I$, *be IF-filters of a pseudo-BL-algebra* A. *Then* $\bigcap_{i\in I} \mathcal{B}_i$ *is an IF-filter of* A.

Proof. Let $\mathcal{B}_i = (\nu_{\mathcal{B}_i}, \mu_{\mathcal{B}_i})$ for $i \in I$, be IF-filters of a pseudo-BL-algebra A and $\mathcal{B} = \bigcap_{i\in I} \mathcal{B}_i = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$. We use Proposition 4 to show that $\mathcal{B}$ is an IF-filter of A. Let $x, y, z \in A$ be such that $x \to (y \to z) = 1$. Hence

$$\nu_{\mathcal{B}}(z) = \bigwedge_{i\in I} \nu_{\mathcal{B}_i}(z) \geq \bigwedge_{i\in I}(\nu_{\mathcal{B}_i}(x) \wedge \nu_{\mathcal{B}_i}(y)) = \bigwedge_{i\in I} \nu_{\mathcal{B}_i}(x) \wedge \bigwedge_{i\in I} \nu_{\mathcal{B}_i}(y) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y),$$
$$\mu_{\mathcal{B}}(z) = \bigvee_{i\in I} \mu_{\mathcal{B}_i}(z) \leq \bigvee_{i\in I}(\mu_{\mathcal{B}_i}(x) \vee \mu_{\mathcal{B}_i}(y)) = \bigvee_{i\in I} \mu_{\mathcal{B}_i}(x) \vee \bigvee_{i\in I} \mu_{\mathcal{B}_i}(y) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y).$$

The proof is closed. ∎

Remark 5. The set of IF-filters of a pseudo-BL-algebra A forms a complete distributive lattice with relation $\leq$.

Proof. Since $[0,1]$ is a complete distributive lattice with usual ordering and by Theorem 2, the proof is completed. ■

Theorem 3. *A lattice of IF-filters of a pseudo-BL-algebra A is bounded.*

Proof. It is easily seen that $0_\sim$ and $1_\sim$ are IF-filters. Since $0_\sim \leq \mathcal{B} \leq 1_\sim$ for every IF-filter $\mathcal{B}$, then a lattice of IF-filters is bounded. ■

Theorem 4. *The lattice of IF-filters of a pseudo-BL-algebras has no atoms.0*

Proof. Let $\mathcal{B}$ be an IF-filter of pseudo-BL-algebra A and $\mathcal{B} \neq 0_\sim$. Let us define an IF-set $\mathcal{D}$ as follows

$$\mathcal{D} = \left(\frac{1}{2}\nu_{\mathcal{B}}, \frac{1}{2}\mu_{\mathcal{B}}\right).$$

It is obvious that $\mathcal{D}$ is an IF-filter of A and $0_\sim < \mathcal{D} < \mathcal{B}$. Hence there are no atoms in a lattice of IF-filters of A. ■

Let $\mathcal{B}$ be an IF-set of a pseudo-BL-algebra A and $\alpha, \beta \in [0,1]$ be such that $\alpha + \beta \leq 1$. Then we can define a set

$$A_{\mathcal{B}}^{(\alpha,\beta)} = \{x \in A : \nu_{\mathcal{B}}(x) \geq \alpha, \mu_{\mathcal{B}}(x) \leq \beta\}$$

called an (α, β) −level of $\mathcal{B}$.

Let us notice that $A_{\mathcal{B}}^{(\alpha,\beta)} = U(\nu_{\mathcal{B}}, \alpha) \cap L(\mu_{\mathcal{B}}, \beta)$.

Theorem 5. Let $\mathcal{B}$ be an IF-set of a pseudo-BL-algebra A. If $\mathcal{B}$ is an IF-filter of A, then $A_{\mathcal{B}}^{(\alpha,\beta)} = \emptyset$ or $A_{\mathcal{B}}^{(\alpha,\beta)}$ is a filter of A for all $\alpha \in [0, \nu_{\mathcal{B}}(1)]$, $\beta \in [\mu_{\mathcal{B}}(1), 1]$ such that $\alpha + \beta \leq 1$.

Proof. By Theorem 3.10 of [13] and Theorem 3.6 of [11] $\nu_{\mathcal{B}}$ is a fuzzy filter and $\mu_{\mathcal{B}}$ is an anti fuzzy filter iff $U(\nu_{\mathcal{B}}, \alpha)$ and $L(\mu_{\mathcal{B}}, \beta)$ are filters or empty. According to fact that the intersection of filters is a filter and by Remark 4 we have the thesis. ■

Corollary 1. *If $\mathcal{B}$ is an IF-filter of a pseudo-BL-algebra A, then the set*

$$A_b = \{x \in A : \nu_{\mathcal{B}}(x) \geq \nu_{\mathcal{B}}(b), \mu_{\mathcal{B}}(x) \leq \mu_{\mathcal{B}}(b)\}$$

is a filter of A for every $b \in A$ such that $\nu_{\mathcal{B}}(b) + \mu_{\mathcal{B}}(b) \leq 1$.

4. Prime IF-filters

In this section we introduce and study prime IF-filters and their connection with pseudo-BL-chains.

Definition 9. An IF-filter $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ of a pseudo-BL-algebra A is said to be prime IF-filter if $\nu_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$ are non-constant and satisfies following conditions for all $x, y \in A$:

$$\nu_{\mathcal{B}}(x \vee y) = \nu_{\mathcal{B}}(x) \vee \nu_{\mathcal{B}}(y) \text{ and } \mu_{\mathcal{B}}(x \vee y) = \mu_{\mathcal{B}}(x) \wedge \mu_{\mathcal{B}}(y).$$

Remark 6. An IF-filter $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ of a pseudo-BL-algebra A is said to be prime IF-filter iff $\nu_{\mathcal{B}}$ is a fuzzy prime filter and $\mu_{\mathcal{B}}$ is an anti fuzzy prime filter of A.

Theorem 6. *Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ be a non-constant IF-filter of a pseudo-BL-algebra A. Then the following are equivalent:*

(i) *$\mathcal{B}$ is a prime IF-filter of A;*

(ii) *for all $x, y \in A$, if ($\nu_{\mathcal{B}}(x \vee y) = \nu_{\mathcal{B}}(1)$ and $\mu_{\mathcal{B}}(x \vee y) = \mu_{\mathcal{B}}(1)$), then*

$$\begin{aligned}(\nu_{\mathcal{B}}(x) &= \nu_{\mathcal{B}}(1) \text{ or } \nu_{\mathcal{B}}(y) = \nu_{\mathcal{B}}(1)) \text{ and}\\ (\mu_{\mathcal{B}}(x) &= \mu_{\mathcal{B}}(1) \text{ or } \mu_{\mathcal{B}}(y) = \mu_{\mathcal{B}}(1));\end{aligned}$$

(iii) *for all $x, y \in A$,*

$$\begin{aligned}(\nu_{\mathcal{B}}(x \to y) &= \nu_{\mathcal{B}}(1) \text{ or } \nu_{\mathcal{B}}(y \to x) = \nu_{\mathcal{B}}(1)) \text{ and}\\ (\mu_{\mathcal{B}}(x \to y) &= \mu_{\mathcal{B}}(1) \text{ or } \mu_{\mathcal{B}}(y \to x) = \mu_{\mathcal{B}}(1));\end{aligned}$$

(iv) *for all $x, y \in A$,*

$$\begin{aligned}(\nu_{\mathcal{B}}(x \rightsquigarrow y) &= \nu_{\mathcal{B}}(1) \text{ or } \nu_{\mathcal{B}}(y \rightsquigarrow x) = \nu_{\mathcal{B}}(1)) \text{ and}\\ (\mu_{\mathcal{B}}(x \rightsquigarrow y) &= \mu_{\mathcal{B}}(1) \text{ or } \mu_{\mathcal{B}}(y \rightsquigarrow x) = \mu_{\mathcal{B}}(1)).\end{aligned}$$

Proof. By Theorem 4.1 of [11] and Theorem 4.3 of [13]. ∎

Theorem 7. *Let A be a pseudo-BL-algebra and $\mathcal{B}$ be an IF-filter of A. Then $\mathcal{B}$ is a prime IF-filter iff $M_{\nu_{\mathcal{B}}}$ and $A_{\mu_{\mathcal{B}}}$ are prime filters of A.*

Proof. By Remark 3. ∎

Theorem 8. *Let A be a pseudo-BL-algebra, P be a filter of A and $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$. Then P is a prime filter of A if and only if $\mathcal{B}(P) = (\mu_P(\alpha, \beta)$, $\mu_P^C(1 - \alpha, 1 - \beta))$ define as in Example 1, is a prime IF-filter of A.*

Proof. By Theorem 4.2 of [11] and Theorem 4.6 of [13]. ∎

Theorem 9. *Let $\mathcal{B}$ be an IF-set of a pseudo-BL-algebra A such that $\nu_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$ are non-constant. Then the following are equivalent:*

(i) *$\mathcal{B}$ is a prime IF-filter of A;*

(ii) *for every $\alpha \in [0,1]$, if $U(\nu_{\mathcal{B}},\alpha), L(\mu_{\mathcal{B}},\alpha) \neq \emptyset$ and $U(\nu_{\mathcal{B}},\alpha), L(\mu_{\mathcal{B}},\alpha) \neq A$, then $U(\nu_{\mathcal{B}},\alpha), L(\mu_{\mathcal{B}},\alpha)$ are prime filters of A.*

Proof. By Theorem 4.4 of [11] and Theorem 4.7 of [13]. ∎

Theorem 10. *Let A be a non-trivial pseudo-BL-algebra. The following are equivalent:*

(i) *A is a pseudo-BL-chain;*

(ii) *every IF-filter $\mathcal{B}$ such that $\nu_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$ are non-constant is a prime IF-filter of A;*

(iii) *every IF-filter $\mathcal{B}$ such that $\nu_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$are non-constant, $\nu_{\mathcal{B}}(1) = 1$ and $\mu_{\mathcal{B}}(1) = 0$ is a prime IF-filter of A;*

(iv) *the IF-filter $\left(\chi_{\{1\}}, \chi_{\{1\}}^{C}\right)$ is a prime IF-filter of A.*

Proof. By Theorem 4.6 of [11] and Theorem 4.9 of [13]. ∎

5. Homomorphism and IF-filters

Let A, B be pseudo-BL-algebras. Following [3] we define a homomorphism of pseudo-BL-algebras as a mapping $h : A \to B$ such that the following conditions hold for all $x, y \in A$:

(H1) $h(x \odot y) = h(x) \odot h(y)$;

(H2) $h(x \to y) = h(x) \to h(y)$;

(H3) $h(x \rightsquigarrow y) = h(x) \rightsquigarrow h(y)$;

(H4) $h(0) = 0$.

Recall that if $h : A \to B$ is a homomorphism of pseudo-BL-algebras, then

(H5) $h(1) = 1$;

(H6) $h(x \wedge y) = h(x) \wedge h(y)$;

(H7) $h(x \vee y) = h(x) \vee h(y)$.

Definition 10. Let $\mathcal{B}$ be an IF-filer of a pseudo-BL-algebra B and $f : A \to B$ be a homomorphism of pseudo-BL-algebras. The preimage of $\mathcal{B}$ is the IF-set $\mathcal{B}^f = (\nu_{\mathcal{B}}^f, \mu_{\mathcal{B}}^f)$ defined by

$$\nu_{\mathcal{B}}^f(x) = \nu_{\mathcal{B}}(f(x)) \text{ and } \mu_{\mathcal{B}}^f(x) = \mu_{\mathcal{B}}(f(x))$$

for all $x \in A$.

Theorem 11. *Let $\mathcal{B}$ be an IF-filter of B and $f : A \to B$ be a homomorphism of pseudo-BL-algebras. Then $\mathcal{B}^f$ is an IF-filter of A.*

Proof. Suppose that $f : A \to B$ is a homomorphism of pseudo-BL-algebras and $\mathcal{B}$ be an IF-filter of B. Let $x, y \in A$. Then

$$\begin{aligned}\nu_{\mathcal{B}}^f(x \odot y) &= \nu_{\mathcal{B}}(f(x \odot y)) = \nu_{\mathcal{B}}(f(x) \odot f(y)) \\ &\geq \nu_{\mathcal{B}}(f(x)) \wedge \nu_{\mathcal{B}}(f(y)) = \nu_{\mathcal{B}}^f(x) \wedge \nu_{\mathcal{B}}^f(y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\mathcal{B}}^f(x \odot y) &= \mu_{\mathcal{B}}(f(x \odot y)) = \mu_{\mathcal{B}}(f(x) \odot f(y)) \\ &\leq \mu_{\mathcal{B}}(f(x)) \vee \mu_{\mathcal{B}}(f(y)) = \mu_{\mathcal{B}}^f(x) \vee \mu_{\mathcal{B}}^f(y).\end{aligned}$$

Hence (IF1) and (IF2) hold.

Now let $x, y \in A$ be such that $x \leq y$. Therefore,

$$\begin{aligned}\nu_{\mathcal{B}}^f(x) &= \nu_{\mathcal{B}}^f(x \wedge y) = \nu_{\mathcal{B}}(f(x \wedge y)) \\ &= \nu_{\mathcal{B}}(f(x) \wedge f(y)) \leq \nu_{\mathcal{B}}(f(y)) = \nu_{\mathcal{B}}^f(y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\mathcal{B}}^f(x) &= \mu_{\mathcal{B}}^f(x \wedge y) = \mu_{\mathcal{B}}(f(x \wedge y)) \\ &= \mu_{\mathcal{B}}(f(x) \wedge f(y)) \geq \mu_{\mathcal{B}}(f(y)) = \mu_{\mathcal{B}}^f(y).\end{aligned}$$

Thus, (IF3) holds.

Concluding, $\mathcal{B}^f$ is an IF-filter of A. ■

Theorem 12. *Let $\mathcal{B}$ be an IF-set of B, $\mathcal{B}^f$ be an IF-filter of A, where $f : A \to B$ is an epimorphism of pseudo-BL-algebras. Then $\mathcal{B}$ is an IF-filter of A.*

Proof. Let $f : A \to B$ be an epimorphism of pseudo-BL-algebras. Then, for any $x, y \in B$, there exist $a, b \in A$ such that $x = f(a)$ and $y = f(b)$. Therefore,

$$\begin{aligned}\nu_{\mathcal{B}}(x \odot y) &= \nu_{\mathcal{B}}(f(a) \odot f(b)) = \nu_{\mathcal{B}}(f(a \odot b)) \\ &= \nu_{\mathcal{B}}^{f}(a \odot b) \geq \nu_{\mathcal{B}}^{f}(a) \wedge \nu_{\mathcal{B}}^{f}(b) \\ &= \nu_{\mathcal{B}}(f(a)) \wedge \nu_{\mathcal{B}}(f(b)) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\mathcal{B}}(x \odot y) &= \mu_{\mathcal{B}}(f(a) \odot f(b)) = \mu_{\mathcal{B}}(f(a \odot b)) \\ &= \mu_{\mathcal{B}}^{f}(a \odot b) \leq \mu_{\mathcal{B}}^{f}(a) \vee \mu_{\mathcal{B}}^{f}(b) \\ &= \mu_{\mathcal{B}}(f(a)) \vee \mu_{\mathcal{B}}(f(b)) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y).\end{aligned}$$

Hence (IF1) and (IF2) hold.

Now let $x, y \in B$ be such that $x \leq y$. Then, there exist $a, b \in A$ such that $x = f(a)$ and $y = f(b)$. Therefore,

$$\begin{aligned}\nu_{\mathcal{B}}(x) &= \nu_{\mathcal{B}}(x \wedge y) = \nu_{\mathcal{B}}(f(a) \wedge f(b)) = \nu_{\mathcal{B}}(f(a \wedge b)) \\ &= \nu_{\mathcal{B}}^{f}(a \wedge b) \leq \nu_{\mathcal{B}}^{f}(b) = \nu_{\mathcal{B}}(f(b)) = \nu_{\mathcal{B}}(y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\mathcal{B}}(x) &= \mu_{\mathcal{B}}(x \wedge y) = \mu_{\mathcal{B}}(f(a) \wedge f(b)) = \mu_{\mathcal{B}}(f(a \wedge b)) \\ &= \mu_{\mathcal{B}}^{f}(a \wedge b) \geq \mu_{\mathcal{B}}^{f}(b) = \mu_{\mathcal{B}}(f(b)) = \mu_{\mathcal{B}}(y).\end{aligned}$$

Thus, (IF3) holds.

Concluding, $\mathcal{B}$ is an IF-filter of B. ■

Now let us denote the set of all filters of pseudo-BL-algebra A by $Fil(A)$ and the set of all IF-filters of A by $IFil(A)$. Let $\alpha \in (0,1)$. We define maps $f_\alpha : IFil(A) \to Fil(A) \cup \{\emptyset\}$ and $g_\alpha : IFil(A) \to Fil(A) \cup \{\emptyset\}$ by

$$\begin{aligned}f_\alpha(\mathcal{B}) &= U(\nu_{\mathcal{B}}, \alpha), \\ g_\alpha(\mathcal{B}) &= L(\mu_{\mathcal{B}}, \alpha)\end{aligned}$$

for all $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}}) \in IFil(A)$.

Theorem 13. *For any $\alpha \in (0,1)$, the maps f_α and g_α are surjective from $IFil(A)$ onto $Fil(A) \cup \{\emptyset\}$.*

Proof. It is obvious, that

$$f_\alpha(0_\sim) = U(0,\alpha) = \emptyset = L(1,\alpha) = g_\alpha(0_\sim).$$

Now let $\emptyset \neq F \in Fil(A)$. Then $\left(\chi_F, \chi_F^C\right)$ is an IF-filter of A. Hence,

$$f_\alpha\left(\left(\chi_F, \chi_F^C\right)\right) = U(\chi_F, \alpha) = F = L(\chi_F^C, \alpha) = g_\alpha\left(\left(\chi_F, \chi_F^C\right)\right).$$

Therefore, f_α and g_α are surjective. ■

6. Direct product of IF-filters

Let us define a direct product $\prod_{i\in I}^{n} A_i$ of pseudo-BL-algebras as usually.

Definition 11. Let A be a pseudo-BL-algebra. Then we define an IF-relation on A as a mapping $\mathcal{R} = (\nu'_{\mathcal{R}}, \mu'_{\mathcal{R}}) : A \times A \to [0,1] \times [0,1]$ such that $\nu'_{\mathcal{R}}(x,y) + \mu'_{\mathcal{R}}(x,y) \leq 1$ for all $x, y \in A$.

Now define a direct product of IF-sets of pseudo-BL-algebra A.

Definition 12. Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ and $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$ be IF-sets of A. We define a direct product $\mathcal{B} \times \mathcal{G}$ by

$$\mathcal{B} \times \mathcal{G} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}}) \times (\nu_{\mathcal{G}}, \mu_{\mathcal{G}}) = (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}}, \mu_{\mathcal{B}} \times \mu_{\mathcal{G}}),$$

where $(\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x,y) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{G}}(y)$ and $(\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(x,y) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{G}}(y)$ for all $x, y \in A$.

Proposition 6. *Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ and $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$ be IF-sets of a pseudo-BL-algebra A, then $\mathcal{B} \times \mathcal{G}$ is an IF-set of $A \times A$.*

Proof. Let $\mathcal{B}, \mathcal{G}$ be IF-sets of A. Then for every $x \in A$ we have $\nu_{\mathcal{B}}(x) + \mu_{\mathcal{B}}(x) \leq 1$ and $\nu_{\mathcal{G}}(x) + \mu_{\mathcal{G}}(x) \leq 1$. Suppose that $\nu_{\mathcal{B}}(x) \leq \nu_{\mathcal{G}}(y)$ for some $x, y \in A$. Then $(\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x,y) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{G}}(y) = \nu_{\mathcal{B}}(x)$. Let us consider two cases:

Case 1. $\mu_{\mathcal{B}}(x) \leq \mu_{\mathcal{G}}(y)$
Hence $(\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(x,y) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{G}}(y) = \mu_{\mathcal{G}}(y)$ and then $(\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x,y) + (\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(x,y) = \nu_{\mathcal{B}}(x) + \mu_{\mathcal{G}}(y) \leq \nu_{\mathcal{G}}(y) + \mu_{\mathcal{G}}(y) \leq 1$.

Case 2. $\mu_{\mathcal{B}}(x) > \mu_{\mathcal{G}}(y)$
Therefore $(\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(x,y) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{G}}(y) = \mu_{\mathcal{B}}(x)$ and then $(\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x,y) + (\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(x,y) = \nu_{\mathcal{B}}(x) + \mu_{\mathcal{B}}(x) \leq 1$. Hence $\mathcal{B} \times \mathcal{G}$ is an IF-set of $A \times A$.

Analogously when $\nu_{\mathcal{B}}(x) > \nu_{\mathcal{G}}(y)$. ■

Now we give a trivial Proposition without a proof:

Proposition 7. *Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ and $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$ be IF-sets of a pseudo-BL-algebra A, then*

(i) *$\mathcal{B} \times \mathcal{G}$ is an IF-relation of A;*

(ii) *$U(\nu_{\mathcal{B}} \times \nu_{\mathcal{G}}; \alpha) = U(\nu_{\mathcal{B}}; \alpha) \times U(\nu_{\mathcal{G}}; \alpha)$ and $L(\mu_{\mathcal{B}} \times \mu_{\mathcal{G}}; \alpha) = L(\mu_{\mathcal{B}}; \alpha) \times L(\mu_{\mathcal{G}}; \alpha)$ for all $\alpha \in [0, 1]$.*

Theorem 14. *Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ and $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$ be IF-filters of a pseudo-BL-algebra A. Then $\mathcal{B} \times \mathcal{G}$ is an IF-filter of $A \times A$.*

Proof. Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ and $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$ be IF-filters of a pseudo-BL-algebra A. Suppose that $x, y \in A$. Then by (IF1) and (IF2), $\nu_{\mathcal{B}}(x \odot y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$, $\nu_{\mathcal{G}}(x \odot y) \geq \nu_{\mathcal{G}}(x) \wedge \nu_{\mathcal{G}}(y)$ and $\mu_{\mathcal{B}}(x \odot y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$, $\mu_{\mathcal{G}}(x \odot y) \leq \mu_{\mathcal{G}}(x) \vee \mu_{\mathcal{G}}(y)$. Let $(x_1, x_2), (y_1, y_2) \in A \times A$. Then,

$$\begin{aligned}
(\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})\left((x_1, x_2) \odot (y_1, y_2)\right) &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})\left(x_1 \odot y_1, x_2 \odot y_2\right) \\
&= \nu_{\mathcal{B}}\left(x_1 \odot y_1\right) \wedge \nu_{\mathcal{G}}\left(x_2 \odot y_2\right) \\
&\geq \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{G}}(x_2) \wedge \nu_{\mathcal{G}}(y_2) \\
&= (\nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{G}}(x_2)) \wedge (\nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{G}}(y_2)) \\
&= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x_1, x_2) \wedge (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(y_1, y_2).
\end{aligned}$$

Similarly, we can prove that $(\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})\left((x_1, x_2) \odot (y_1, y_2)\right) \leq (\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(x_1, x_2) \vee (\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(y_1, y_2)$.

It is proved that (IF1) and (IF2) hold.

Now let $(x_1, x_2), (y_1, y_2) \in A \times A$ be such that $(x_1, x_2) \leq (y_1, y_2)$. Then

$$\begin{aligned}
(\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x_1, x_2) &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})\left((x_1, x_2) \wedge (y_1, y_2)\right) \\
&= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})\left(x_1 \wedge y_1, x_2 \wedge y_2\right) \\
&= \nu_{\mathcal{B}}\left(x_1 \wedge y_1\right) \wedge \nu_{\mathcal{G}}\left(x_2 \wedge y_2\right) \\
&\leq \nu_{\mathcal{B}}\left(y_1\right) \wedge \nu_{\mathcal{G}}\left(y_2\right) \\
&= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(y_1, y_2).
\end{aligned}$$

and similarly $(\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(x_1, x_2) \geq (\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(y_1, y_2)$.

The proof is completed. ■

Theorem 15. *Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ be IF-set of a pseudo-BL-algebra A. Then $\mathcal{B}$ is an IF-filter of A if and only if $\mathcal{B} \times \mathcal{B}$ is an IF-filter of $A \times A$.*

Proof. $\Rightarrow$: By Theorem 14.
$\Leftarrow$: Let $\mathcal{B} \times \mathcal{B}$ be an IF-filter of $A \times A$. Let $(x_1, x_2), (y_1, y_2) \in A \times A$. Hence

$$\begin{aligned} \nu_{\mathcal{B}}(x_1 \odot y_1) \wedge \nu_{\mathcal{B}}(x_2 \odot y_2) &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(x_1 \odot y_1, x_2 \odot y_2) \\ &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})((x_1, x_2) \odot (y_1, y_2)) \\ &\geq (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(x_1, x_2) \wedge (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(y_1, y_2) \\ &= \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(x_2) \wedge \nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{B}}(y_2). \end{aligned}$$

Putting $x_1 = x_2$ and $y_1 = y_2$ we have

$$\nu_{\mathcal{B}}(x_1 \odot y_1) \geq \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{B}}(y_1) = \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(y_1).$$

Similarly, $\mu_{\mathcal{B}}(x_1 \odot y_1) \leq \mu_{\mathcal{B}}(x_1) \vee \mu_{\mathcal{B}}(y_1)$.

Let $x, y \in A$ be such that $x \leq y$. Then by (IF3),

$$\nu_{\mathcal{B}}(x) = (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(x, x) \leq (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(y, y) = \nu_{\mathcal{B}}(y).$$

Analogously, $\mu_{\mathcal{B}}(x) \geq \mu_{\mathcal{B}}(y)$.

Hence $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ is an IF-filter of A. ∎

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