

IF-FILTERS OF PSEUDO-BL-ALGEBRAS

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Abstract

Characterizations of IF-filters of a pseudo-BL-algebra are established. Some related properties are investigated. The notation of prime IF-filters and a characterization of a pseudo-BL-chain are given. Homomorphisms of IF-filters and direct product of IF-filters are studied.

Keywords: pseudo-BL-algebra, filter, IF-filter, prime IF-filters, pseudo-BL-chain, homomorphism, direct product.

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1. INTRODUCTION

In 1958, Chang [2] gave a notation and a characterization of MV-algebras. In 1998, Hájek [8] introduced BL-algebras, which contain the class of MV-algebras. Georgescu and Iorgulescu [5] and independently Rachůnek [10] introduced pseudo MV-algebras as a noncommutative extension of MV-algebras. Finally, in 2000 there were given a notion of pseudo-BL-algebras, which are a noncommutative extension of BL-algebras. Some important properties of pseudo-BL-algebras were studied in [3, 4] and [7].

Zadeh [14] introduced fuzzy sets. Fuzzy sets and filters of pseudo-BL-algebras were studied in [11] and anti fuzzy filters were investigated in [13]. In 1983, Atanassov [1] gave a notion of intuitionistic fuzzy sets as a generalization of fuzzy sets. Takeuti and Titants [12] introduced a intuitionistic fuzzy logic.

In this paper, we introduce a notation of intuitionistic fuzzy filters of pseudo-BL-algebras and study their properties. We introduce prime intuitionistic fuzzy filters and using them we give a characterization of a pseudo-BL-chain. We investigate a homomorphism of intuitionistic fuzzy filters. Finally, we study a direct product of intuitionistic fuzzy filters. We will write shortly IF-filters instead of intuitionistic fuzzy filters.

2. PRELIMINARIES

Definition 1. In [6], there were introduced a pseudo-BL-algebra A as an algebra $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ satisfying the following axioms for all $x, y, z \in A$:

- (C1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;
- (C2) $(A, \odot, 1)$ is a monoid;
- (C3) $x \odot y \leq z \Leftrightarrow x \leq y \rightarrow z \Leftrightarrow y \leq x \rightsquigarrow z$;
- (C4) $x \wedge y = (x \rightarrow y) \odot x = x \odot (x \rightsquigarrow y)$;
- (C5) $(x \rightarrow y) \vee (y \rightarrow x) = (x \rightsquigarrow y) \vee (y \rightsquigarrow x) = 1$.

Lemma 1 ([7]). *Let $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ be a pseudo-BL-algebra. Then for all $x, y, z \in A$:*

- (i) $y \leq x \rightarrow y$ and $y \leq x \rightsquigarrow y$;
- (ii) $x \odot y \leq x \wedge y$;
- (iii) $x \odot y \leq x$ and $x \odot y \leq y$;
- (iv) $x \rightarrow 1 = x \rightsquigarrow 1 = 1$;
- (v) $x \leq y \Leftrightarrow x \rightarrow y = x \rightsquigarrow y = 1$;
- (vi) $x \rightarrow x = x \rightsquigarrow x = 1$;
- (vii) $x \rightarrow (y \rightarrow z) = (x \odot y) \rightarrow z$ and $x \rightsquigarrow (y \rightsquigarrow z) = (y \odot x) \rightsquigarrow z$.

We will write shortly A instead of $(A, \vee, \wedge, \odot, \rightarrow, \rightsquigarrow, 0, 1)$.

Definition 2. A nonempty subset F of a pseudo-BL-algebra A is called a filter if it satisfies the following two conditions:

- (F1) if $x, y \in F$, then $x \odot y \in F$;
- (F2) if $x \in F$ and $x \leq y$, then $y \in F$.

A filter F of a pseudo-BL-algebra A is called *proper* if $F \neq A$. The proper filter F is prime if for all $x, y \in A$

$$x \vee y \in F \text{ implies } (x \in F \text{ or } y \in F).$$

Now, we give definitions of a fuzzy filter and an anti fuzzy filter of a pseudo-BL-algebra A and their some properties.

Recall that a *fuzzy set* of A is a function $\nu : A \rightarrow [0, 1]$. For any fuzzy set ν and real number $\alpha \in [0, 1]$ there are defined two sets:

$$U(\nu, \alpha) = \{x \in A : \nu(x) \geq \alpha\};$$

$$L(\nu, \alpha) = \{x \in A : \nu(x) \leq \alpha\};$$

which are called an upper and a lower α -level set of ν .

Definition 3. Let ν be a fuzzy set of pseudo-BL-algebra A . A *complement* of ν is the fuzzy set ν^C defined as follows

$$\nu^C(x) = 1 - \nu(x)$$

for any $x \in A$.

A fuzzy set μ is called:

1. a *fuzzy filter*, if for all $x, y \in A$
 - (ff1) $\mu(x \odot y) \geq \mu(x) \wedge \mu(y)$;
 - (ff2) $x \leq y \Rightarrow \mu(x) \leq \mu(y)$.
2. an *anti fuzzy filter*, if for all $x, y \in A$
 - (af1) $\mu(x \odot y) \leq \mu(x) \vee \mu(y)$;
 - (af2) $x \leq y \Rightarrow \mu(y) \leq \mu(x)$.

Remark 1. Let μ and ν be a fuzzy sets of a pseudo-BL-algebra A . Then:

- (i) μ is a fuzzy filter of A iff μ^C is an anti fuzzy filter of A ;
- (ii) ν is an anti fuzzy filter of A iff ν^C is a fuzzy filter of A .

Definition 4 ([11]). Let F be a filter of a pseudo-BL-algebra A and $\alpha, \beta \in [0, 1]$ such that $\alpha > \beta$. Let us define a fuzzy filter $\mu_F(\alpha, \beta)$ as follows

$$\mu_F(\alpha, \beta)(x) = \begin{cases} \alpha & \text{if } x \in F, \\ \beta & \text{otherwise.} \end{cases}$$

Remark 2 ([13]). A fuzzy set $\mu_F^C(\alpha, \beta)$ is an anti fuzzy filter of A .

We denote by χ_F the characteristic function of F and by χ_F^C the complement of the characteristic function of F .

Definition 5. Let A be a pseudo-BL-algebra and ν be a fuzzy filter of A . Then ν is called a *fuzzy prime filter* if

$$\nu(x \vee y) = \nu(x) \vee \nu(y)$$

for all $x, y \in A$.

Definition 6. Let A be a pseudo-BL-algebra and μ be an anti fuzzy filter of A . Then μ is called an anti *fuzzy prime filter* if

$$\mu(x \vee y) = \mu(x) \wedge \mu(y)$$

for all $x, y \in A$.

For a fuzzy filter ν of pseudo-BL-algebra A we define a set

$$M_\nu = \{x \in A : \nu(x) = \nu(1)\}$$

and similarly, for an anti fuzzy filter μ we define a set

$$A_\mu = \{x \in A : \mu(x) = \mu(1)\}.$$

Remark 3. It is proved in [11] and [13] that a fuzzy filter ν of A is a fuzzy prime filter (an anti fuzzy filter μ of A is an anti fuzzy prime filter) iff M_ν (A_μ) is a prime filter of A .

3. IF-FILTERS

Definition 7. A mapping $\mathcal{B} : A \rightarrow [0, 1] \times [0, 1]$ such that $\mathcal{B}(x) = (\nu_{\mathcal{B}}(x), \mu_{\mathcal{B}}(x))$, in which $\nu_{\mathcal{B}}(x) + \mu_{\mathcal{B}}(x) \leq 1$ for any $x \in A$, is called an IF-set of A .

In particular, we use 0_\sim and 1_\sim to denote the IF-empty set and the IF-whole set in a set A such that $0_\sim(x) = (0; 1)$ and $1_\sim(x) = (1; 0)$ for each $x \in A$, respectively.

For IF-sets $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ and $\mathcal{C} = (\nu_{\mathcal{C}}, \mu_{\mathcal{C}})$ we define a relation \leq as follows:

$$\mathcal{B} \leq \mathcal{C} \Leftrightarrow (\nu_{\mathcal{B}}(x) < \nu_{\mathcal{C}}(x) \text{ or } (\nu_{\mathcal{B}}(x) = \nu_{\mathcal{C}}(x) \text{ and } \mu_{\mathcal{B}}(x) < \mu_{\mathcal{C}}(x)) \text{ for any } x \in A).$$

Now, we give the definition of an IF-filter of a pseudo-BL-algebra. From this place an IF-set $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ will be denoted by \mathcal{B} .

Definition 8. An IF-set \mathcal{B} of pseudo-BL-algebra A is an IF-filter of A if it satisfies the following conditions for all $x, y \in A$:

$$(IF1) \quad \nu_{\mathcal{B}}(x \odot y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y);$$

$$(IF2) \quad \mu_{\mathcal{B}}(x \odot y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y);$$

$$(IF3) \quad x \leq y \Rightarrow (\nu_{\mathcal{B}}(x) \leq \nu_{\mathcal{B}}(y) \text{ and } \mu_{\mathcal{B}}(x) \geq \mu_{\mathcal{B}}(y)).$$

Remark 4. An IF-set \mathcal{B} of a pseudo-BL-algebra A is an IF-filter of A iff $\nu_{\mathcal{B}}$ is a fuzzy filter and $\mu_{\mathcal{B}}$ is an anti fuzzy filter of A .

It is easy to see, that (IF3) implies

(IF4) $\nu_{\mathcal{B}}(x) \leq \nu_{\mathcal{B}}(1)$ and $\mu_{\mathcal{B}}(x) \geq \mu_{\mathcal{B}}(1)$ for every $x \in A$;

(IF4') $\nu_{\mathcal{B}}(0) \leq \nu_{\mathcal{B}}(x)$ and $\mu_{\mathcal{B}}(0) \geq \mu_{\mathcal{B}}(x)$ for every $x \in A$.

Proposition 1. *Let \mathcal{B} be an IF-set of a pseudo-BL-algebra A . Then \mathcal{B} is an IF-filter of A iff $\mathcal{B}_C = (\nu_{\mathcal{B}}, \nu_{\mathcal{B}}^C)$ and ${}_C\mathcal{B} = (\mu_{\mathcal{B}}^C, \mu_{\mathcal{B}})$ are IF-filters of A .*

Proof. \Rightarrow : Let \mathcal{B} be an IF-set of a pseudo-BL-algebra A . By Remark 4 $\nu_{\mathcal{B}}$ is a fuzzy filter and $\mu_{\mathcal{B}}$ is an anti fuzzy filter of A . Then $\nu_{\mathcal{B}}^C$ is an anti fuzzy filter and $\mu_{\mathcal{B}}^C$ is a fuzzy filter of A . Using Remark 4 once again we obtain that $\mathcal{B}_C = (\nu_{\mathcal{B}}, \nu_{\mathcal{B}}^C)$ and ${}_C\mathcal{B} = (\mu_{\mathcal{B}}^C, \mu_{\mathcal{B}})$ are IF-filters of A .

\Leftarrow : By Remark 4. ■

Example 1. Let F be a filter of a pseudo-BL-algebra A and $\mathcal{B}(F) = (\nu_{\mathcal{B}(F)}, \mu_{\mathcal{B}(F)})$ be an IF-set of A defined as follows

$$\nu_{\mathcal{B}(F)}(x) := \begin{cases} \alpha & \text{if } x \in F; \\ \beta & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_{\mathcal{B}(F)}(x) := \begin{cases} \alpha_1 & \text{if } x \in F; \\ \beta_1 & \text{otherwise.} \end{cases}$$

where $\alpha, \alpha_1, \beta, \beta_1 \in [0, 1]$, $\alpha > \beta, \alpha_1 < \beta_1$ and $\alpha + \alpha_1, \beta + \beta_1 \leq 1$.

By Definition 4 and Remark 2, $\nu_{\mathcal{B}(F)}$ is a fuzzy filter of A and $\mu_{\mathcal{B}(F)}$ is an anti fuzzy filter of A . Hence, by Remark 4, $\mathcal{B}(F)$ is an IF-filter of A .

Proposition 2. *Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ be an IF-filter of a pseudo-BL-algebra A , then for all $x, y \in A$:*

- (i) $\nu_{\mathcal{B}}(x \vee y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$;
- (ii) $\nu_{\mathcal{B}}(x \wedge y) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$;
- (iii) $\nu_{\mathcal{B}}(x \odot y) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$;
- (iv) $\mu_{\mathcal{B}}(x \wedge y) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$;
- (v) $\mu_{\mathcal{B}}(x \odot y) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$;
- (vi) $\mu_{\mathcal{B}}(x \vee y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$.

Proof. By Lemma 1 (ii) $x \odot y \leq x \wedge y \leq x \vee y$. Then, by definition of an IF-filter, $\nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y) \leq \nu_{\mathcal{B}}(x \odot y) \leq \nu_{\mathcal{B}}(x \wedge y) \leq \nu_{\mathcal{B}}(x \vee y)$ and $\mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y) \geq \mu_{\mathcal{B}}(x \odot y) \geq \mu_{\mathcal{B}}(x \wedge y) \geq \mu_{\mathcal{B}}(x \vee y)$. (i) and (vi) are proved. Applying Lemma 1 (iii), we have $\nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y) \leq \nu_{\mathcal{B}}(x \odot y) \leq \nu_{\mathcal{B}}(x \wedge y) \leq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$ and $\mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y) \geq \mu_{\mathcal{B}}(x \odot y) \geq \mu_{\mathcal{B}}(x \wedge y) \geq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$. The proofs for (ii), (iii), (iv) and (v) are finished. ■

Proposition 3. *An IF-set \mathcal{B} of a pseudo-BL-algebra A is an IF-filter of A if and only if it satisfies (IF1), (IF2) and*

(IF5) $\nu_{\mathcal{B}}(x \vee y) \geq \nu_{\mathcal{B}}(x)$ and $\mu_{\mathcal{B}}(x \vee y) \leq \mu_{\mathcal{B}}(x)$ for all $x, y \in A$.

Proof. \Rightarrow : Let us suppose that \mathcal{B} is an IF-filter of A . Then, by (IF3), $\nu_{\mathcal{B}}(x \vee y) \geq \nu_{\mathcal{B}}(x)$ and $\mu_{\mathcal{B}}(x \vee y) \leq \mu_{\mathcal{B}}(x)$ for all $x, y \in A$.

\Leftarrow : Conversely, let \mathcal{B} satisfies (IF1), (IF2) and (IF5). We need to show that \mathcal{B} satisfies (IF3). Let $x, y \in A$ be such that $x \leq y$. By (IF5) we have $\nu_{\mathcal{B}}(y) = \nu_{\mathcal{B}}(x \vee y) \geq \nu_{\mathcal{B}}(x)$ and $\mu_{\mathcal{B}}(y) = \mu_{\mathcal{B}}(x \vee y) \leq \mu_{\mathcal{B}}(x)$. Hence (IF3) is satisfied. ■

Theorem 1. *Let \mathcal{B} be an IF-set of a pseudo-BL-algebra A . The following are equivalent:*

(i) \mathcal{B} is an IF-filter;

(ii) \mathcal{B} satisfies (IF3) and for all $x, y \in A$

$$(1) \quad \nu_{\mathcal{B}}(y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(x \rightarrow y),$$

$$(2) \quad \mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(x \rightarrow y),$$

(iii) \mathcal{B} satisfies (IF3) and for all $x, y \in A$

$$(3) \quad \nu_{\mathcal{B}}(y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(x \rightsquigarrow y),$$

$$(4) \quad \mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(x \rightsquigarrow y).$$

Proof. Using Remark 4 of this paper, Proposition 3.3 and Corollary 3.4 of [13] and Theorem 3.3 of [11] we have the thesis. ■

Proposition 4. *Let \mathcal{B} be an IF-set of a pseudo-BL-algebra A . The following are equivalent:*

(i) \mathcal{B} is an IF-filter;

(ii) for all $x, y, z \in A$

$$(5) \quad x \rightarrow (y \rightarrow z) = 1 \Rightarrow \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y),$$

$$(6) \quad x \rightarrow (y \rightarrow z) = 1 \Rightarrow \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y).$$

(iii) for all $x, y, z \in A$

$$(7) \quad x \rightsquigarrow (y \rightsquigarrow z) = 1 \Rightarrow \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y),$$

$$(8) \quad x \rightsquigarrow (y \rightsquigarrow z) = 1 \Rightarrow \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y).$$

Proof. (i) \Rightarrow (ii) Suppose that \mathcal{B} is an IF-filter of a pseudo-BL-algebra A . Let $x, y, z \in A$ be such that $x \rightarrow (y \rightarrow z) = 1$. By Theorem 1 (ii)

$$(9) \quad \nu_{\mathcal{B}}(y \rightarrow z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(x \rightarrow (y \rightarrow z)) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(1) = \nu_{\mathcal{B}}(x),$$

$$(10) \quad \mu_{\mathcal{B}}(y \rightarrow z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(x \rightarrow (y \rightarrow z)) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(1) = \mu_{\mathcal{B}}(x).$$

Applying Theorem 1 (ii) the second time we obtain

$$(11) \quad \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(y) \wedge \nu_{\mathcal{B}}(y \rightarrow z),$$

$$(12) \quad \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(y) \vee \mu_{\mathcal{B}}(y \rightarrow z).$$

(9), (10), (11) and (12) force $\nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$ and $\mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$.

(ii) \Rightarrow (i) Let \mathcal{B} be an IF-set of a pseudo-BL-algebra A which satisfies (3). Let $x, y \in A$ be such that $x \leq y$. By Lemma 1 (iv) and (v),

$$x \rightarrow (x \rightarrow y) = 1,$$

hence applying (5) and (6) we have

$$\nu_{\mathcal{B}}(y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(x) = \nu_{\mathcal{B}}(x),$$

$$\mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(x) = \mu_{\mathcal{B}}(x),$$

that is, (IF3) holds.

Now we prove that (1) and (2) hold. By Lemma 1 (vi), $(x \rightarrow y) \rightarrow (x \rightarrow y) = 1$. Thus, applying (5) and (6) we get

$$\nu_{\mathcal{B}}(y) \geq \nu_{\mathcal{B}}(x \rightarrow y) \wedge \nu_{\mathcal{B}}(x) \text{ and}$$

$$\mu_{\mathcal{B}}(y) \leq \mu_{\mathcal{B}}(x \rightarrow y) \vee \mu_{\mathcal{B}}(x).$$

Hence by Theorem 1, \mathcal{B} is an IF-filter.

(iii) \Leftrightarrow (i) Analogously. ■

Proposition 5. *Let \mathcal{B} be an IF-set of a pseudo-BL-algebra A . The following are equivalent:*

(i) \mathcal{B} is an IF-filter;

(ii) for all $x, y, z \in A$

$$(x \odot y) \rightarrow z = 1 \Rightarrow \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y),$$

$$(x \odot y) \rightarrow z = 1 \Rightarrow \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y),$$

(iii) for all $x, y, z \in A$

$$\begin{aligned}(x \odot y) \rightsquigarrow z = 1 &\Rightarrow \nu_{\mathcal{B}}(z) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y), \\ (x \odot y) \rightsquigarrow z = 1 &\Rightarrow \mu_{\mathcal{B}}(z) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y).\end{aligned}$$

Proof. By Proposition 4 and Lemma 1 (vii). ■

Let $\mathcal{B}_i = (\nu_{\mathcal{B}_i}, \mu_{\mathcal{B}_i})$ be IF-filters of a pseudo-BL-algebra A for every $i \in I$. We define fuzzy sets $\bigwedge_{i \in I} \nu_{\mathcal{B}_i}$ and $\bigvee_{i \in I} \mu_{\mathcal{B}_i}$ as follows:

$$\begin{aligned}\left(\bigwedge_{i \in I} \nu_{\mathcal{B}_i}\right)(x) &= \bigwedge \{\nu_{\mathcal{B}_i}(x) : i \in I\}, \\ \left(\bigvee_{i \in I} \mu_{\mathcal{B}_i}\right)(x) &= \bigvee \{\mu_{\mathcal{B}_i}(x) : i \in I\}.\end{aligned}$$

For any IF-filters $\mathcal{B}_i = (\nu_{\mathcal{B}_i}, \mu_{\mathcal{B}_i})$ for $i \in I$, of a pseudo-BL-algebra A we define the IF-set $\bigcap_{i \in I} \mathcal{B}_i$ of A by

$$\bigcap_{i \in I} \mathcal{B}_i = \left(\bigwedge_{i \in I} \nu_{\mathcal{B}_i}, \bigvee_{i \in I} \mu_{\mathcal{B}_i}\right).$$

Theorem 2. Let $\mathcal{B}_i = (\nu_{\mathcal{B}_i}, \mu_{\mathcal{B}_i})$ for $i \in I$, be IF-filters of a pseudo-BL-algebra A . Then $\bigcap_{i \in I} \mathcal{B}_i$ is an IF-filter of A .

Proof. Let $\mathcal{B}_i = (\nu_{\mathcal{B}_i}, \mu_{\mathcal{B}_i})$ for $i \in I$, be IF-filters of a pseudo-BL-algebra A and $\mathcal{B} = \bigcap_{i \in I} \mathcal{B}_i = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$. We use Proposition 4 to show that \mathcal{B} is an IF-filter of A .

Let $x, y, z \in A$ be such that $x \rightarrow (y \rightarrow z) = 1$. Hence

$$\begin{aligned}\nu_{\mathcal{B}}(z) &= \bigwedge_{i \in I} \nu_{\mathcal{B}_i}(z) \geq \bigwedge_{i \in I} (\nu_{\mathcal{B}_i}(x) \wedge \nu_{\mathcal{B}_i}(y)) = \bigwedge_{i \in I} \nu_{\mathcal{B}_i}(x) \wedge \bigwedge_{i \in I} \nu_{\mathcal{B}_i}(y) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y), \\ \mu_{\mathcal{B}}(z) &= \bigvee_{i \in I} \mu_{\mathcal{B}_i}(z) \leq \bigvee_{i \in I} (\mu_{\mathcal{B}_i}(x) \vee \mu_{\mathcal{B}_i}(y)) = \bigvee_{i \in I} \mu_{\mathcal{B}_i}(x) \vee \bigvee_{i \in I} \mu_{\mathcal{B}_i}(y) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y).\end{aligned}$$

The proof is closed. ■

Remark 5. The set of IF-filters of a pseudo-BL-algebra A forms a complete distributive lattice with relation \leq .

Proof. Since $[0, 1]$ is a complete distributive lattice with usual ordering and by Theorem 2, the proof is completed. ■

Theorem 3. *A lattice of IF-filters of a pseudo-BL-algebra A is bounded.*

Proof. It is easily seen that 0_{\sim} and 1_{\sim} are IF-filters. Since $0_{\sim} \leq \mathcal{B} \leq 1_{\sim}$ for every IF-filter \mathcal{B} , then a lattice of IF-filters is bounded. ■

Theorem 4. *The lattice of IF-filters of a pseudo-BL-algebra has no atoms.*

Proof. Let \mathcal{B} be an IF-filter of pseudo-BL-algebra A and $\mathcal{B} \neq 0_{\sim}$. Let us define an IF-set \mathcal{D} as follows

$$\mathcal{D} = \left(\frac{1}{2}\nu_{\mathcal{B}}, \frac{1}{2}\mu_{\mathcal{B}} \right).$$

It is obvious that \mathcal{D} is an IF-filter of A and $0_{\sim} < \mathcal{D} < \mathcal{B}$. Hence there are no atoms in a lattice of IF-filters of A . ■

Let \mathcal{B} be an IF-set of a pseudo-BL-algebra A and $\alpha, \beta \in [0, 1]$ be such that $\alpha + \beta \leq 1$. Then we can define a set

$$A_{\mathcal{B}}^{(\alpha, \beta)} = \{x \in A : \nu_{\mathcal{B}}(x) \geq \alpha, \mu_{\mathcal{B}}(x) \leq \beta\}$$

called an (α, β) –level of \mathcal{B} .

Let us notice that $A_{\mathcal{B}}^{(\alpha, \beta)} = U(\nu_{\mathcal{B}}, \alpha) \cap L(\mu_{\mathcal{B}}, \beta)$.

Theorem 5. Let \mathcal{B} be an IF-set of a pseudo-BL-algebra A . If \mathcal{B} is an IF-filter of A , then $A_{\mathcal{B}}^{(\alpha, \beta)} = \emptyset$ or $A_{\mathcal{B}}^{(\alpha, \beta)}$ is a filter of A for all $\alpha \in [0, \nu_{\mathcal{B}}(1)]$, $\beta \in [\mu_{\mathcal{B}}(1), 1]$ such that $\alpha + \beta \leq 1$.

Proof. By Theorem 3.10 of [13] and Theorem 3.6 of [11] $\nu_{\mathcal{B}}$ is a fuzzy filter and $\mu_{\mathcal{B}}$ is an anti fuzzy filter iff $U(\nu_{\mathcal{B}}, \alpha)$ and $L(\mu_{\mathcal{B}}, \beta)$ are filters or empty. According to fact that the intersection of filters is a filter and by Remark 4 we have the thesis. ■

Corollary 1. *If \mathcal{B} is an IF-filter of a pseudo-BL-algebra A , then the set*

$$A_b = \{x \in A : \nu_{\mathcal{B}}(x) \geq \nu_{\mathcal{B}}(b), \mu_{\mathcal{B}}(x) \leq \mu_{\mathcal{B}}(b)\}$$

is a filter of A for every $b \in A$ such that $\nu_{\mathcal{B}}(b) + \mu_{\mathcal{B}}(b) \leq 1$.

4. PRIME IF-FILTERS

In this section we introduce and study prime IF-filters and their connection with pseudo-BL-chains.

Definition 9. An IF-filter $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ of a pseudo-BL-algebra A is said to be prime IF-filter if $\nu_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$ are non-constant and satisfies following conditions for all $x, y \in A$:

$$\nu_{\mathcal{B}}(x \vee y) = \nu_{\mathcal{B}}(x) \vee \nu_{\mathcal{B}}(y) \text{ and } \mu_{\mathcal{B}}(x \vee y) = \mu_{\mathcal{B}}(x) \wedge \mu_{\mathcal{B}}(y).$$

Remark 6. An IF-filter $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ of a pseudo-BL-algebra A is said to be prime IF-filter iff $\nu_{\mathcal{B}}$ is a fuzzy prime filter and $\mu_{\mathcal{B}}$ is an anti fuzzy prime filter of A .

Theorem 6. Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ be a non-constant IF-filter of a pseudo-BL-algebra A . Then the following are equivalent:

(i) \mathcal{B} is a prime IF-filter of A ;

(ii) for all $x, y \in A$, if $(\nu_{\mathcal{B}}(x \vee y) = \nu_{\mathcal{B}}(1) \text{ and } \mu_{\mathcal{B}}(x \vee y) = \mu_{\mathcal{B}}(1))$, then

$$\begin{aligned} &(\nu_{\mathcal{B}}(x) = \nu_{\mathcal{B}}(1) \text{ or } \nu_{\mathcal{B}}(y) = \nu_{\mathcal{B}}(1)) \text{ and} \\ &(\mu_{\mathcal{B}}(x) = \mu_{\mathcal{B}}(1) \text{ or } \mu_{\mathcal{B}}(y) = \mu_{\mathcal{B}}(1)); \end{aligned}$$

(iii) for all $x, y \in A$,

$$\begin{aligned} &(\nu_{\mathcal{B}}(x \rightarrow y) = \nu_{\mathcal{B}}(1) \text{ or } \nu_{\mathcal{B}}(y \rightarrow x) = \nu_{\mathcal{B}}(1)) \text{ and} \\ &(\mu_{\mathcal{B}}(x \rightarrow y) = \mu_{\mathcal{B}}(1) \text{ or } \mu_{\mathcal{B}}(y \rightarrow x) = \mu_{\mathcal{B}}(1)); \end{aligned}$$

(iv) for all $x, y \in A$,

$$\begin{aligned} &(\nu_{\mathcal{B}}(x \rightsquigarrow y) = \nu_{\mathcal{B}}(1) \text{ or } \nu_{\mathcal{B}}(y \rightsquigarrow x) = \nu_{\mathcal{B}}(1)) \text{ and} \\ &(\mu_{\mathcal{B}}(x \rightsquigarrow y) = \mu_{\mathcal{B}}(1) \text{ or } \mu_{\mathcal{B}}(y \rightsquigarrow x) = \mu_{\mathcal{B}}(1)). \end{aligned}$$

Proof. By Theorem 4.1 of [11] and Theorem 4.3 of [13]. ■

Theorem 7. Let A be a pseudo-BL-algebra and \mathcal{B} be an IF-filter of A . Then \mathcal{B} is a prime IF-filter iff $M_{\nu_{\mathcal{B}}}$ and $A_{\mu_{\mathcal{B}}}$ are prime filters of A .

Proof. By Remark 3. ■

Theorem 8. Let A be a pseudo-BL-algebra, P be a filter of A and $\alpha, \beta \in [0, 1]$ with $\alpha > \beta$. Then P is a prime filter of A if and only if $\mathcal{B}(P) = (\mu_P(\alpha, \beta), \mu_P^C(1 - \alpha, 1 - \beta))$ define as in Example 1, is a prime IF-filter of A .

Proof. By Theorem 4.2 of [11] and Theorem 4.6 of [13]. ■

Theorem 9. *Let \mathcal{B} be an IF-set of a pseudo-BL-algebra A such that $\nu_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$ are non-constant. Then the following are equivalent:*

- (i) \mathcal{B} is a prime IF-filter of A ;
- (ii) for every $\alpha \in [0, 1]$, if $U(\nu_{\mathcal{B}}, \alpha), L(\mu_{\mathcal{B}}, \alpha) \neq \emptyset$ and $U(\nu_{\mathcal{B}}, \alpha), L(\mu_{\mathcal{B}}, \alpha) \neq A$, then $U(\nu_{\mathcal{B}}, \alpha), L(\mu_{\mathcal{B}}, \alpha)$ are prime filters of A .

Proof. By Theorem 4.4 of [11] and Theorem 4.7 of [13]. ■

Theorem 10. *Let A be a non-trivial pseudo-BL-algebra. The following are equivalent:*

- (i) A is a pseudo-BL-chain;
- (ii) every IF-filter \mathcal{B} such that $\nu_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$ are non-constant is a prime IF-filter of A ;
- (iii) every IF-filter \mathcal{B} such that $\nu_{\mathcal{B}}$ and $\mu_{\mathcal{B}}$ are non-constant, $\nu_{\mathcal{B}}(1) = 1$ and $\mu_{\mathcal{B}}(1) = 0$ is a prime IF-filter of A ;
- (iv) the IF-filter $(\chi_{\{1\}}, \chi_{\{1\}}^C)$ is a prime IF-filter of A .

Proof. By Theorem 4.6 of [11] and Theorem 4.9 of [13]. ■

5. HOMOMORPHISM AND IF-FILTERS

Let A, B be pseudo-BL-algebras. Following [3] we define a homomorphism of pseudo-BL-algebras as a mapping $h : A \rightarrow B$ such that the following conditions hold for all $x, y \in A$:

- (H1) $h(x \odot y) = h(x) \odot h(y)$;
- (H2) $h(x \rightarrow y) = h(x) \rightarrow h(y)$;
- (H3) $h(x \rightsquigarrow y) = h(x) \rightsquigarrow h(y)$;
- (H4) $h(0) = 0$.

Recall that if $h : A \rightarrow B$ is a homomorphism of pseudo-BL-algebras, then

- (H5) $h(1) = 1$;
- (H6) $h(x \wedge y) = h(x) \wedge h(y)$;

$$(H7) \quad h(x \vee y) = h(x) \vee h(y).$$

Definition 10. Let \mathcal{B} be an IF-filter of a pseudo-BL-algebra B and $f : A \rightarrow B$ be a homomorphism of pseudo-BL-algebras. The preimage of \mathcal{B} is the IF-set $\mathcal{B}^f = (\nu_{\mathcal{B}}^f, \mu_{\mathcal{B}}^f)$ defined by

$$\nu_{\mathcal{B}^f}^f(x) = \nu_{\mathcal{B}}(f(x)) \text{ and } \mu_{\mathcal{B}^f}^f(x) = \mu_{\mathcal{B}}(f(x))$$

for all $x \in A$.

Theorem 11. Let \mathcal{B} be an IF-filter of B and $f : A \rightarrow B$ be a homomorphism of pseudo-BL-algebras. Then \mathcal{B}^f is an IF-filter of A .

Proof. Suppose that $f : A \rightarrow B$ is a homomorphism of pseudo-BL-algebras and \mathcal{B} be an IF-filter of B . Let $x, y \in A$. Then

$$\begin{aligned} \nu_{\mathcal{B}^f}^f(x \odot y) &= \nu_{\mathcal{B}}(f(x \odot y)) = \nu_{\mathcal{B}}(f(x) \odot f(y)) \\ &\geq \nu_{\mathcal{B}}(f(x)) \wedge \nu_{\mathcal{B}}(f(y)) = \nu_{\mathcal{B}^f}^f(x) \wedge \nu_{\mathcal{B}^f}^f(y) \end{aligned}$$

and

$$\begin{aligned} \mu_{\mathcal{B}^f}^f(x \odot y) &= \mu_{\mathcal{B}}(f(x \odot y)) = \mu_{\mathcal{B}}(f(x) \odot f(y)) \\ &\leq \mu_{\mathcal{B}}(f(x)) \vee \mu_{\mathcal{B}}(f(y)) = \mu_{\mathcal{B}^f}^f(x) \vee \mu_{\mathcal{B}^f}^f(y). \end{aligned}$$

Hence (IF1) and (IF2) hold.

Now let $x, y \in A$ be such that $x \leq y$. Therefore,

$$\begin{aligned} \nu_{\mathcal{B}^f}^f(x) &= \nu_{\mathcal{B}^f}^f(x \wedge y) = \nu_{\mathcal{B}}(f(x \wedge y)) \\ &= \nu_{\mathcal{B}}(f(x) \wedge f(y)) \leq \nu_{\mathcal{B}}(f(y)) = \nu_{\mathcal{B}^f}^f(y) \end{aligned}$$

and

$$\begin{aligned} \mu_{\mathcal{B}^f}^f(x) &= \mu_{\mathcal{B}^f}^f(x \wedge y) = \mu_{\mathcal{B}}(f(x \wedge y)) \\ &= \mu_{\mathcal{B}}(f(x) \wedge f(y)) \geq \mu_{\mathcal{B}}(f(y)) = \mu_{\mathcal{B}^f}^f(y). \end{aligned}$$

Thus, (IF3) holds.

Concluding, \mathcal{B}^f is an IF-filter of A . ■

Theorem 12. Let \mathcal{B} be an IF-set of B , \mathcal{B}^f be an IF-filter of A , where $f : A \rightarrow B$ is an epimorphism of pseudo-BL-algebras. Then \mathcal{B} is an IF-filter of A .

Proof. Let $f : A \rightarrow B$ be an epimorphism of pseudo-BL-algebras. Then, for any $x, y \in B$, there exist $a, b \in A$ such that $x = f(a)$ and $y = f(b)$. Therefore,

$$\begin{aligned}\nu_{\mathcal{B}}(x \odot y) &= \nu_{\mathcal{B}}(f(a) \odot f(b)) = \nu_{\mathcal{B}}(f(a \odot b)) \\ &= \nu_{\mathcal{B}}^f(a \odot b) \geq \nu_{\mathcal{B}}^f(a) \wedge \nu_{\mathcal{B}}^f(b) \\ &= \nu_{\mathcal{B}}(f(a)) \wedge \nu_{\mathcal{B}}(f(b)) = \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\mathcal{B}}(x \odot y) &= \mu_{\mathcal{B}}(f(a) \odot f(b)) = \mu_{\mathcal{B}}(f(a \odot b)) \\ &= \mu_{\mathcal{B}}^f(a \odot b) \leq \mu_{\mathcal{B}}^f(a) \vee \mu_{\mathcal{B}}^f(b) \\ &= \mu_{\mathcal{B}}(f(a)) \vee \mu_{\mathcal{B}}(f(b)) = \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y).\end{aligned}$$

Hence (IF1) and (IF2) hold.

Now let $x, y \in B$ be such that $x \leq y$. Then, there exist $a, b \in A$ such that $x = f(a)$ and $y = f(b)$. Therefore,

$$\begin{aligned}\nu_{\mathcal{B}}(x) &= \nu_{\mathcal{B}}(x \wedge y) = \nu_{\mathcal{B}}(f(a) \wedge f(b)) = \nu_{\mathcal{B}}(f(a \wedge b)) \\ &= \nu_{\mathcal{B}}^f(a \wedge b) \leq \nu_{\mathcal{B}}^f(b) = \nu_{\mathcal{B}}(f(b)) = \nu_{\mathcal{B}}(y)\end{aligned}$$

and

$$\begin{aligned}\mu_{\mathcal{B}}(x) &= \mu_{\mathcal{B}}(x \wedge y) = \mu_{\mathcal{B}}(f(a) \wedge f(b)) = \mu_{\mathcal{B}}(f(a \wedge b)) \\ &= \mu_{\mathcal{B}}^f(a \wedge b) \geq \mu_{\mathcal{B}}^f(b) = \mu_{\mathcal{B}}(f(b)) = \mu_{\mathcal{B}}(y).\end{aligned}$$

Thus, (IF3) holds.

Concluding, \mathcal{B} is an IF-filter of B . ■

Now let us denote the set of all filters of pseudo-BL-algebra A by $Fil(A)$ and the set of all IF-filters of A by $IFil(A)$. Let $\alpha \in (0, 1)$. We define maps $f_{\alpha} : IFil(A) \rightarrow Fil(A) \cup \{\emptyset\}$ and $g_{\alpha} : IFil(A) \rightarrow Fil(A) \cup \{\emptyset\}$ by

$$\begin{aligned}f_{\alpha}(\mathcal{B}) &= U(\nu_{\mathcal{B}}, \alpha), \\ g_{\alpha}(\mathcal{B}) &= L(\mu_{\mathcal{B}}, \alpha)\end{aligned}$$

for all $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}}) \in IFil(A)$.

Theorem 13. *For any $\alpha \in (0, 1)$, the maps f_{α} and g_{α} are surjective from $IFil(A)$ onto $Fil(A) \cup \{\emptyset\}$.*

Proof. It is obvious, that

$$f_\alpha(0_\sim) = U(0, \alpha) = \emptyset = L(1, \alpha) = g_\alpha(0_\sim).$$

Now let $\emptyset \neq F \in \text{Fil}(A)$. Then (χ_F, χ_F^C) is an IF-filter of A . Hence,

$$f_\alpha((\chi_F, \chi_F^C)) = U(\chi_F, \alpha) = F = L(\chi_F^C, \alpha) = g_\alpha((\chi_F, \chi_F^C)).$$

Therefore, f_α and g_α are surjective. ■

6. DIRECT PRODUCT OF IF-FILTERS

Let us define a direct product $\prod_{i \in I}^n A_i$ of pseudo-BL-algebras as usually.

Definition 11. Let A be a pseudo-BL-algebra. Then we define an IF-relation on A as a mapping $\mathcal{R} = (\nu'_\mathcal{R}, \mu'_\mathcal{R}) : A \times A \rightarrow [0, 1] \times [0, 1]$ such that $\nu'_\mathcal{R}(x, y) + \mu'_\mathcal{R}(x, y) \leq 1$ for all $x, y \in A$.

Now define a direct product of IF-sets of pseudo-BL-algebra A .

Definition 12. Let $\mathcal{B} = (\nu_\mathcal{B}, \mu_\mathcal{B})$ and $\mathcal{G} = (\nu_\mathcal{G}, \mu_\mathcal{G})$ be IF-sets of A . We define a direct product $\mathcal{B} \times \mathcal{G}$ by

$$\mathcal{B} \times \mathcal{G} = (\nu_\mathcal{B}, \mu_\mathcal{B}) \times (\nu_\mathcal{G}, \mu_\mathcal{G}) = (\nu_\mathcal{B} \times \nu_\mathcal{G}, \mu_\mathcal{B} \times \mu_\mathcal{G}),$$

where $(\nu_\mathcal{B} \times \nu_\mathcal{G})(x, y) = \nu_\mathcal{B}(x) \wedge \nu_\mathcal{G}(y)$ and $(\mu_\mathcal{B} \times \mu_\mathcal{G})(x, y) = \mu_\mathcal{B}(x) \vee \mu_\mathcal{G}(y)$ for all $x, y \in A$.

Proposition 6. Let $\mathcal{B} = (\nu_\mathcal{B}, \mu_\mathcal{B})$ and $\mathcal{G} = (\nu_\mathcal{G}, \mu_\mathcal{G})$ be IF-sets of a pseudo-BL-algebra A , then $\mathcal{B} \times \mathcal{G}$ is an IF-set of $A \times A$.

Proof. Let \mathcal{B}, \mathcal{G} be IF-sets of A . Then for every $x \in A$ we have $\nu_\mathcal{B}(x) + \mu_\mathcal{B}(x) \leq 1$ and $\nu_\mathcal{G}(x) + \mu_\mathcal{G}(x) \leq 1$. Suppose that $\nu_\mathcal{B}(x) \leq \nu_\mathcal{G}(y)$ for some $x, y \in A$. Then $(\nu_\mathcal{B} \times \nu_\mathcal{G})(x, y) = \nu_\mathcal{B}(x) \wedge \nu_\mathcal{G}(y) = \nu_\mathcal{B}(x)$. Let us consider two cases:

Case 1. $\mu_\mathcal{B}(x) \leq \mu_\mathcal{G}(y)$

Hence $(\mu_\mathcal{B} \times \mu_\mathcal{G})(x, y) = \mu_\mathcal{B}(x) \vee \mu_\mathcal{G}(y) = \mu_\mathcal{G}(y)$ and then $(\nu_\mathcal{B} \times \nu_\mathcal{G})(x, y) + (\mu_\mathcal{B} \times \mu_\mathcal{G})(x, y) = \nu_\mathcal{B}(x) + \mu_\mathcal{G}(y) \leq \nu_\mathcal{G}(y) + \mu_\mathcal{G}(y) \leq 1$.

Case 2. $\mu_\mathcal{B}(x) > \mu_\mathcal{G}(y)$

Therefore $(\mu_\mathcal{B} \times \mu_\mathcal{G})(x, y) = \mu_\mathcal{B}(x) \vee \mu_\mathcal{G}(y) = \mu_\mathcal{B}(x)$ and then $(\nu_\mathcal{B} \times \nu_\mathcal{G})(x, y) + (\mu_\mathcal{B} \times \mu_\mathcal{G})(x, y) = \nu_\mathcal{B}(x) + \mu_\mathcal{B}(x) \leq 1$. Hence $\mathcal{B} \times \mathcal{G}$ is an IF-set of $A \times A$.

Analogously when $\nu_\mathcal{B}(x) > \nu_\mathcal{G}(y)$. ■

Now we give a trivial Proposition without a proof:

Proposition 7. *Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ and $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$ be IF-sets of a pseudo-BL-algebra A , then*

- (i) $\mathcal{B} \times \mathcal{G}$ is an IF-relation of A ;
- (ii) $U(\nu_{\mathcal{B} \times \mathcal{G}}; \alpha) = U(\nu_{\mathcal{B}}; \alpha) \times U(\nu_{\mathcal{G}}; \alpha)$ and $L(\mu_{\mathcal{B} \times \mathcal{G}}; \alpha) = L(\mu_{\mathcal{B}}; \alpha) \times L(\mu_{\mathcal{G}}; \alpha)$ for all $\alpha \in [0, 1]$.

Theorem 14. *Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ and $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$ be IF-filters of a pseudo-BL-algebra A . Then $\mathcal{B} \times \mathcal{G}$ is an IF-filter of $A \times A$.*

Proof. Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ and $\mathcal{G} = (\nu_{\mathcal{G}}, \mu_{\mathcal{G}})$ be IF-filters of a pseudo-BL-algebra A . Suppose that $x, y \in A$. Then by (IF1) and (IF2), $\nu_{\mathcal{B}}(x \odot y) \geq \nu_{\mathcal{B}}(x) \wedge \nu_{\mathcal{B}}(y)$, $\nu_{\mathcal{G}}(x \odot y) \geq \nu_{\mathcal{G}}(x) \wedge \nu_{\mathcal{G}}(y)$ and $\mu_{\mathcal{B}}(x \odot y) \leq \mu_{\mathcal{B}}(x) \vee \mu_{\mathcal{B}}(y)$, $\mu_{\mathcal{G}}(x \odot y) \leq \mu_{\mathcal{G}}(x) \vee \mu_{\mathcal{G}}(y)$. Let $(x_1, x_2), (y_1, y_2) \in A \times A$. Then,

$$\begin{aligned} (\nu_{\mathcal{B} \times \mathcal{G}})((x_1, x_2) \odot (y_1, y_2)) &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x_1 \odot y_1, x_2 \odot y_2) \\ &= \nu_{\mathcal{B}}(x_1 \odot y_1) \wedge \nu_{\mathcal{G}}(x_2 \odot y_2) \\ &\geq \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{G}}(x_2) \wedge \nu_{\mathcal{G}}(y_2) \\ &= (\nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{G}}(x_2)) \wedge (\nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{G}}(y_2)) \\ &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x_1, x_2) \wedge (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(y_1, y_2). \end{aligned}$$

Similarly, we can prove that $(\mu_{\mathcal{B} \times \mathcal{G}})((x_1, x_2) \odot (y_1, y_2)) \leq (\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(x_1, x_2) \vee (\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(y_1, y_2)$.

It is proved that (IF1) and (IF2) hold.

Now let $(x_1, x_2), (y_1, y_2) \in A \times A$ be such that $(x_1, x_2) \leq (y_1, y_2)$. Then

$$\begin{aligned} (\nu_{\mathcal{B} \times \mathcal{G}})(x_1, x_2) &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})((x_1, x_2) \wedge (y_1, y_2)) \\ &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(x_1 \wedge y_1, x_2 \wedge y_2) \\ &= \nu_{\mathcal{B}}(x_1 \wedge y_1) \wedge \nu_{\mathcal{G}}(x_2 \wedge y_2) \\ &\leq \nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{G}}(y_2) \\ &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{G}})(y_1, y_2). \end{aligned}$$

and similarly $(\mu_{\mathcal{B} \times \mathcal{G}})(x_1, x_2) \geq (\mu_{\mathcal{B}} \times \mu_{\mathcal{G}})(y_1, y_2)$.

The proof is completed. ■

Theorem 15. *Let $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ be IF-set of a pseudo-BL-algebra A . Then \mathcal{B} is an IF-filter of A if and only if $\mathcal{B} \times \mathcal{B}$ is an IF-filter of $A \times A$.*

Proof. \Rightarrow : By Theorem 14.

\Leftarrow : Let $\mathcal{B} \times \mathcal{B}$ be an IF-filter of $A \times A$. Let $(x_1, x_2), (y_1, y_2) \in A \times A$. Hence

$$\begin{aligned} \nu_{\mathcal{B}}(x_1 \odot y_1) \wedge \nu_{\mathcal{B}}(x_2 \odot y_2) &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(x_1 \odot y_1, x_2 \odot y_2) \\ &= (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})((x_1, x_2) \odot (y_1, y_2)) \\ &\geq (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(x_1, x_2) \wedge (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(y_1, y_2) \\ &= \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(x_2) \wedge \nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{B}}(y_2). \end{aligned}$$

Putting $x_1 = x_2$ and $y_1 = y_2$ we have

$$\nu_{\mathcal{B}}(x_1 \odot y_1) \geq \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(y_1) \wedge \nu_{\mathcal{B}}(y_1) = \nu_{\mathcal{B}}(x_1) \wedge \nu_{\mathcal{B}}(y_1).$$

Similarly, $\mu_{\mathcal{B}}(x_1 \odot y_1) \leq \mu_{\mathcal{B}}(x_1) \vee \mu_{\mathcal{B}}(y_1)$.

Let $x, y \in A$ be such that $x \leq y$. Then by (IF3),

$$\nu_{\mathcal{B}}(x) = (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(x, x) \leq (\nu_{\mathcal{B}} \times \nu_{\mathcal{B}})(y, y) = \nu_{\mathcal{B}}(y).$$

Analogously, $\mu_{\mathcal{B}}(x) \geq \mu_{\mathcal{B}}(y)$.

Hence $\mathcal{B} = (\nu_{\mathcal{B}}, \mu_{\mathcal{B}})$ is an IF-filter of A . ■

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