

ON THE SUBSEMIGROUP GENERATED BY ORDERED IDEMPOTENTS OF A REGULAR SEMIGROUP

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Abstract

An element e of an ordered semigroup S is called an ordered idempotent if $e \leq e^2$. Here we characterize the subsemigroup $\langle E_{\leq}(S) \rangle$ generated by the set of all ordered idempotents of a regular ordered semigroup S . If S is a regular ordered semigroup then $\langle E_{\leq}(S) \rangle$ is also regular. If S is a regular ordered semigroup generated by its ordered idempotents then every ideal of S is generated as a subsemigroup by ordered idempotents.

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1. INTRODUCTION

Idempotents play an important role in the theory of semigroups as well as in ring theory. Particularly, in case of different major subclasses of the regular semigroups S , the set $E(S)$ of all idempotents of S is like the nucleus of a cell, that possesses several properties of S in an encrypted form. Unfortunately, the set of all idempotents $E(S)$ of a semigroup (without order) does not form a subsemigroup in general. A regular semigroup S such that $E(S)$ is a subsemigroup is called orthodox. Hall [3] – [5], Meakin [10, 11], Yamada [13] and many others like Milles [12], McAlister [9] have studied orthodox semigroups and characterized such semigroup S by $E(S)$. Another approach, introduced by C. Eberhart, W. Williams and L. Kinch [2] is to study a semigroup S by the subsemigroup generated by $E(S)$. They considered the subsemigroup $\langle E \rangle = \bigcup_{n=1}^{\infty} E^n$. (where $E = E(S)$ is not necessarily a subsemigroup of S) and established a connection between the regularity of S and $\langle E \rangle$. T.E. Hall [5] studied subsemigroups of

an idempotent generated regular semigroup. He showed that a regular semigroup is generated by its idempotents if and only if each principal factor is generated by its own idempotents.

In [1], we introduced the notion of ordered idempotents and characterized ordered semigroups S such that every element is an ordered idempotent. If T is a subsemigroup of S , then the set of ordered regular elements of T is denoted by $Reg_{\leq}(T)$ [7]. Thus, if $T = \langle E_{\leq}(S) \rangle$ then $Reg_{\leq}(T) = T = Reg_{\leq}(S) \cap T$, in general. In [7], Hansda proved several equivalent conditions so that $Reg_{\leq}(T) = T = Reg_{\leq}(S) \cap T$ for $T = (Se], (eS]$ and $(eSf]$, where e, f are ordered idempotents. It is the purpose of this paper to characterize an ordered semigroup S by the subsemigroup generated by ordered idempotents of S . We show that in a regular ordered semigroup S the subsemigroup $\langle E_{\leq}(S) \rangle$ generated by all ordered idempotents of S is also regular. Similar result holds for completely regular ordered semigroups.

The article is organized as follows. The basic definitions and properties of ordered semigroups are presented in Section 2. Section 3 is devoted to characterize the regular ordered semigroups generated by their ordered idempotents.

2. PRELIMINARIES

In this paper \mathbb{N} denotes the set of all natural numbers. An ordered semigroup S is a partially ordered set, and at the same time a semigroup (S, \cdot) such that $(\forall a, b, x \in S) a \leq b \Rightarrow xa \leq xb$ and $ax \leq bx$. It is denoted by (S, \cdot, \leq) . For an ordered semigroup S and $\emptyset \neq H \subseteq S$, denote

$$(H) = \{t \in S : t \leq h, \text{ for some } h \in H\}.$$

H is called downward closed if $H = (H]$.

Let I be a nonempty subset of an ordered semigroup S . I is called a left (right) ideal of S , if $SI \subseteq I$ ($IS \subseteq I$) and $(I) = I$. I is an ideal of S if it is both a left and a right ideal of S . S is left (right) simple if it has no non-trivial proper left (right) ideal. Similarly we define simple ordered semigroups. S is called a t-simple ordered semigroup if it is both left and right simple.

An element $a \in S$ is called ordered regular if $a \leq axa$ for some $x \in S$. If every element of an ordered semigroup is ordered regular then S is called regular. Thus S is a regular ordered semigroup if and only if $a \in (aSa]$ for all $a \in S$. An element $b \in S$ is said to be an ordered inverse of a if $a \leq aba$ and $b \leq bab$. In a regular ordered semigroup every element has an ordered inverse. For, if $a \leq axa$ then $a \leq a(xax)a$ and $xax \leq (xax)a(xax)$ shows that xax is an ordered inverse of a . We denote the set of all ordered inverse of a by $V_{\leq}(a)$. If A is a nonempty subset of S , then we denote $\cup_{a \in A} V_{\leq}(a)$ by $V_{\leq}(A)$. An element $e \in S$ is defined to

be an ordered idempotent if $e \leq e^2$ [1]. Ordered idempotents take a determining role in characterizing regular ordered semigroups [7], completely regular ordered semigroups, Clifford ordered semigroups [1], etc. If $a \leq axa$ then both ax and xa are ordered idempotents. The set of all ordered idempotents of S is denoted by $E_{\leq}(S)$. Kehayopulu [8] defined an ordered semigroups S to be completely regular if $a \in (a^2Sa^2)$ for all $a \in S$. Thus an ordered semigroup S is a completely regular ordered semigroup if and only if for every $a \in S$, $a \leq a^2xa^2$ for some $x \in S$.

3. REGULAR ORDERED SEMIGROUPS GENERATED BY ORDERED IDEMPOTENTS

In this section we show that the subsemigroup generated by all ordered idempotents in a regular ordered semigroup S is always a regular ordered subsemigroup.

We denote the subsemigroup generated by the set $E_{\leq}(S)$ of all ordered idempotents of S by $\langle E_{\leq}(S) \rangle$ or simply by $\langle E_{\leq} \rangle$. Let E_{\leq}^n be the set of all elements of S which can be written as the product of n (not necessarily distinct) ordered idempotents of S . Then $\langle E_{\leq}(S) \rangle = \bigcup_{n=1}^{\infty} E_{\leq}^n$.

Lemma 3.1. *Let S be a regular ordered semigroup and $n \in \mathbb{N}$ be such that $n > 1$*

- (1) *Then $x \in E_{\leq}^n$ implies that $V_{\leq}(x) \cap E_{\leq}^{n-1} \neq \phi$.*
- (2) *Let E_{\leq}^n be downward closed in S for every $n \in \mathbb{N}$. Then $x \in E_{\leq}^n$ if and only if $V_{\leq}(x) \cap E_{\leq}^{n-1} \neq \phi$.*

Proof. (1) We prove this results by induction on n . Consider $x \in E_{\leq}^2$. Then for some $e_1, e_2 \in E_{\leq}$, $x = e_1e_2$. Let $y \in V_{\leq}(x)$. Then $x \leq xyx$ and $y \leq yxy$. Now take $f = e_2ye_1$. Then it follows that $f = e_2ye_1 \leq e_2yxye_1 \leq e_2ye_1e_2ye_1 = f^2$. Thus $f \in E_{\leq}(S)$.

Now $x \leq xyx$ implies that $x \leq e_1e_2e_2ye_1e_1e_2 = xfx$. Furthermore $f = e_2ye_1 \leq e_2yxye_1 \leq e_2ye_1(e_1e_2)e_2ye_1 = fxf$. Hence $f \in V_{\leq}(x) \cap E_{\leq}$.

Suppose that the result holds for all $k < n$. Let $x \in E_{\leq}^n$. Then $x = e_1e_2 \dots e_n$ where $e_1, e_2, \dots, e_n \in E_{\leq}(S)$. Let $y = e_2 \dots e_n$. Then $y \in E_{\leq}^{n-1}$. So by the induction hypothesis we have $z \in V_{\leq}(y) \cap E_{\leq}^{n-2}$. Consider $w \in V_{\leq}(x)$. Let $f = z(e_2 \dots e_nwe_1)$. Now $e_2 \dots e_nwe_1 \leq e_2 \dots e_n(wxw)e_1 \leq (e_2 \dots e_nwe_1)(e_2 \dots e_nwe_1)$, since $x = e_1e_2 \dots e_n$. Thus $e_2 \dots e_nwe_1 \in E_{\leq}(S)$. Since $z \in E_{\leq}^{n-2}$, so $f = z(e_2 \dots e_nwe_1) \in E_{\leq}^{n-1}$.

Now we have

$$\begin{aligned}
f &= z(e_2 \dots e_n w e_1) \\
&\leq z[e_2 \dots e_n (x w) e_1] && [w \in V_{\leq}(x)] \\
&\leq z[(e_2 \dots e_n w)(e_1 e_2 \dots e_n) w e_1] && [x = e_1 e_2 \dots e_n] \\
&\leq z(e_2 \dots e_n w e_1)(e_2 \dots e_n z e_2 \dots e_n) w e_1 && [z \in V_{\leq}(e_2 e_3 \dots e_n)] \\
&\leq z(e_2 \dots e_n w e_1) e_1 e_2 \dots e_n (z e_2 \dots e_n) w e_1 && [e_1 \leq e_1^2] \\
&\leq f(e_1 e_2 \dots e_n) f && [f = z(e_2 \dots e_n w e_1)] \\
&\leq f x f.
\end{aligned}$$

Since $w \in V_{\leq}(x)$, so $x \leq x w x$. This implies that $x \leq e_1 e_2 \dots e_n w x \leq e_1 y w e_1 (e_1 e_2 \dots e_n) \leq e_1 (y w e_1) x \leq e_1 (z y w e_1) x \leq e_1 e_2 \dots e_n z e_2 \dots e_n w e_1 x \leq x (z e_2 \dots e_n w e_1) x = x f x$. Thus $f \in V_{\leq}(x) \cap E_{\leq}^{n-1}$, that is, $V_{\leq}(x) \cap E_{\leq}^{n-1} \neq \phi$. So the result follows by induction.

(2) The necessary part follows from (1).

Let $V_{\leq}(x) \cap E_{\leq}^{n-1} \neq \phi$. Then there is $y \in V_{\leq}(x) \cap E_{\leq}^{n-1}$. Now $y \in E_{\leq}^{n-1}$ implies that there is $z \in V_{\leq}(y) \cap E_{\leq}^{n-2}$, by (1). Then $x \leq x y x \leq (x y) z (y x)$. Now $x y, y x \in E_{\leq}(S)$ and $z \in E_{\leq}^{n-2}$ implies that $x y z y x \in E_{\leq}^n$; and hence $x \in (E_{\leq}^n) = E_{\leq}^n$, since E_{\leq}^n is downward closed. Thus the result follows. ■

Now we show that regularity of an ordered semigroup S ensures the same for $\langle E_{\leq}(S) \rangle$.

Theorem 3.2. *Let S be a regular ordered semigroup and E_{\leq}^n is downward closed in S for every $n > 1$. Then $V_{\leq}(E_{\leq}^{n-1}) = E_{\leq}^n$; and hence the subsemigroup $\langle E_{\leq} \rangle$ of S generated by the ordered idempotents of S is also regular.*

Proof. Let $z \in V_{\leq}(E_{\leq}^{n-1})$. Then there is $w \in E_{\leq}^{n-1}$ such that $z \in V_{\leq}(w)$, which again implies that $w \in E_{\leq}^{n-1} \cap V_{\leq}(z)$. Then $z \in E_{\leq}^n$, by Lemma 3.1 and so $V_{\leq}(E_{\leq}^{n-1}) \subseteq E_{\leq}^n$.

Now let $x \in E_{\leq}^n$. Then $V_{\leq}(x) \cap E_{\leq}^{n-1} \neq \phi$. Consider $y \in V_{\leq}(x) \cap E_{\leq}^{n-1}$. Then $x \in V_{\leq}(y)$ implies that $x \in V_{\leq}(E_{\leq}^{n-1})$. Thus $V_{\leq}(E_{\leq}^{n-1}) \subseteq E_{\leq}^n$. Hence $V_{\leq}(E_{\leq}^{n-1}) = E_{\leq}^n$. This completes the proof. ■

The following lemma has been given in [1]. For the sake of completeness, we would like to include a short proof here also.

Lemma 3.3. *Let S be a completely regular ordered semigroup. Then for every $a \in S$ there is $a' \in V_{\leq}(a)$ such that $a \leq a a' a$, $a \leq a^2 a'$, and $a \leq a' a^2$.*

Proof. Let $a \in S$. Then there is $t \in S$ such that $a \leq a^2 t a^2$. Then $a \leq a^3 t a^2 t a^2 \leq a^3 t a^2 t a^2 t a^3 \leq a a' a$, where $a' = a^2 t a^2 t a^2 t a^2$. Similarly we have $a' \leq a' a a'$. Thus $a' \in V_{\leq}(a)$. Also $a \leq a^2 t a^2 \leq a^4 t a^2 t a^2 t a^2 = a^2 a'$. Similarly it can be shown that $a \leq a' a^2$. ■

Now we have the following results on completely regular ordered semigroups.

Theorem 3.4. *Let S be a completely regular ordered semigroup and E_{\leq}^n is downward closed in S , for every $n \in \mathbb{N}$. Then $\langle E_{\leq}(S) \rangle$ is a completely regular ordered subsemigroup.*

Proof. Consider $a \in \langle E_{\leq}(S) \rangle$. Then $a = e_1 e_2 \dots e_m$ for some $m \in \mathbb{N}$, and $e_1, e_2, \dots, e_m \in E_{\leq}(S)$. Since a is completely regular, there is $h \in V_{\leq}(a)$ such that $a \leq aha$, $a \leq a^2 h$ and $a \leq ha^2$, by Lemma 3.3. Now $a \in V_{\leq}(h) \cap E_{\leq}^m$ implies that $h \in E_{\leq}^{m+1}$, by Lemma 3.1 and hence $h^3 \in \langle E_{\leq}(S) \rangle$. Then $a \leq a^2 h^3 a^2$ implies that $\langle E_{\leq}(S) \rangle$ is completely regular. ■

Following Eberhart, Williams and Kinch [2] let us define an equivalence relation γ on S in the following way: for $a, b \in S$,

$$a \gamma b \text{ if and only if there exists a sequence } x_1, x_2, \dots, x_n \text{ of elements of } S \text{ such that } x_1 \in V_{\leq}(a), x_i \in V(x_{i-1}); i = 2, \dots, n \text{ and } b \in V(x_n).$$

In the following theorem we show that $\langle E_{\leq}(S) \rangle$ is the union of all γ -equivalence classes of ordered idempotents.

Theorem 3.5. *Let S be a regular ordered semigroup and E_{\leq}^n be downward closed in S for every $n \in \mathbb{N}$. Then for any $x \in S$, $x \in \langle E_{\leq} \rangle$ if and only if there is an ordered idempotent $e \in E_{\leq}$ such that $x \gamma e$.*

Proof. Consider $x \in \langle E_{\leq} \rangle$. Then $x \in E_{\leq}^m$; for some $m \in \mathbb{N}$. If $m = 1$, then the result follows trivially. Let $m > 1$. Then $V_{\leq}(x) \cap E_{\leq}^{m-1} \neq \phi$, by Lemma 3.1. Consider $x_1 \in V_{\leq}(x) \cap E_{\leq}^{m-1}$. Repeated application of this process yields a sequence of elements x_1, x_2, \dots, x_{m-1} of S such that $x_1 \in V_{\leq}(x)$, $x_i \in V(x_{i-1}) \cap E_{\leq}^{m-i}$ for $i = 2, \dots, m-1$, whence $x_{m-1} \in E_{\leq}$. Say $e = x_{m-1}$. Thus $x \gamma e$.

Conversely assume that $x \in S$ and there is $e \in E_{\leq}(S)$ such that $x \gamma e$. Then there are elements $x_1, x_2, \dots, x_n \in S$ such that $x_1 \in V_{\leq}(x), x_i \in V_{\leq}(x_{i-1})$ and $e \in V(x_n)$ so that $x \leq x x_1 x \leq (x x_1)(x_2 x_1) x \leq \dots \leq (x x_1) x_2 \dots x_n e x_n \dots x_2 x_1 x$ which can be rearranged as:

$$(3.1) \quad x \leq \begin{cases} (x x_1)(x_2 x_3) \dots (x_n e)(e x_n) \dots (x_3 x_2)(x_1 x) & \text{if } n \text{ is even} \\ (x x_1)(x_2 x_3) \dots (x_{n-1} x_n) e (x_n x_{n-1}) \dots (x_3 x_2)(x_1 x) & \text{if } n \text{ is odd.} \end{cases}$$

Since $x x_1, x_1 x, x_n e, e x_n$ and $x_i x_{i-1}, x_{j-1} x_j (i = 2, \dots, n; j = 2, \dots, n)$ are all ordered idempotents of S , so $x \in (E_{\leq}^{n+2}(S))$ in both the cases. This implies that $x \in E_{\leq}^{n+2}(S)$ and hence $x \in \langle E_{\leq}(S) \rangle$. ■

Definition 3.6. An ordered semigroup S is said to be generated by the ordered idempotents if $S = \langle E_{\leq}(S) \rangle$.

Let T be a subsemigroup of S and $A \subseteq S$. We say that T is generated as an ordered subsemigroup by A if for every $t \in T$ there are $n \in \mathbb{N}$ and $a_1, a_2, \dots, a_n \in A$ such that $t \leq a_1 a_2 \dots a_n$. It is denoted by $T = \langle A \rangle_{\leq}$. Now we show that every ideal I of an ordered idempotent generated ordered semigroup is also generated as an ordered subsemigroup by the ordered idempotents $E_{\leq}(I)$.

Theorem 3.7. Let S be a regular ordered semigroup generated by its ordered idempotents and for every $n > 1$, let E_{\leq}^n be downward closed in S . Then every ideal of S is also a regular ordered semigroup generated by ordered idempotents as an ordered subsemigroup.

Proof. Suppose that $S = \langle E_{\leq}(S) \rangle$. Consider an ideal I of S and $x \in I$. Since $x \in \langle E_{\leq}(S) \rangle$, so $x\gamma e$ for some $e \in E_{\leq}(S)$, by Theorem 3.5; and hence there is a sequence of elements $x_1, x_2, \dots, x_n \in S$ such that $x_1 \in V_{\leq}(x)$, $x_i \in V_{\leq}(x_{i-1})$; $i = 2, 3, \dots, n$ and $e \in V(x_n)$. Then it follows that $x \leq x x_1 x \leq x x_1 x_2 x_1 x \leq x(x_1 x_2 \dots x_n) e(x_n \dots x_2 x_1) x$ which can be rearranged as:

$$(3.2) \quad x \leq \begin{cases} x((x_1 x_2)(x_2 x_3) \dots (x_{n-1} x_n) e(x_n x_{n-1}) \dots (x_3 x_2)(x_2 x_1)) x; & \text{if } n \text{ is even} \\ x((x_1 x_2)(x_2 x_3) \dots (x_n e)(e x_n) \dots (x_3 x_2)(x_2 x_1)) x; & \text{if } n \text{ is odd.} \end{cases}$$

Since I is an ideal of S , $x x_1, x_1 x \in I$. Now for $r = 1, 2, \dots, n - 1$; $x_{r+1} x_r \leq x_{r+1} x_r x_{r+1} x_r \leq x_{r+1} x_r (x_{r-1} x_r) x_{r+1} x_r \leq \dots \leq x_{r+1} x_r \dots x_2 (x_1 x) x_1 x_2 \dots x_{r+1} x_r$. Since $x_1 x \in I$ the above inequality implies that $x_{r+1} x_r \in I$ for $r = 1, 2, \dots, n - 1$. Similarly $e, x_n e, e x_n$ and $x_r x_{r+1}$ for $r = 1, 2, \dots, n - 1$ all belong to I . Also for $r = 1, 2, \dots, n - 1$ the elements $x_r x_{r+1}, x_{r+1} x_r, x_n e$ and $e x_n$ are all ordered idempotents in S . Thus $x_r x_{r+1}, x_{r+1} x_r, x_n e$ and $e x_n \in E_{\leq}(I)$ for all $r = 1, 2, \dots, n - 1$. So $(x_1 x_2)(x_2 x_3) \dots (x_{n-1} x_n) e(x_n x_{n-1}) \dots (x_3 x_2)(x_2 x_1)$ and $(x_1 x_2)(x_2 x_3) \dots (x_n e)(e x_n) \dots (x_3 x_2)(x_2 x_1) \in \langle E_{\leq}(I) \rangle$. Therefore from (3.2) it follows that $x \in \langle E_{\leq}(I) \rangle_{\leq}$. Hence $\langle E_{\leq}(I) \rangle_{\leq} = I$, in other words I is generated as an ordered subsemigroup by its ordered idempotents. Also every ideal of a regular ordered semigroup is also regular. This completes the proof. ■

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