

## SEMIGROUPS DERIVED FROM $(\Gamma, N)$ -SEMIHYPERGROUPS AND $T$ -FUNCTOR

S. OSTADHADI-DEHKORDI

*Department of Mathematics, Hormozgan University*  
*Bandar Abbas, Iran*

**e-mail:** Ostadhadi@hormozgan.ac.ir

### Abstract

The main purpose of this paper is to introduce the concept of  $(\Gamma, n)$ -semihypergroups as a generalization of hypergroups, as a generalization of  $n$ -ary hypergroups and obtain an exact covariant functor between the category  $(\Gamma, n)$ -semihypergroups and the category semigroups. Moreover, we introduce and study complete part. Finally, we obtain some new results and some fundamental theorems in this respect.

**Keywords:**  $(\Gamma, n)$ -semihypergroup,  $\Theta$ -relation,  $T$ -factor, fundamental semigroup.

**2010 Mathematics Subject Classification:** 20N15.

### 1. INTRODUCTION

Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. The hypergroup notion was introduced in 1934 by the French mathematician F. Marty [16], at the 8<sup>th</sup> Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non commutative groups. Since then, hundreds of papers and several books have been written on this topic and several kinds of hypergroups have been intensively studied, such as: regular hypergroups, reversible regular hypergroups, canonical hypergroups, cogroups, cyclic hypergroups, associativity hypergroups.

A recent book on hyperstructures [4] points out on their applications in fuzzy and rough set theory, cryptography, codes, automata, probability, geometry, lattices, binary relations, graphs and hypergraphs. Hypergraph theory is a useful

toll for discrete optimization problems. A comprehensive review of the theory of hypergraph appears in [2].

Let  $H$  be a nonempty set and  $\circ : H \times H \longrightarrow \wp^*(H)$ , be a map such that  $\wp^*(H)$  be the set of all nonempty subset of  $H$ . The couple  $(H, \circ)$  is called *hypergroupoid*. If  $A$  and  $B$  are nonempty subset of  $H$ , then we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad x \circ A = \{x\} \circ A, \quad A \circ x = A \circ \{x\}.$$

A hypergroupoid  $(H, \circ)$  is called a *semihypergroup* if for all  $x, y, z \in H$ , we have  $x \circ (y \circ z) = (x \circ y) \circ z$ . A semihypergroup  $(H, \circ)$  is called *hypergroup* if for every  $x \in H$ , we have  $x \circ H = H \circ x = H$ . Several books have been written on hyperstructure theory, see [3, 4, 7]. A *regular hypergroup*  $(H, \circ)$  is a hypergroup which has at least an identity and any element of  $H$  has at least an inverse. In other words, there exists  $e \in H$ , such that for all  $x \in H$ , we have  $x \in x \circ e \cap e \circ x$  and there exists  $x' \in H$  such that  $e \in x \circ x' \cap x' \circ x$ .

An *n-ary structure* generalizations of algebraic structures is the most natural way for further development and deeper understanding of their fundamental properties. The notion of *n-ary group*, which is a generalization of the notion of a group, was introduced by W. Dörnte in 1928 [10]. Since then many papers concerning various *n-ary algebra* have appeared in the literature [8, 9, 11, 12].

The notion of  $\Gamma$ -*semigroups* was introduced by Sen in [17, 18]. Let  $G$  and  $\Gamma$  be two nonempty sets. Then,  $G$  is called a  $\Gamma$ -*semigroup* if there exists a mapping  $G \times \Gamma \times G \longrightarrow G$ , written  $(a, \alpha, b)$  by  $a\alpha b$ , such that it satisfies the identities  $a\alpha(b\beta c) = (a\alpha b)\beta c$ , for all  $a, b, c \in G$  and  $\alpha, \beta \in \Gamma$ . The concept of  $\Gamma$ -*semihypergroups* was introduced by Davvaz *et al.* [13]. Let  $G$  and  $\Gamma$  be two nonempty sets. Then,  $G$  is called a  $\Gamma$ -*semihypergroup* if each  $\alpha \in \Gamma$  be a hyperoperation on  $G$ , i.e.,  $a\alpha b \subseteq G$ , and for every  $a, b, c \in G$ , and for every  $\alpha, \beta \in \Gamma$  we have the associative property that is  $a\alpha(b\beta c) = (a\alpha b)\beta c$ . Let  $G_1$  be a  $\Gamma_1$ -*semihypergroup* and  $G_2$  be a  $\Gamma_2$ -*semigroup*. If there exists a map  $\varphi : G_1 \longrightarrow G_2$  and a bijection  $f : \Gamma_1 \longrightarrow \Gamma_2$  such that  $\varphi(x\alpha y) \subseteq \varphi(x)f(\alpha)\varphi(y)$ , for every  $x, y \in G_1$  and  $\alpha \in \Gamma_1$ , then  $\varphi$  a *homomorphism* between  $G_1$  and  $G_2$ .

In 1964, Nobusawa introduced  $\Gamma$ -*rings* as a generalization of rings. Barnes [1] weakened slightly the conditions in the definition of  $\Gamma$ -ring in the sense of Nobusawa. Barnes [1], Luh [15] and Kyuno [14] studied the structure of  $\Gamma$ -rings and obtained various generalization analogous to corresponding parts in ring theory. After that, Dehkordi *et. al.* [5, 6] investigated the ideals, rough ideals, homomorphisms and regular relations of  $\Gamma$ -*semihyperrings*.

The aim of this research work is to define a new class of *n-ary multialgebras* that we call  $(\Gamma, n)$ -*semihypergroups* that is a generalization of *n-ary semihypergroups*, a generalization of  $\Gamma$ -*semihypergroups*, a generalization of *semihypergroups* and a generalization of *semigroups*. Also, we define complete part and

regular relation. Moreover, we introduce an exact covariant functor between the category  $(\Gamma, n)$ -semihypergroups and the category semigroups.

## 2. $(\Gamma, \mathbb{N})$ -SEMIHYPERGROUP

In this section, we present some definitions and results concerning. First of all, let us introduced  $(\Gamma, n)$ -semihypergroup. Let  $G, \Gamma$  be nonempty sets and  $n \in \mathbb{N}$ ,  $n \geq 2$ . A map  $\alpha : G^n \rightarrow \wp^*(G)$  is called  $n$ -ary hyperoperation on  $G$ , where  $\wp^*(G)$  is the set of all nonempty subsets of  $G$  and  $\alpha \in \Gamma$ . Then,  $(G, \Gamma)$  is called  $(\Gamma, n)$ -hypergroupoid. If  $G_1, G_2, \dots, G_n$  are subsets of  $G$ , then we define

$$\alpha(G_1, G_2, \dots, G_n) = \bigcup \{ \alpha(x_1, x_2, \dots, x_n) : x_i \in G_i, 1 \leq i \leq n \},$$

$$\Gamma(G_1, G_2, \dots, G_n) = \bigcup \{ \alpha(x_1, x_2, \dots, x_n) : x_i \in G_i, \alpha \in \Gamma, 1 \leq i \leq n \}.$$

The sequence  $x_i, x_{i+1}, \dots, x_j$ , will be denoted by  $x_i^j$ . For  $j \leq i$ ,  $x_i^j$  is empty. In the case when  $x_{i+1} = \dots = x_j = x$  will be written in the form  $x^{j-i}$ .

A  $(\Gamma, n)$ -hypergroupoid is called  $(\Gamma, n)$ -semihypergroup if for every  $\alpha, \beta \in \Gamma$  and  $x_1, x_2, \dots, x_{2n-1} \in G$

$$\alpha \left( x_1^{i-1}, \beta \left( x_1^{n+i-1} \right), x_{n+i}^{2n-1} \right) = \beta \left( x_1^{j-1}, \alpha \left( x_j^{n+j-1} \right), x_{n+j}^{2n-1} \right).$$

A  $(\Gamma, n)$ -hypergroupoid  $(G, \Gamma)$  in which for every  $\alpha \in \Gamma$  the equation

$$y \in \alpha \left( y_1^{i-1}, x_i, y_{i+1}^n \right),$$

has the solution  $x_i \in G$  for every  $y_1^{i-1}, y_{i+1}^n, y \in G$  is called  $(\Gamma, n)$ -quasihypergroup. A  $(\Gamma, n)$ -hypergroup is both a  $(\Gamma, n)$ -semihypergroup and  $(\Gamma, n)$ -quasihypergroup. A  $(\Gamma, n)$ -hypergroup  $G$  is commutative if for every  $x_1^n$  of  $G$  and any permutation  $\delta$  of  $\{1, 2, \dots, n\}$  and for all  $\alpha \in \Gamma$  we have

$$\alpha(x_1^n) = \alpha(x_{\delta(1)}, x_{\delta(2)}, \dots, x_{\delta(n)}).$$

An element  $e$  of a  $(\Gamma, n)$ -hypergroup  $G$  is called an  $n$ -ary identity or a neutral element, if there exist  $\alpha \in \Gamma$  such that

$$x = \alpha(e^{i-1}, x, e^{n-1}).$$

Let  $G$  be a  $(\Gamma, n)$ -semihypergroup and  $\alpha \in \Gamma$  be a fixed element. We define  $f(a_1, a_2, \dots, a_n) = \alpha(a_1, a_2, \dots, a_n)$ . It is easy to see that  $(G, f)$  is an  $n$ -ary semihypergroup and when  $n = 2$ ,  $(G, f)$  is a semihypergroup. We denote this  $n$ -ary semihypergroup by  $G[\alpha]$ .

**Proposition 2.1.** *Let  $G$  be a  $(\Gamma, n)$ -semihypergroup and for every  $\alpha \in \Gamma$ , the element  $e \in G$  be neutral element. Then, for every  $\alpha_1, \alpha_2 \in \Gamma$  and  $x_1^n \in G$ , we have  $\alpha_1(x_1^n) = \alpha_2(x_1^n)$ .*

**Proof.** Suppose that  $e$  is a neutral element for every  $\alpha \in \Gamma$ . Then for  $x_1^n \in G$ , we have  $x_1 = \alpha(x_1, e^{n-1})$  and  $x_1 = \beta(x_1, e^{n-1})$ . Hence

$$\alpha(x_1, x_2^n) = \alpha(\beta(x_1, e^{n-1}), x_2^n) = \beta(x_1, \alpha(x_2, e^{n-1}), x_3^n) = \beta(x_1, x_2, x_3^n).$$

This completes the proof. ■

By Proposition 2.1, if for every  $\alpha, \beta \in \Gamma$ ,  $e$  is a neutral element, then  $G[\alpha] = G[\beta]$ . This implies that  $(\Gamma, n)$ -semihypergroup  $G$  is an  $n$ -ary hypergroup.

**Definition 2.2.** Let  $(G, \Gamma)$  be a  $(\Gamma, n)$ -hypergroup and  $H$  be a nonempty subset of  $G$ . We say that  $H$  is a  $(\Gamma, n)$ -subhypergroup of  $G$  if following conditions hold:

1. For every  $\alpha \in \Gamma$ ,  $H$  is closed under the  $n$ -ary hyperoperation  $\alpha$ ,
2. For all  $x_0, x_1, \dots, x_n \in H$ ,  $\alpha \in \Gamma$  and fixed  $i \in \{1, 2, \dots, n\}$  there exists  $x \in H$  such that  $x_0 \in \alpha(x_1^{i-1}, x, x_{i+1}^n)$ .

**Definition 2.3.** A nonempty subset  $I$  of a  $(\Gamma, n)$ -semihypergroup is said to be a  $k$ -ideal of  $G$  if

1.  $I$  is a  $(\Gamma, n)$ -subsemihypergroup of  $G$ ,
2.  $\Gamma(G_1^{k-1}, I, G_{k+1}^n) \subseteq I$ .

If for every  $1 \leq k \leq n$ ,  $I$  is a  $k$ -ideal, then we say that  $I$  is an ideal.

**Definition 2.4.** Let  $G$  be a semigroup and  $I$  be a nonempty subset of  $G$ . We say that  $I$  is a left ideal if  $I$  is a subsemigroup of  $G$  and  $GI \subseteq I$ . In the same way can define right ideal.

**Definition 2.5.** Let  $G_1$  and  $G_2$  be  $(\Gamma_1, n)$  and  $(\Gamma_2, n)$ -semihypergroup, respectively. A map  $(\varphi, f) : G_1 \times \Gamma_1 \rightarrow G_2 \times \Gamma_2$  is called a homomorphism if for every  $x_1^n \in G_1$

$$\varphi(\alpha(x_1^n)) = f(\alpha)(\varphi(x_1^n)).$$

Also, if  $\varphi$  and  $f$  are onto, then  $(\varphi, f)$  is called an epimorphism.

**Example 1.** Let  $G$  be a group and  $\Gamma = \{\alpha_n : n \in \mathbb{N}\}$ . Then, for every  $x_1^n \in G$ , we define

$$\alpha_n(x_1^n) = G.$$

Then,  $G$  is a  $(\Gamma, n)$ -hypergroup.

**Example 2.** Let  $X$  be a totally ordered set and  $\Gamma$  be a nonempty subset of  $X$ . We define

$$\alpha(x_i^n) = \{x \in G : x \geq \max\{x_i^n, \alpha\}\},$$

for every  $\alpha \in \Gamma$  and  $x_i^n \in X$ . Then,  $(X, \Gamma)$  is a  $(\Gamma, n)$ -semihypergroup.

**Example 3.** Let  $H$  be a semigroup and  $\{X_h\}_{h \in H}$  be a collection of disjoint sets. Consider  $G = \bigcup_{h \in H} X_h$  and  $\Gamma = Z(H)$ . For every  $g \in G$  there exist  $h \in H$  such that  $g \in X_h$ . We define

$$\alpha(x_i^n) = X_{\alpha x_1, x_2, \dots, x_n},$$

where  $x_i \in X_{h_i}$ , for  $1 \leq i \leq n$ . Then  $G$  is a  $(\Gamma, n)$ -hypergroup and is called  $(Z(H), n)$ -hypergroup.

**Example 4.** Let  $A_n = [n, n+1)$ ,  $\Gamma_1 = 2\mathbb{Z}$ ,  $\Gamma_2 = 2\mathbb{Z}+1$  and  $G_1, G_2$  be  $(Z(2\mathbb{Z}), n)$ ,  $(Z(2\mathbb{Z}+1), n)$ -semihypergroup, respectively. Then,  $(\varphi, f)$  is a homomorphism.

$$\varphi : G_1 \longrightarrow G_2, \quad \varphi(x) = x + 1$$

$$f : \Gamma_1 \longrightarrow \Gamma_2, \quad f(\alpha) = \alpha.$$

**Example 5.** Let  $(H, \circ)$  be a hypergroup and  $\Gamma \subseteq H$  be a nonempty set. We define for every  $x_i \in H$  and  $\alpha \in \Gamma$

$$\alpha(x_i) = \alpha \circ x_1 \circ \dots \circ x_n.$$

Then  $H$  is a  $(\Gamma, n)$ -semihypergroup.

**Example 6.** Let  $G$  be a group and  $H_n$  be a normal subgroups of  $G$  such that  $H_n \subseteq H_{n+1}$ . We define  $n$ -ary hyperoperation on  $G$  as follows:

$$\alpha_n(x_i) = H_n \circ x_1 \circ x_2, \dots, \circ x_n.$$

Then  $G$  is a  $(\Gamma, n)$ -hypergroup.

### 3. FUNDAMENTAL RELATION AND COMPLETE PART

By using a certain type of equivalence relations, we can connect  $(\Gamma, n)$ -semihypergroup to semigroups and  $(\Gamma, n)$ -hypergroups to groups. These equivalence relations are called strong regular relations. More exactly, starting with a  $(\Gamma, n)$ -semihypergroup (hypergroup) and using a strong regular relation, we can construct semigroup (group). In this section, we introduce a strong regular relation  $\beta^*$  and complete part such that has an important role in the study of  $(\Gamma, n)$ -semihypergroups.

Let  $G$  be a  $(\Gamma, n)$ -hypergroup. We define

$$\begin{aligned}\alpha_{1[1]} &= \{\alpha_1(x_1^n) : x_i \in G, 1 \leq i \leq n\} \\ \alpha_{2[2]} &= \{\alpha_2(x_1^{i-1}, \alpha_{1[1]}, x_{i+1}^n), x_i \in G, 2 \leq i \leq n\} \\ \alpha_{3[3]} &= \{\alpha_3(x_1^{i-1}, \alpha_{2[2]}, x_{i+1}^n), x_i \in G, 2 \leq i \leq n\} \\ &\vdots \\ \alpha_{n[n]} &= \{\alpha_n(x_1^{i-1}, \alpha_{n-1[n-1]}, x_{i+1}^n), x_i \in G, 2 \leq i \leq n\},\end{aligned}$$

for every  $\alpha_1, \alpha_2, \dots, \alpha_n \in \Gamma$ . Let  $\mathcal{U} = \bigcup_{k \geq 1, \alpha \in \Gamma} \mathcal{U}_{k[\alpha]}$ . We define

$$x\beta_n y \Leftrightarrow \exists \alpha_{n[n]} \in \mathcal{U}, \text{ such that } \{x, y\} \subseteq \alpha_{n[n]}.$$

We have  $\beta = \bigcup_{n \geq 1} \beta_n$  is reflexive and symmetric. Let  $\beta^*$  be the transitive closure of  $\beta$ . This relation is called *fundamental relation*.

Let  $G$  be a  $(\Gamma, n)$ -semihypergroup and  $\rho$  be an equivalence relation on  $G$ . If  $A$  and  $B$  are nonempty subset of  $G$ , then

$$\begin{aligned}A\bar{\rho}B &\iff \forall a \in A, \exists b \in B \text{ such that } a\rho b \\ &\quad \forall b \in B, \exists a \in A \text{ such that } a\rho b.\end{aligned}$$

and

$$A\bar{\bar{\rho}}B \iff \forall a \in A, \text{ and } b \in B, a\bar{\bar{\rho}}b.$$

The equivalence relation  $\rho$  is called *k-regular* if from  $a\rho b$ , it follows that

$$\alpha(x_1^{k-1}, a, x_{k+1}^n) \bar{\rho} \alpha(x_1^{k-1}, b, x_{k+1}^n),$$

for every  $\alpha \in \Gamma$  and is called *k-strongly regular* if from  $a\rho b$ ,

$$\alpha(x_1^{k-1}, a, x_{k+1}^n) \bar{\bar{\rho}} \alpha(x_1^{k-1}, b, x_{k+1}^n).$$

for every  $\alpha \in \Gamma$ .  $\rho$  is called *regular (strongly regular)* if it is *k-regular (strongly regular)* for every  $1 \leq k \leq n$ .

**Proposition 3.1.** *Let  $G$  be a  $(\Gamma, n)$ -semihypergroup and  $\beta^*$  be a fundamental relation on  $G$ . Then,  $\beta^*$  is the smallest strongly regular relation on  $G$ .*

**Proof.** Suppose that  $a\beta^*b$  and  $x$  is an arbitrary element of  $G$ . It follows that there exists  $x_0 = a, x_1, \dots, x_n = b$  such that for every  $i \in \{0, 1, \dots, n-1\}$  such that  $x_i\beta x_{i+1}$ . Let  $u_1 \in \alpha(a, y_2^n)$  and  $u_2 \in \alpha(b, y_2^n)$ . It follows that there exist  $\xi_{n[n]}$  such that  $\{x_i, y_2^n\} \subseteq \xi_{n[n]}$ . Hence  $\alpha(x_i, y_2^n) \subseteq \alpha(\xi_{n[n]}, y_2^n)$  and  $\alpha(x_{i+1}, y_2^n)$

$\subseteq \alpha(\xi_{n[n]}, y_2^n)$ . Thus  $\alpha(x_i, y_2^n) \bar{\beta} \alpha(x_{i+1}, y_2^n)$ . This implies that for every  $i \in \{0, 1, 2, \dots, n-1\}$  and for all  $t_i \in \alpha(x_i, y_2^n)$ , we have  $t_i \beta t_{i+1}$ . If we consider  $t_0 = u_1$  and  $t_n = u_2$ , then we obtain  $u_1 \beta^* u_2$ . Then  $\beta^*$  is 1-strongly regular and similarly, it is  $j$ -strongly regular for  $2 \leq j \leq n$ .

Let  $R$  be a strongly regular relation on  $G$ . Since  $R$  is reflexive, we have  $\beta \subseteq R$ . Suppose that  $\beta_{n-1} \subseteq R$ . If  $a \beta_n b$ , then  $\{a, b\} \subseteq \alpha_{n[n]}$ . Since  $\alpha_{n[n]} = \alpha(y_1^{i-1}, \xi_{n-1[n-1]}, y_{i+1}^n)$  there exists  $u, v \in \xi_{n-1[n-1]}$  such that  $a \in \alpha_n(y_1^{i-1}, u, y_{i+1}^n)$  and  $b \in \alpha_n(y_1^{i-1}, v, y_{i+1}^n)$  and according to the hypothesis since  $u \beta_{n-1} v$ , we have  $u R v$ . Since  $R$  is strongly regular it follows that  $a R b$  and by induction, it follows that  $\beta \subseteq R$ , whence  $\beta^* \subseteq R$ .  $\blacksquare$

**Proposition 3.2.** *Let  $G$  be a  $(\Gamma, n)$ -semihypergroup and  $\rho$  be an equivalence relation on  $G$ . Then,  $\rho$  is regular if and only if  $[G : \rho]$  is a  $(\widehat{\Gamma}, n)$ -semihypergroup with respect the following operation:*

$$\widehat{\alpha}(\rho(a_1), \rho_2(a_2), \dots, \rho(a_n)) = \{\rho(a) : a \in \alpha(a_1, a_2, \dots, a_n)\}.$$

**Proof.** First we check that the hyperoperation  $\widehat{\alpha}$  is well defined. Let  $\rho(a_i) = \rho(b_i)$ , for  $1 \leq i \leq n$ . Then, we have  $a_i \rho b_i$ . Since  $\rho$  is regular, it follows that

$$\begin{aligned} \alpha(a_1, a_2, \dots, a_n) \bar{\rho} \alpha(b_1, a_2, \dots, a_n), \\ \alpha(b_1, a_2, \dots, a_n) \bar{\rho} \widehat{\alpha}(b_1, b_2, \dots, a_n), \\ \vdots \\ \alpha(b_1, b_2, \dots, a_{n-1}, a_n) \bar{\rho} \alpha(b_1, b_2, \dots, b_n). \end{aligned}$$

Hence for every  $u_1 \in \alpha(a_1, a_2, \dots, a_n)$  there exists  $u_2 \in \alpha(b_1, b_2, \dots, b_n)$  such that  $\rho(u_1) = \rho(u_2)$ . It follows that

$$\widehat{\alpha}(\rho(a_1), \rho_2(a_2), \dots, \rho(a_n)) \subseteq \widehat{\alpha}(\rho(b_1), \rho_2(b_2), \dots, \rho(b_n)),$$

and similarly we obtain the converse inclusion. Now, we check the associativity of  $n$ -ary hyperoperation  $\alpha$ . Let

$$\rho(u) \in \widehat{\alpha} \left( \rho(x_i)_{i=1}^{k-1}, \widehat{\beta}(\rho(y_i))_{i=1}^n, \rho(x_i)_{i=k+1}^n \right).$$

This means that there exists  $\rho(v) \in \widehat{\beta}(\rho(y_i))_{i=1}^n$  such that

$$\rho(u) \in \widehat{\alpha} \left( \rho(x_i)_{i=1}^{k-1}, \rho(v), \rho(x_i)_{i=k+1}^n \right).$$

Hence there exist  $u_1 \in \alpha(x_i^{k-1}, v, x_{i=k+1}^n)$  such that  $\rho(u) = \rho(u_1)$  and there exist  $v_1 \in \beta(y_i)_{i=1}^n$  such that  $\rho(v) = \rho(v_1)$ . Since  $\rho$  is regular there exist

$$u_2 \in \alpha(x_i^{k-1}, v_1, x_{k+1}^n) \subseteq \alpha(x_i^{k-1}, \beta(y_i)_1^n, x_{k+1}^n) = \beta(x_i^{k-1}, \alpha(y_i)_1^n, x_{k+1}^n),$$

such that  $\rho(u_2) = \rho(u)$ . Hence we obtain that there exists  $u_3 \in \alpha(y_i)_1^n$  such that  $u_2 \in \beta(x_i^{k-1}, u_3, x_{k+1}^n)$ . We have

$$\rho(u) = \rho(u_3) \in \widehat{\beta}\left(\rho(x_i)_{i=1}^{k-1}, \widehat{\alpha}(\rho(y_i))_{i=1}^n, \rho(x_i)_{i=k+1}^n\right).$$

It follows that

$$\widehat{\alpha}\left(\rho(x_i)_{i=1}^{k-1}, \widehat{\beta}(\rho(y_i))_{i=1}^n, \rho(x_i)_{i=k+1}^n\right) \subseteq \widehat{\beta}\left(\rho(x_i)_{i=1}^{k-1}, \widehat{\alpha}(\rho(y_i))_{i=1}^n, \rho(x_i)_{i=k+1}^n\right).$$

Similarly, we obtain the converse inclusion.

Let  $[G : \rho]$  be a  $(\widehat{\Gamma}, n)$ -semihypergroup,  $a\rho b$  and  $x_i \in G$ , for  $1 \leq i \leq n-1$ . Since  $\rho$  is well-defined. If  $u \in \alpha(x_i^{k-1}, a, x_{k+1}^n)$ , then

$$\begin{aligned} \rho(u) \in \widehat{\alpha}\left(\rho(x_1^{k-1}), \rho(a), \rho(x_{k+1}^n)\right) &= \widehat{\alpha}\left(\rho(x_1^{k-1}), \rho(b), \rho(x_{k+1}^n)\right) \\ &= \left\{ \rho(v) : v \in \alpha(x_1^{k-1}, b, x_{k+1}^n) \right\}. \end{aligned}$$

Hence there exists  $v \in \alpha(x_1^{k-1}, b, x_{k+1}^n)$  such that  $u\rho v$ , whence

$$\alpha(x_1^{k-1}, a, x_{k+1}^n) \bar{\rho} \alpha(x_1^{k-1}, b, x_{k+1}^n).$$

This completes the proof. ■

**Definition 3.3.** Let  $G$  be a  $(\Gamma, n)$ -semihypergroup and  $C$  be a nonempty subset of  $G$ . We say that  $C$  is an  $\alpha$ -complete part of  $G$  if for any nonzero number  $n$ , the following implication holds:

$$C \cap \alpha_{n[n]} \neq \emptyset \implies \alpha_{n[n]} \subseteq C.$$

If for every  $\alpha \in \Gamma$ ,  $C$  is an  $\alpha$ -complete part, then  $C$  is *complete part*.

**Example 7.** Let  $A_n = [n, n+1]$ ,  $\Gamma = \mathbb{Z}$ . Then,  $\mathbb{R}$  is a  $(\mathbb{Z}, n)$ -semihypergroup by  $n$ -ary hyperoperation defined in the Example 3. For every  $n \in \mathbb{Z}$ ,  $A_n$  is a complete part but  $C = \mathbb{N}$  is not complete part.



**Proposition 3.4.** *Let  $G$  be a  $(\Gamma, n)$ -semihypergroup and  $\rho$  is a strongly regular relation on  $G$ , then for every  $a \in G$ , the equivalence class  $\rho(a)$  is a complete part of  $G$ .*

**Proof.** Suppose that for  $n \in \mathbb{N}$ ,  $\rho(a) \cap \alpha_{n[n]} \neq \emptyset$ . This implies that there exists  $b \in \alpha_{n[n]}$  such that  $\rho(a) = \rho(b)$ . Let  $\pi : G \rightarrow [G : \rho]$  be a natural homomorphism. Then, we have

$$\begin{aligned} \pi(a) &= \pi(\alpha_{n[n]}) = \pi(\alpha(x_1^{i-1}, \beta_{n-1[(n-1)]}, x_{i+1}^n)) \\ &= \hat{\alpha}(\pi(x_1^{i-1}), \pi(\beta_{n-1[n-1]}), \pi(x_{i+1}^n)). \end{aligned}$$

This means that  $\alpha_{n[n]} \subseteq \pi(a) = \rho(a)$ . ■

**Definition 3.5.** Let  $A$  be a subset of  $G$ . Then, the smallest complete part of  $G$  that contain  $A$  denoted by  $C(A)$ .

Denote  $K_1^\alpha(A) = A$  and for every  $n \geq 1$  denote

$$K_{n+1}^\alpha(A) = \{x \in G : \exists m \in \mathbb{N}, x \in \alpha_{m[m]}, K_n^\alpha(A) \cap \alpha_{m[m]} \neq \emptyset\},$$

and  $K_\alpha(A) = \bigcup_{n \geq 1} K_n^\alpha(A)$ . Let  $K(A) = \bigcup_{\alpha \in \Gamma} K_\alpha(A)$ .

**Proposition 3.6.** *Let  $G$  be a  $(\Gamma, n)$ -hypergroup. Then*

- (1) *For every  $n \geq 2$ , we have  $K_n^\alpha(K_2^\alpha(x)) = K_{n+1}^\alpha(x)$ ,*
- (2) *the following relation is equivalence*

$$x \sim y \iff \exists n \geq 1, x \in K_n(y).$$

**Proof.** Suppose that  $n = 2$ . We have

$$\begin{aligned} K_2^\alpha(K_2^\alpha(x)) &= \{y \in G : \exists n \in \mathbb{N}, y \in \alpha_{n[n]}, K_2^\alpha(x) \cap \alpha_{n[n]} \neq \emptyset\} \\ &= K_3^\alpha(x). \end{aligned}$$

Let  $K_{n-1}^\alpha(K_2^\alpha(x)) = K_n^\alpha(x)$ . Then,

$$\begin{aligned} K_n^\alpha(K_2^\alpha(x)) &= \{y \in G : \exists n \in \mathbb{N}, y \in \alpha_{n[n]}, K_{n-1}^\alpha(K_2^\alpha(x)) \cap \alpha_{n[n]} \neq \emptyset\} \\ &= K_{n+1}^\alpha(x). \end{aligned}$$

(2) Suppose that  $n = 2$  and  $x \in K_2^\alpha(y)$ . Then,

$$x \in K_2^\alpha(y) = \{z \in G : \exists m \in \mathbb{N}, z \in \alpha_{m[m]}, K_1^\alpha(y) \cap \alpha_{m[m]} \neq \emptyset\}.$$

Hence  $\{x, y\} \subseteq \alpha_{n[n]}$  which implies that  $y \in K_2^\alpha(x)$ . Let

$$x \in K_{n-1}(y) \iff y \in K_{n-1}(x),$$

and  $x \in K_n^\alpha(y)$ . Then, there exists  $n \in \mathbb{N}$  such that  $x \in \alpha_{n[n]}$  and  $b \in \alpha_{n[n]} \cap K_{n-1}^\alpha(y)$ . This implies that  $b \in K_2^\alpha(x)$  and  $y \in K_{n-1}^\alpha(b)$ . Hence  $y \in K_{n-1}^\alpha(K_2^\alpha(x)) = K_n^\alpha(x)$ . In the same way, the converse implication holds. ■

**Proposition 3.7.** *Let  $G$  be a  $(\Gamma, n)$ -hypergroup and  $A$  be a nonempty subset of  $G$ . Then,  $C(A) = K(A)$ .*

**Proof.** Suppose that  $A$  is a nonempty subset of  $G$  and  $K(A) \cap \alpha_{n[n]} \neq \emptyset$ . Then there exist  $m \geq 1$  and  $\alpha \in \Gamma$  such that  $K_m^\alpha(A) \cap \alpha_{n[n]} \neq \emptyset$ . Hence  $\alpha_{n[n]} \subseteq K_{m+1}^\alpha(A)$ , which means that  $\alpha_{n[n]} \subseteq K(A)$ .

Let  $A \subseteq B$  and  $B$  is a complete part of  $G$ , then we show that  $K(A) \subseteq B$ . We have  $K_1^\alpha(A) \subseteq B$  and suppose that  $K_n^\alpha(A) \subseteq B$ . We check that  $K_{n+1}^\alpha(A) \subseteq B$ . Let  $b \in K_{n+1}^\alpha(A)$ . Then there exists  $m \in \mathbb{N}$ , such that  $b \in \alpha_{m[m]}$  and  $K_n^\alpha(A) \cap \alpha_{m[m]} \neq \emptyset$ . Hence  $B \cap \alpha_{m[m]} \neq \emptyset$  and we obtain  $\alpha_{m[m]} \subseteq B$ . Therefore,  $b \in B$  and  $K(A) = C(A)$ . ■

**Proposition 3.8.** *Let  $G$  be a  $(\Gamma, n)$ -hypergroup. Then, the relation  $\sim$  and  $\beta^*$  are coincide.*

**Proof.** Suppose that  $x\beta y$ . Then there exists  $\alpha \in \Gamma$  such that  $x \in K_2^\alpha(y) \subseteq K(y)$ . This implies that  $x \sim y$ . Now, if we have  $x \sim y$ , then there exists  $\alpha \in \Gamma$  and  $n \geq 1$  such that  $xK_{n+1}^\alpha y$  which implies that  $\alpha_{n[n]} \cap K_n^\alpha(y) \neq \emptyset$ . Let  $a \in \alpha_{n[n]} \cap K_n^\alpha(y)$ . Hence  $a\beta x$ . Since  $a_1 \in K_n^\alpha(y)$ , it follows that there exists  $\alpha_{n-1[n-1]}$  such that  $\alpha_{n-1[n-1]} \cap K_{n-1}^\alpha(y) \neq \emptyset$ . Let  $a_2 \in \alpha_{n-1[n-1]} \cap K_{n-1}^\alpha(y)$ . Hence  $a_1\beta a_2$  and  $a_2 \in K_{n-1}^\alpha(y)$ . After finite number of steps, we obtain that there exists  $a_{n-1}$  and  $a_n$  such that  $a_{n-1}\beta a_n$  and  $a_n \in K_{n-(n-1)}^\alpha(y) = \{y\}$ . Therefore,  $x\beta^* y$ . ■

#### 4. $\Theta$ RELATION AND $T$ FUNCTOR

The category  $CTH$  in which the objects are  $(\Gamma, n)$ -semihypergroups. For  $(\Gamma, n)$ -semihypergroups  $G_1$  and  $G_2$   $Mor(G_1, G_2)$ , are epimorphism from  $G_1$  to  $G_2$  and  $CG$  is the category of all semigroups. The purpose of this section is to introduce the concept of  $T$  - functor. First we shall present the fundamental definitions.

We denote the equivalence class of element  $x \in G$  by  $\beta^*(x)$ . Let

$$[G : \Gamma] = \{(\beta^*(x_i))_{i=1}^{n-1}, \alpha\}.$$

We define the relation  $\Theta$  as follows:

$$\left( (\beta^*(x_i))_{i=1}^{n-1}, \alpha \right), \left( (\beta^*(y_j))_{j=1}^{n-1}, \beta \right) \in \Theta$$

if and only if

$$\widehat{\alpha} \left( (\beta^*(x_i))_{i=1}^{n-1}, \beta^*(x) \right) = \widehat{\beta} \left( (\beta^*(y_j))_{j=1}^{n-1}, \beta^*(x) \right),$$

for every  $\beta^*(x) \in [G : \beta^*]$ . Obviously,  $\Theta$  is an equivalence relation. Let  $\Theta \left( (\beta^*(x_i))_{i=1}^{n-1}, \alpha \right)$  denote the equivalence class contain  $(\beta^*(x_i))_{i=1}^{n-1}, \alpha$ . Let  $\Delta[G]$  be the set of all equivalence classes on  $[G : \Gamma]$ . We define operation as follows:

$$\begin{aligned} & \Theta \left( (\beta^*(x_i))_{i=1}^{n-1}, \alpha \right) \circ \Theta \left( (\beta^*(y_i))_{i=1}^{n-1}, \beta \right) \\ &= \Theta \left( \widehat{\alpha} \left( (\beta^*(x_i))_{i=1}^{n-1}, \beta^*(y_1) \right), \beta^*(y_i)_{i=2}^{n-1}, \beta \right), \end{aligned}$$

for every  $\beta^*(x_i), \beta^*(y_i) \in [G : \beta^*]$ ,  $1 \leq i \leq n-1$  and  $\alpha, \beta \in \Gamma$ . This operation is well-defined. Indeed,

$$\begin{aligned} \Theta \left( (\beta^*(x_i))_{i=1}^{n-1}, \alpha_1 \right) &= \Theta \left( (\beta^*(a_i))_{i=1}^{n-1}, \alpha_2 \right), \\ \Theta \left( (\beta^*(y_i))_{i=1}^{n-1}, \beta_1 \right) &= \Theta \left( (\beta^*(b_i))_{i=1}^{n-1}, \beta_2 \right). \end{aligned}$$

Hence

$$\begin{aligned} \widehat{\alpha}_1 \left( (\beta^*(x_i))_{i=1}^{n-1}, \beta^*(x) \right) &= \widehat{\alpha}_2 \left( (\beta^*(a_i))_{i=1}^{n-1}, \beta^*(x) \right), \\ \widehat{\beta}_1 \left( (\beta^*(y_i))_{i=1}^{n-1}, \beta^*(y) \right) &= \widehat{\beta}_2 \left( (\beta^*(b_i))_{i=1}^{n-1}, \beta^*(y) \right), \end{aligned}$$

for every  $\beta^*(x), \beta^*(y) \in [G : \beta^*]$ . We have

$$\begin{aligned} & \widehat{\beta}_2 \left( \widehat{\alpha}_2 \left( (\beta^*(a_i))_{i=1}^{n-1}, \beta^*(b_1) \right), \beta^*(b_i)_{i=2}^{n-1}, \beta^*(y) \right) \\ &= \widehat{\alpha}_2 \left( \beta^*(a_i)_{i=1}^{n-1}, \widehat{\beta}_2 \left( (\beta^*(b_i))_{i=1}^{n-1}, \beta^*(y) \right) \right) \\ &= \widehat{\alpha}_2 \left( \beta^*(a_i)_{i=1}^{n-1}, \widehat{\beta}_1 \left( (\beta^*(y_i))_{i=1}^{n-1}, \beta^*(y) \right) \right) \\ &= \widehat{\beta}_1 \left( \widehat{\alpha}_2 \left( (\beta^*(a_i))_{i=1}^{n-1}, \beta^*(y_1) \right), \beta^*(y_i)_{i=2}^{n-1}, \beta^*(y) \right) \\ &= \widehat{\beta}_1 \left( \widehat{\alpha}_1 \left( (\beta^*(x_i))_{i=1}^{n-1}, \beta^*(y_1) \right), \beta^*(y_i)_{i=2}^{n-1}, \beta^*(y) \right). \end{aligned}$$

This implies that

$$\begin{aligned} & \Theta \left( \widehat{\alpha}_2 \left( (\beta^*(a_i))_{i=1}^{n-1}, \beta^*(b_1) \right), \beta^*(b_i)_{i=2}^{n-1}, \beta_2 \right) \\ &= \Theta \left( \widehat{\alpha}_1 \left( (\beta^*(x_i))_{i=1}^{n-1}, \beta^*(y_1) \right), \beta^*(y_i)_{i=2}^{n-1}, \beta_1 \right). \end{aligned}$$

Hence

$$\begin{aligned}
& \Theta((\beta^*(x_i))_{i=1}^{n-1}, \alpha_1) \circ \Theta((\beta^*(y_i))_{i=1}^{n-1}, \beta_1) \\
&= \Theta((\widehat{\alpha}_2(\beta^*(a_i))_{i=1}^{n-1}, \beta^*(b_1)), \beta^*(b_i)_{i=2}^{n-2}, \beta_2) \\
&= \Theta((\widehat{\alpha}_1(\beta^*(x_i))_{i=1}^{n-1}, \beta^*(y_1)), \beta^*(y_i)_{i=2}^{n-2}, \beta_1) \\
&= \Theta((\beta^*(a_i))_{i=1}^{n-1}, \alpha_2) \circ \Theta((\beta^*(b_i))_{i=1}^{n-1}, \beta_2).
\end{aligned}$$

Thus  $\circ$  is well-defined.

Moreover, the function  $\circ$  is associative. Indeed,

$$\begin{aligned}
& \Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha) \circ (\Theta(\beta^*(y_i)_{i=1}^{n-1}, \beta) \circ \Theta(\beta^*(z_i)_{i=1}^{n-1}, \gamma)) \\
&= \Theta((\beta^*(x_i)_{i=1}^{n-1}, \alpha) \circ \Theta(\widehat{\beta}(\beta^*(y_i)_{i=1}^{n-1}, \beta^*(z_1)), \beta^*(z_2), \dots, \beta^*(z_{n-1}), \gamma)) \\
&= \Theta\left(\left(\widehat{\alpha}(\beta^*(x_i)_{i=1}^{n-1}, \widehat{\beta}(\beta^*(y_i)_{i=1}^{n-1}, \beta^*(z_1))), \beta^*(z_i)_{i=2}^{n-1}\right), \gamma\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& (\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha) \circ \Theta(\beta^*(y_i)_{i=1}^{n-1}, \beta)) \circ \Theta(\beta^*(z_i)_{i=1}^{n-1}, \gamma) \\
&= \left(\Theta\left(\widehat{\alpha}(\beta^*(x_i)_{i=1}^{n-1}, \beta^*(y_1)), \beta^*(y_i)_{i=2}^{(n-1)}, \beta\right)\right) \circ \Theta(\beta^*(z_i)_{i=1}^{n-1}, \gamma) \\
&= \Theta\left(\widehat{\beta}(\widehat{\alpha}(\beta^*(x_i)_{i=1}^{n-1}, \beta^*(y_1)), \beta^*(y_i)_{i=2}^{n-1}, \beta^*(z_1)), \beta^*(z_i)_{i=2}^{n-1}, \gamma\right).
\end{aligned}$$

Hence  $(\Delta[G], \circ)$  is a semigroup.

Let  $G$  be a  $(\Gamma, n)$ -semihypergroup. Then, for  $\Delta_1 \subseteq \Delta$  and  $G_1 \subseteq G$  we define

$$[\Delta_1] = \{\beta^*(x) \in [G : \beta^*] : \Theta(\beta^*(x), \beta^*(y)_{i=2}^{n-1} \alpha) \in \Delta_1, \forall \alpha \in \Gamma, \beta^*(y) \in [G : \beta^*]\},$$

$$[[G_1]] = \{\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha_i) \in \Delta : \widehat{\alpha}_i(\beta^*(x_i)_{i=1}^{n-1}, \beta^*(x)) \in G_1, \forall \beta^*(x) \in [G : \beta^*]\}.$$

**Proposition 4.1.** *Let  $G$  be a commutative  $(\Gamma, n)$ -semihypergroup. Then, the following statements are true:*

1. *If  $\Delta_1 \subseteq \Delta[G]$  is an ideal, then  $[\Delta_1]$  is a  $(\Gamma, n)$ -ideal of  $[G : \beta^*]$ .*
2. *If  $G_1$  is a  $(\Gamma, n)$ -ideal of  $[G : \beta^*]$ , then  $[[G_1]]$  is an ideal of  $\Delta[G]$ .*

**Proof.** (i) Suppose that  $\Delta_1$  is an ideal of  $\Delta[G]$  and  $\beta^*(x) \in \Delta_1$ . This implies that  $\Theta(\beta^*(x), \beta^*(y)_{i=2}^{n-1} \alpha) \in \Delta_1$ , for every  $\alpha \in \Gamma$  and  $\beta^*(y) \in [G : \beta^*]$ . Let  $\Theta(\beta^*(y_i)_{i=1}^{n-1}, \beta) \in \Delta[G]$ . Since  $\Delta_1$  is an ideal of  $\Delta[G]$ , thus

$$\begin{aligned} & \Theta(\beta^*(y_i)_{i=1}^{n-1}, \beta) \circ \Theta(\beta^*(x), \beta^*(y)_{i=2}^{n-1}, \alpha) \in \Delta_1 \\ \implies & \Theta(\widehat{\beta}(\beta^*(y_i)_{i=1}^{n-1}, \beta^*(x)), \beta^*(y)_{i=2}^{n-1}, \alpha) \in \Delta_1. \end{aligned}$$

So for every  $\alpha, \beta \in \Gamma$  and  $\beta^*(y_i)_{i=1}^{n-1} \in [G : \beta^*]$ , we have  $\widehat{\beta}(\beta^*(y_i)_{i=1}^{n-1}, \beta^*(x)) \in \Delta_1$ . Therefore,  $\Delta_1$  is an ideal of  $[G : \beta^*]$ .

(ii) Let  $\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha) \in [[G_1]]$  and  $\Theta(\beta^*(y_i)_{i=1}^{n-1}, \beta) \in \Delta[G]$ . Hence for every  $\beta^*(x) \in [G : \beta^*]$ ,

$$\widehat{\alpha}(\beta^*(x_1), \beta^*(x_2), \dots, \beta^*(x_{n-1}), \beta^*(x)) \in G_1.$$

On the other hand

$$\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha) \circ \Theta(\beta^*(y_i)_{i=1}^{n-1}, \beta) = \Theta(\widehat{\alpha}(\beta^*(x_i)_{i=1}^{n-1}, \beta^*(y_1)), \beta^*(y_i)_{i=2}^{n-1}, \beta).$$

Since  $G_1$  is an ideal of  $[G : \beta^*]$ , this implies that

$$\Theta(\widehat{\alpha}(\beta^*(x_i)_{i=1}^{n-1}, \beta^*(y_1)), \beta^*(y_i)_{i=2}^{n-1}, \beta) \in G_1.$$

Therefore,  $[[G_1]]$  is a right ideal of  $\Delta[G]$ . This completes the proof.  $\blacksquare$

Let  $G$  be a  $(\Gamma, n)$ -semihypergroup and  $e$  is a natural element of  $G$ . It is easy to see that  $\Theta(\beta^*(e)_{i=1}^{n-1}, \alpha)$  is a left unity of  $\Delta[G]$ .

**Proposition 4.2.** *Let  $G$  be a  $(\Gamma, n)$ -semihypergroup with natural element and  $I$  be an ideal of  $G$ .  $[[[I]]] = I$ .*

**Proof.** Suppose that  $\beta^*(x) \in [[[I]]]$ . Hence  $\Theta(\beta^*(x), \beta^*(y_i)_{i=1}^{n-1}, \alpha) \in [[I]]$ , for every  $\alpha \in \Gamma$  and  $\beta^*(y_i)_{i=1}^{n-1} \in [G : \beta^*]$ . So  $\widehat{\alpha}(\beta^*(x), \beta(e)_{i=1}^n) \subseteq I$ . Since  $G$  has a left unity, thus  $\beta^*(x) \in I$ . Therefore,  $I = [[I]]$ .  $\blacksquare$

**Theorem 4.3.** *Let  $G_1$  and  $G_2$  be  $(\Gamma_1, n)$  and  $(\Gamma_2, n)$ -semihypergroups and  $(\varphi, f) : G_1 \times \Gamma_1 \rightarrow G_2 \times \Gamma_2$  be an epimorphism. Then, there exists a homomorphism  $\widehat{(\varphi, f)} : \Delta[G_1] \rightarrow \Delta[G_2]$ . Moreover, if  $(\varphi, f)$  is an isomorphism then,  $\widehat{(\varphi, f)}$  is isomorphism.*

**Proof.** We define

$$\widehat{(\varphi, f)}(\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha)) = \Theta(\beta^*(\varphi(x_i))_{i=1}^{n-1}, f(\alpha)).$$

First we prove that  $\widehat{(\varphi, f)}$  is well-defined. Let

$$\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha) = \Theta(\beta^*(y_i)_{i=1}^{n-1}, \beta).$$

This implies that

$$(\varphi, f) (\widehat{\alpha}(\beta^*(x_i)_{i=1}^{n-1}, \beta^*(x))) = (\varphi, f) (\widehat{\beta}(\beta^*(y_i)_{i=1}^{n-1}, \beta^*(x))),$$

for every  $\beta^*(x) \in [G : \beta^*]$ . Hence

$$\widehat{f(\alpha)} (\beta^*(\varphi(x_i)_{i=1}^{n-1}, \beta^*(\varphi(x)))) = \widehat{f(\beta)} (\beta^*(\varphi(y_i)_{i=1}^{n-1}, \beta^*(\varphi(x)))).$$

Since  $\varphi$  is onto,

$$\widehat{f(\alpha)} (\beta^*(\varphi(x_i)_{i=1}^{n-1}, \beta^*(y))) = \widehat{f(\beta)} (\beta^*(\varphi(y_i)_{i=1}^{n-1}, \beta^*(y))),$$

for every  $\beta^*(y) \in [G_2 : \beta^*]$ . Hence the function  $\widehat{(\varphi, f)}$  is well defined. Let  $\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha), \Theta(\beta^*(y_i)_{i=1}^{n-1}, \beta) \in \Delta[G_1]$ . Then,

$$\begin{aligned} & \widehat{(\varphi, f)} (\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha) \circ \Theta(\beta^*(y_i)_{i=1}^{n-1}, \beta)) \\ &= \widehat{(\varphi, f)} (\Theta(\widehat{\alpha}(\beta^*(x_i)_{i=1}^{n-1}, \beta^*(y_1)), \beta^*(y_i)_{i=2}^{n-1}, \beta)) \\ &= \Theta(\widehat{f(\alpha)}(\beta^*(\varphi(x_i)_{i=1}^{n-1}, \beta^*(\varphi(y_1))), \beta^*(\varphi(y_i)_{i=2}^{n-1}, f(\beta))) \\ &= \Theta(\beta^*(\varphi(x_i)_{i=1}^{n-1}, f(\alpha)) \circ \Theta(\beta^*(\varphi(y_i)_{i=1}^{n-1}, f(\beta))) \\ &= \widehat{(\varphi, f)} (\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha)) \circ \widehat{(\varphi, f)} (\Theta(\beta^*(y_i)_{i=1}^{n-1}, \beta)). \end{aligned}$$

Hence  $\widehat{(\varphi, f)}$  is homomorphism.

Let  $(\varphi, f)$  be one to one and  $\widehat{(\varphi, f)} (\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha)) = \widehat{(\varphi, f)} (\Theta(\beta^*(y_i)_{i=1}^{n-1}, \beta))$ . Then, we have

$$\begin{aligned} & \Theta(\beta^*(\varphi(x_i)_{i=1}^{n-1}, f(\alpha)) = \Theta(\beta^*(\varphi(y_i)_{i=1}^{n-1}, f(\beta)) \\ & \implies \widehat{f(\alpha)}(\beta^*(\varphi(x_i)_{i=1}^{n-1}, \beta^*(y))) = \widehat{f(\beta)} (\beta^*(\varphi(y_i)_{i=1}^{n-1}, \beta^*(y))) \\ & \implies \widehat{f(\alpha)} (\beta^*(\varphi(x_i)_{i=1}^{n-1}, \beta^*(\varphi(x))) = \widehat{f(\beta)}(\beta^*(\varphi(y_i)_{i=1}^{n-1}, \beta^*(\varphi(x))) \\ & \implies (\varphi, f) (\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha)) = (\varphi, f) (\Theta(\beta^*(y_i)_{i=1}^{n-1}, \alpha)) \\ & \implies \Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha) = \Theta(\beta^*(y_i)_{i=1}^{n-1}, \alpha), \end{aligned}$$

where  $y = \varphi(x)$ . One can see that if  $(\varphi, f)$  is an onto, then  $\widehat{(\varphi, f)}$  is an onto. This completes the proof.  $\blacksquare$

**Proposition 4.4.** *Let  $G_1$  and  $G_2$  be  $(\Gamma_1, n)$  and  $(\Gamma_2, n)$ -semihypergroups, respectively. Then,*

$$\Delta[G_1 \times G_2] \cong \Delta[G_1] \times \Delta[G_2].$$

**Proof.** Suppose that  $\beta^*$ ,  $\beta_1^*$  and  $\beta_2^*$  be fundamental relations on  $G_1 \times G_2$ ,  $G_1$  and  $G_2$ , respectively. It is easy to see that

$$[G_1 \times G_2 : \beta^*] \cong [G_1 : \beta_1^*] \times [G_2 : \beta_2^*].$$

We define

$$\begin{aligned} \psi : \Delta[G_1 \times G_2] &\longrightarrow \Delta[G_1] \times \Delta[G_2] \\ \Theta(\beta^*((x_i, y_i)_{i=1}^{n-1}), (\alpha_1, \alpha_2)) &\longrightarrow (\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha_1), \Theta(\beta^*(y_i)_{i=1}^{n-1}, \alpha_2)). \end{aligned}$$

Obviously, this function is well-defined. We proof  $\psi$  is a homomorphism.

$$\begin{aligned} &\psi(\Theta(\beta^*(x_i, y_i)_{i=1}^{n-1}, (\alpha_1, \alpha_2))\Theta(\beta^*(z_i, w_i)_{i=1}^{n-1}, (\beta_1, \beta_2))) \\ &= \psi(\Theta(\alpha_1(\beta^*(x_i)_{i=1}^{n-1}, \beta^*(z_1)), \beta^*(z_i)_{i=2}^{n-1}), \\ &\quad \Theta(\alpha_2(\beta^*(y_i)_{i=1}^{n-1}, \beta^*(w_1)), \beta^*(w_i)_{i=2}^{n-1}), (\beta_1, \beta_2)) \\ &= (\Theta(\alpha_1(\beta^*(x_i)_{i=1}^{n-1}, \beta^*(z_1)), \beta^*(z_i)_{i=2}^{n-1}, \beta_1), \\ &\quad \Theta(\alpha_2(\beta^*(y_i)_{i=1}^{n-1}, \beta^*(w_1)), \beta^*(w_i)_{i=2}^{n-1}), \beta_2) \\ &= \psi(\Theta(\beta^*(x_i, y_i)_{i=1}^{n-1}, (\alpha_1, \alpha_2))) \circ \psi(\Theta(\beta^*(z_i, w_i)_{i=1}^{n-1}, (\beta_1, \beta_2))). \end{aligned}$$

It is easy to see that  $\psi$  is onto and one to one. ■

**Theorem 4.5.** *There exists an exact covariant functor between the category of  $(\Gamma, n)$ -semihypergroup and the category of semigroups.*

**Proof.** Suppose that  $G_1$ ,  $G_2$  and  $G_3$  are  $(\Gamma_1, n)$ ,  $(\Gamma_2, n)$  and  $(\Gamma_3, n)$ -semihypergroups, respectively and  $(\varphi_1, f_1) : (G_1, \Gamma_1) \longrightarrow (G_2, \Gamma_2)$ ,  $(\varphi_2, f_2) : (G_2, \Gamma_2) \longrightarrow (G_3, \Gamma_3)$  are epimorphisms. We define

$$T(G_1, \Gamma_1) = \Delta[G_1], T(G_2, \Gamma_2) = \Delta[G_2], T(G_3, \Gamma_3) = \Delta[G_3].$$

$$T((\varphi_1, f_1)) = \widehat{(\varphi_1, f_1)}, T((\varphi_2, f_2)) = \widehat{(\varphi_2, f_2)}.$$

$$\begin{aligned} T((\varphi_2, f_2) \circ (\varphi_1, f_1)) \Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha) &= T((\varphi_2 \circ \varphi_1, f_2 \circ f_1)) \Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha) \\ &= \Theta(\varphi_2 \circ \varphi_1(\beta^*(x_i)_{i=1}^{n-1}), f_2 \circ f_1(\alpha)) \\ &= T((\varphi_2, f_2)) \circ T((\varphi_1, f_1)) \Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha). \end{aligned}$$

for every  $\Theta(\beta^*(x_i)_{i=1}^{n-1}, \alpha) \in \Delta[G_1]$ . On the other hand, if  $Id$  is an identity homomorphism, then  $T(Id)$  is an identity homomorphism. Therefore,  $T$  is a covariant functor. By Theorem 4.3,  $T$  is an exact functor. This complete the proof. ■

## REFERENCES

- [1] W.E. Barnes, *On the  $\Gamma$ -rings of Nobusawa*, Pacific J. Math. **18** (3) (1966) 411–422. doi:10.2140/pjm.1966.18.411
- [2] C. Berge, *Graphes et Hypergraphes* (Dunod, Paris, 1970).
- [3] P. Corsini, *Prolegomena of Hypergroup Theory*, Second Edition (Aviani Editore, 1993).
- [4] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, Advances in Mathematics, Kluwer Academic Publishers (Dordrecht, 2003). doi:10.1007/978-1-4757-3714-1
- [5] S.O. Dehkordi and B. Davvaz, *A Strong Regular Relation on  $\Gamma$ -Semihyperrings*, J. Sci. I.R. Iran. **22** (3) (2011) 257–266.
- [6] S.O. Dehkordi and B. Davvaz,  *$\Gamma$ -semihyperrings: Approximations and rough ideals*, Bull. Malays. Math. Sci. Soc. **35** (2) (2012) 1035–1047.
- [7] B. Davvaz and V. Leoreanu-Fotea, *Hyperring Theory and Applications* (International Academic Press USA, 2007).
- [8] B. Davvaz and T. Vougiouklis,  *$n$ -ary hypergroups*, Iran. J. Sci. Technol. Trans. A **30** (A2) (2006) 165–174. doi:10.1080/00927870802466835
- [9] B. Davvaz, W.A. Dudek and T. Vougiouklis, *A generalization of  $n$ -ary algebraic systems*, Comm. Algebra **37** (2009) 1248–1263. doi:10.1007/BF01180515
- [10] W. Dörnte, *Untersuchungen über einen verallgemeinerten gruppenbegriff*, Math. Z. **29** (1929) 1–19. doi:10.1007/BF01180515
- [11] W.A. Dudek and K. Glazek, *Around the Hosszu Gluskin theorem for  $n$ -ary groups*, Discrete Math. **308** (2008) 4861–4876. doi:10.1016/j.disc.2007.09.005
- [12] W. A. Dudek and I. Grozdzińska *On ideals in regular  $n$ -semigroups*, Mat. Bilten **29** (1980), 29–30.
- [13] D. Heidari, S.O. Dehkordi and B. Davvaz,  *$\Gamma$ -Semihypergroups and their properties*, U.P.B. Sci. Bull. (A) **72** (2010) 197–210.
- [14] S. Kyuno, *On prime  $\Gamma$ -rings*, Pacific J. Math. **75** (1) (1978) 185–190. doi:10.2140/pjm.1978.75.185
- [15] J. Luh, *On the theory of simple  $\Gamma$ -rings*, Michigan Math. J. **16** (1969) 65–75. doi:10.1307/mmj/1029000167
- [16] F. Marty, *Sur une generalization de la notion de group*, 8<sup>th</sup> Congres Math. Scandinaves (1934), 45–49.



- [17] M.K. Sen, *On  $\Gamma$ -semigroups*, Proc. of the Int. Conf. on Algebra and it's Appl. (1981) 301–308. New York, Decker Publication.
- [18] M.K. Sen and N.K. Saha, *On  $\Gamma$ -semigroup*, I. Bull. Cal. Math. Soc. **78** (1986) 180–186.

Received 22 January 2015

Revised 25 February 2015