

ENUMERATION OF Γ -GROUPS OF FINITE ORDER

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Abstract

The concept of Γ -semigroups is a generalization of semigroups. In this paper, we consider Γ -groups and prove that every Γ -group is derived from a group then, we give the number of Γ -groups of small order.

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1. INTRODUCTION

The concept of Γ -semigroups was introduced by Sen in [14] and [15] that is a generalization of a semigroups. Many classical notions of semigroups have been extended to Γ -semigroups (see, for example, [6, 10, 13, 16] and [17]). Dutta and Adhikari have found operator semigroups of a Γ -semigroup to be a very effective tool in studying Γ -semigroups [5]. Recently, Davvaz *et al.* introduced the notion of Γ -semihypergroups as a generalization of semigroups, a generalization of semihypergroups and a generalization of Γ -semigroups [2, 8, 9].

The determination of all groups of a given order up to isomorphism is a very old question in group theory. It was introduced by Cayley who constructed the groups of order 4 and 6 in 1854, see [4]. In this paper, we prove that a Γ -group is derived from a group. Also, we give the number of Γ -groups of small order.

2. PRELIMINARIES

We begin this section by the definition of a Γ -semigroup.

Definition [14]. Let S and Γ be nonempty sets. Then S is called a Γ -semigroup if there exists a mapping $S \times \Gamma \times S \rightarrow S$, written (a, γ, b) by $a\gamma b$, such that satisfies the identities $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Let S be a Γ -semigroup and α be a fixed element in Γ . We define $a.b = a\alpha b$, for all $a, b \in S$. It is easy to check that $(S, .)$ is a semigroup and we denote this semigroup by S_α .

Let A and B be subsets of a Γ -semigroup S and $\Delta \subseteq \Gamma$. Then $A\Delta B$ is defined as follows

$$A\Delta B = \{a\delta b \mid a \in A, b \in B, \delta \in \Delta\}.$$

For simplicity we write $a\Delta B$ and $A\Delta b$ instead of $\{a\}\Delta B$ and $A\Delta\{b\}$, respectively. Also, we write $A\delta B$ in place of $A\{\delta\}B$.

Let S be an arbitrary semigroup and Γ any nonempty set. Define a mapping $S \times \Gamma \times S \rightarrow S$ by $a\alpha b = ab$, for all $a, b \in S$ and $\alpha \in \Gamma$. It is easy to see that S is a Γ -semigroup. Thus a semigroup can be considered to be a Γ -semigroup.

In the following some examples of Γ -semigroups are presented.

Example 1. Let $S = \{i, 0, -i\}$ and $\Gamma = S$. Then S is a Γ -semigroup under the multiplication over complex number while S is not a semigroup under complex number multiplication.

Example 2. Let S be the set of all $m \times n$ matrices with entries from a field F and Γ be a set of $n \times m$ matrices with entries from F . Then S is a Γ -semigroup with the usual product of matrices.

Example 3. Let (S, \leq) be a totally ordered set and Γ be a nonempty subset of S . We define

$$x\gamma y = \max\{x, \gamma, y\},$$

for every $x, y \in S$ and $\gamma \in \Gamma$. Then S is a Γ -semigroup.

Example 4. Let $S = [0, 1]$ and $\Gamma = \mathbb{N}$. For every $x, y \in S$ and $\gamma \in \Gamma$ we define $x\gamma y = \frac{xy}{\gamma}$. Then, for every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, we have

$$(x\alpha y)\beta z = \frac{xyz}{\alpha\beta} = x\alpha(y\beta z).$$

This means that S is a Γ -semigroup.

A nonempty subset T of a Γ -semigroup S is said to be a Γ -subsemigroup of S if $TTT \subseteq T$.

Definition. A nonempty subset I of Γ -semigroup S is called a left (right) Γ -closed subset if $STI \subseteq I$ ($ITS \subseteq I$). A Γ -semigroup S is called a left (right) simple Γ -semigroup if it has no proper left (right) Γ -closed subset. Also, S is called a simple Γ -semigroup if it has no proper Γ -closed subset both left and right.

3. ENUMERATION OF Γ -GROUPS OF FINITE ORDER

Definition. A Γ -semigroup S is called a Γ -group if S_α is a group, for every $\alpha \in \Gamma$.

Example 5. Let $S = \{a, b, c, d, e, f\}$ and $\Gamma = \{\alpha, \beta\}$. Define the operations α and β as the following tables

| | | | | | | |
|----------|-----|-----|-----|-----|-----|-----|
| α | a | b | c | d | e | f |
| a | b | c | d | e | f | a |
| b | c | d | e | f | a | b |
| c | d | e | f | a | b | c |
| d | e | f | a | b | c | d |
| e | f | a | b | c | d | e |
| f | a | b | c | d | e | f |

| | | | | | | |
|---------|-----|-----|-----|-----|-----|-----|
| β | a | b | c | d | e | f |
| a | c | d | e | f | a | b |
| b | d | e | f | a | b | c |
| c | e | f | a | b | c | d |
| d | f | a | b | c | d | e |
| e | a | b | c | d | e | f |
| f | b | c | d | e | f | a |

Then S is a Γ -group. One can see that f and e are the neutral elements of S_α and S_β , respectively.

Theorem 6. Let S be a Γ -semigroup. Then S is a simple Γ -semigroup if and only if S_α is a group, for every $\alpha \in \Gamma$.

Proof. Let S be a simple Γ -semigroup and $\alpha \in \Gamma$, we show that S_α is a group. Let $I = a\alpha S$, where $a \in S$. Then, I is a right Γ -closed subset of S , indeed

$$ITS = (a\alpha S)\Gamma S \subseteq a\alpha S = I.$$

Since S has no proper right Γ -closed subset, we have $I = a\alpha S = S$. Similarly, we can prove that $S\alpha a = S$. Therefore, S_α is a group.

Conversely, let $I \neq \phi$ be a left Γ -closed subset of S , $s \in S$ and $a \in I$. Since S_α is a group, there exists $t \in S$ such that $s = t\alpha a \subseteq S\alpha I \subseteq I$. So $S = I$. Similarly, we can prove that S has no proper right Γ -closed subset. Therefore, S is simple. ■

Corollary 7. *Let S be a Γ -semigroup. If S_α is a group, for some $\alpha \in \Gamma$, then S_β is a group, for every $\beta \in \Gamma$.*

Proof. Since S_α is a group, previous theorem implies that S is a simple Γ -group. Thus, for every $\beta \in \Gamma$, S_β is a group. ■

Corollary 8. *Let S be a Γ -semigroup. If S_α is a group, for some $\alpha \in \Gamma$, then S is a Γ -group.*

Proof. By Corollary 7, it is trivial. ■

Theorem 9. *Let S be a Γ -group and $\alpha, \beta \in \Gamma$. Then there exists $b \in S$ such that $x\beta y = xab\alpha y$, for every $x, y \in S$.*

Proof. It is sufficient to put $b = e_\alpha \beta e_\alpha$, where e_α is the neutral element of S_α . Then, for every $x, y \in S$, we have

$$\begin{aligned} x\beta y &= (x\alpha e_\alpha)\beta(e_\alpha\alpha y) \\ &= x\alpha(e_\alpha\beta e_\alpha)\alpha y \\ &= xab\alpha y. \end{aligned}$$

■

By the previous theorem, we conclude that every Γ -group is derived from a group. Therefore, if S is a Γ -group, then we can consider (S, \cdot) as a group and $\Gamma \subseteq S$, so $x\alpha y$ is a product in (S, \cdot) , for every $x, y \in S$ and $\alpha \in \Gamma$. Also, Theorem 9 implies that the groups S_α and S_β are isomorphic, for every $\alpha, \beta \in \Gamma$.

Definition. Let S be a Γ -group and S' be a Γ' -group. If there exist mappings $\varphi_\gamma : S \rightarrow S'$, for every $\gamma \in \Gamma$, and $f : \Gamma \rightarrow \Gamma'$ such that

$$\varphi_\gamma(x\gamma y) = \varphi_\gamma(x)f(\gamma)\varphi_\gamma(y),$$

for all $x, y \in S$, then we say $(\{\varphi_\gamma\}_{\gamma \in \Gamma}, f)$ is a homomorphism between S and S' . Also, if f and φ_γ , for every $\gamma \in \Gamma$, are bijections, then $(\{\varphi_\gamma\}_{\gamma \in \Gamma}, f)$ is called an isomorphism, and S and S' are called isomorphic.

Lemma 10. *Let S be a Γ -group and S' be a Γ' -group. Then S and S' are isomorphic if and only if S and S' are isomorphic group and $|\Gamma| = |\Gamma'|$.*

Proof. If S and S' are isomorphic, then by the previous definition, for every $\alpha \in \Gamma$, the groups S_α and $S'_{\alpha'}$ are isomorphic where $f : S \rightarrow S'$ is a bijection and $f(\alpha) = \alpha'$. ■

Theorem 11. *The number of Γ -groups of order n is nk , up to isomorphism, where k is the number of isomorphism classes of groups of order n .*

Proof. Suppose that (S, \cdot) is a group and Γ and Γ' be two subsets of S such that $|\Gamma| = |\Gamma'|$. Then by previous lemma, there exists only one Γ -group derived from (S, \cdot) , up to isomorphism. So, for every $m \leq n$ there exists only one Γ -group, where Γ is a subset of S such that $|\Gamma| = m$. Thus, the number of Γ -groups derived from (S, \cdot) is n , up to isomorphism. Therefore, if there exist k groups of order n , then we have nk Γ -groups of order n , up to isomorphism. ■

Corollary 12. *Suppose that $n > 1$ is an integer with decomposition into primes as $n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$. If n is prime to*

$$\prod_{j=1}^r (p_j^{e_j} - 1)$$

and $e_j \leq 2$, then the number of Γ -groups of order n is $n2^m$, where m is the number of j 's with $e_j = 2$.

Proof. By a result of Rédei [12], all such groups of order n are abelian. Thus, the number of isomorphism types of abelian groups of order n is given by

$$\prod_{j=1}^r p(e_j) = 2^m,$$

where $p(e_j)$ is the number of partitions of $e_j \leq 2$ and $p(1) = 1, p(2) = 2$. The proof is completed by applying Theorem 11. ■

The case $m = 0$ of the Corollary 12 was studied by Szele [18]. In connection with this, Erdős [7] showed that the number of $n \leq x$ such that $(n, \varphi(n)) = 1$ ($\varphi(n)$ is Euler's phi function) is asymptotic to

$$\frac{e^{-\gamma} x}{\log \log \log x}$$

where γ is Euler's constant. For additional results on the asymptotic of $n \leq x$ satisfying Rédei's condition and asymptotic enumeration of finite abelian groups see [1, 11, 19].

In the following table we give the number of Γ -groups of order less than 30.

| <i>Order</i> | <i>Number of Γ – groups</i> | <i>Order</i> | <i>Number of Γ – groups</i> |
|--------------|-----------------------------------------------|--------------|-----------------------------------------------|
| 1 | 1 | 16 | 224 |
| 2 | 2 | 17 | 17 |
| 3 | 3 | 18 | 90 |
| 4 | 8 | 19 | 19 |
| 5 | 5 | 20 | 100 |
| 6 | 12 | 21 | 42 |
| 7 | 7 | 22 | 44 |
| 8 | 40 | 23 | 23 |
| 9 | 18 | 24 | 360 |
| 10 | 20 | 25 | 50 |
| 11 | 11 | 26 | 52 |
| 12 | 60 | 27 | 135 |
| 13 | 13 | 28 | 112 |
| 14 | 28 | 29 | 29 |
| 15 | 15 | 30 | 120 |

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