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COMPLICATED BE-ALGEBRAS AND CHARACTERIZATIONS OF IDEALS

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Abstract

In this paper, using the notion of upper sets, we introduced the notions of complicated BE-Algebras and gave some related properties on complicated, self-distributive and commutative BE-algebras. In a self-distributive and complicated BE-algebra, characterizations of ideals are obtained.

Keywords: BE-algebras, complicated BE-algebras, ideals in BE-algebras.

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1. Introduction

Y. Imai and K. Isėki introduced two classes of abstract algebras called BCK-algebras and BCI-algebras [8, 10]. It is known that the class of BCK-algebras is a proper subclass of BCI-algebras. In [5, 6], Q.P. Hu and X. Li introduced a wide class of abstract algebras called BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of BCH-algebras. J. Neggers and H.S. Kim ([16]) introduced the notion of a d-algebra which is a generalization of BCK-algebras, and also they introduced the notion of B-algebras ([17, 18]). Y.B. Jun, E.H. Roh and H.S. Kim ([11]) introduced a new notion called BH-algebra which is another generalization of BCH/BCI/BCK-algebras. A. Walendziak obtained another equivalent axioms for B-algebras ([20]). C.B. Kim and H.S. Kim ([13]) introduced the notion of BM-algebra which is a specialization of B-algebras. They proved that the class of BM-algebras is a proper subclass of B-algebras and also showed that a BM-algebra is equivalent to a 0-commutative B-algebra. In [14], H.S. Kim and Y.H. Kim introduced the notion of BE-algebra as a generalization of a BCK-algebra. Using the notion of upper sets they gave an equivalent condition of the filter in BE-algebras. In [2] and [3], S.S. Ahn and K.S. So introduced the notion of ideals in BE-algebras, and proved several characterizations of such ideals. Also they generalized the notion of upper sets in BE-algebras and discussed some properties of the characterizations of generalized upper sets related to the structure of ideals in transitive and self distributive BE-algebras. In [4], S.S. Ahn, Y.H. Kim and J.M. Ko are introduced the notion of terminal section of BE-algebras and provided the characterization of a commutative BE-algebras.

B.M. Schein [19] considered systems of the form $(\phi; \circ, \setminus)$, where ϕ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\setminus$" (and hence $(\phi; \setminus)$ is a subtraction algebra in the sence of [1]). B. Zelinka [22] discussed a problem proposed by B.M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. Y.B. Jun *et al.* [12] introduced the complicated subtraction algebras and investigated several properties on it.

In this paper, using the notion of upper sets, we introduced the notions of complicated BE-Algebras and gave some related properties on complicated, self-distrubutive and commutative BE-algebras. In a self-distributive and complicated BE-algebra, characterizations of ideals are obtained.

2. Preliminaries

Definition 2.1 [14]**.** An algebra $(X; *, 1)$ of type (2, 0) is called a BE-algebra if, for all a, b, c in X, the following identities hold:

(BE1) $a * a = 1$,

(BE2) $a * 1 = 1$,

(BE3) $1 * a = a$,

(BE4) $a * (b * c) = b * (a * c)$.

In a BE-algebra X, the relation "$\leq$" is defined by $a \leq b$ if and only if $a * b = 1$.

Proposition 2.2 [14]**.** *If $(X; *, 1)$ is a BE-algebra, then*

(i) $a * (b * a) = 1$,

(ii) $a * ((a * b) * b) = 1$

for any $a, b \in X$.

Example 2.1. [14] Let $X = \{1, a, b, c, d, 0\}$ be a set with the following table:

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(X; *, 1)$ is a BE-algebra.

Definition 2.3 [14]**.** A BE-algebra $(X; *, 1)$ is said to be self-distributive if $a * (b * c) = (a * b) * (a * c)$ for all $a, b, c \in X$.

Example 2.2 [14]**.** Let $X = \{1, a, b, c, d\}$ be a set with the following table:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	d
b	1	a	1	c	c
c	1	1	b	1	b
d	1	1	1	1	1

Then $(X; *, 1)$ is a self-distributive BE-algebra.

Proposition 2.4 ([2, 4])**.** *Let $(X; *, 1)$ be a self-distributive BE-algebra. If $a \leq b$, then, for all a, b, c in X, the following hold:*

(i) $c * a \leq c * b$,

(ii) $b * c \leq a * c$,

(iii) $a * b \leq (b * c) * (a * c)$.

Definition 2.5 [21]**.** Let X be a BE-algebra. We say that X is commutative if

(C) $(a * b) * b = (b * a) * a$

for all $a, b \in X$.

Proposition 2.6 [21]**.** *If $(X; *, 1)$ is a commutative BE-algebra, then for all $a, b \in X$,*

$$a * b = 1 \text{ and } b * a = 1 \text{ imply } a = b.$$

Definition 2.7 [2]**.** Let X be a BE-algebra. A nonempty subset I of X is called an ideal of X if

(I1) $\forall x \in X$ and $\forall a \in I$ imply $x * a \in I$,

(I2) $\forall x \in X$ and $\forall a, b \in I$ imply $(a * (b * x)) * x \in I$.

Corollary 2.8 [2]**.** *Let I be an ideal of X. If $a \in I$ and $a \leq x$, then $x \in I$.*

Corollary 2.9 [2]**.** *Let X be a self-distributive BE-algebra. A nonempty subset I of X is an ideal of X if and only if it satisfies the following conditions*

(I3) $1 \in I$,

(I4) $x * (y * z) \in I$ *and* $y \in I$ *imply* $x * z \in I$ *for all* $x, y, z \in X$.

3. Complicated BE-algebras

Definition 3.1. Let $(X; *, 1)$ be a BE-algebra and $a, b \in X$. The set

$$A(a, b) = \{x \in X : a * (b * x) = 1\}$$

is called an upper set of a and b. It is easy to see that $1, a, b \in A(a, b)$.

Proposition 3.2. *Let $(X; *, 1)$ be a BE-algebra. Then $A(a, b) = A(b, a)$ for all $a, b \in X$.*

Proof. It is clear by (BE4). ■

Example 3.1. Let $X = \{1, a, b, c\}$ be a set with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	1	1	c
c	1	1	1	1

.

It is clear that X is a BE-algebra and $A(1, 1) = \{1\}, A(1, a) = A(a, a) = \{1, a\}, A(1, b) = A(a, b) = A(b, b) = \{1, a, b\}$ and $A(1, c) = A(a, c) = A(b, c) = A(c, c) = X$.

Example 3.2. Let $X = \{1, a, b, c\}$ be a set with the following table:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	1	1

.

It is clear that X is a BE-algebra and $A(1,1) = \{1\}, A(1,a) = A(a,a) = \{1,a\}, A(1,b) = A(b,b) = \{1,b\}, A(a,b) = \{1,a,b\}$ and $A(1,c) = A(a,c) = A(b,c) = A(c,c) = X$.

Definition 3.3. A BE-algebra $(X;*,1)$ is called a complicated BE-algebra (c-BE-algebra, shortly) if for all $a,b \in X$, the set $A(a,b)$ has the smallest element. The smallest element of $A(a,b)$ is denoted by $a \circledS b$.

Example 3.3. The BE-algebra X in Example 3.1 is a c-BE-algebra since $1 \circledS 1 = 1, 1 \circledS a = a, a \circledS a = a, 1 \circledS b = a \circledS b = b \circledS b = b$ and $1 \circledS c = a \circledS c = b \circledS c = c \circledS c = c$. But the BE-algebra in Example 3.2 is not a c-BE-algebra since $A(a,b) = \{1,a,b\}$ has no the smallest element.

Proposition 3.4. *Let $(X;*,1)$ be a c-BE-algebra. Then, for all $a,b \in X$,*

(i) $a \circledS b \leq a$ *and* $a \circledS b \leq b$,

(ii) $a \circledS 1 = a$,

(iii) $a \circledS b = b \circledS a$,

(iv) $a \circledS (a * b) \leq b$.

Proof. (i) and (ii) are easily seen by the definition of the c-BE algebra.

(iii) is clear since $A(a,b) = A(b,a)$.

(iv) From Proposition 2.1 (i), since $a * ((a*b)*b) = 1$, we have $b \in A(a, a*b)$ and hence $a \circledS (a*b) \leq b$. ∎

Proposition 3.5. *Let $(X;*,1)$ be a self-distributive BE-algebra. If, for all $a,b,c \in X$, $a \leq b$ and $b \leq c$ then $a \leq c$.*

Proof. Since $a* c = 1*(a*c) = (a*b)*(a*c) = a*(b*c) = a*1 = 1$, we have $a \leq c$. ∎

Proposition 3.6. *Let $(X;*,1)$ be a self-distributive c-BE-algebra. Then, for all $a,b,c \in X$,*

(i) $a \leq b$ *implies* $a \circledS c \leq b \circledS c$,

(ii) $(a*b) \circledS (b*c) \leq a*c$.

Proof. (i) Let $a \leq b$. Since X is self-distributive, by Proposition 2.4 (ii), we have $b*(b \circledS c) \leq a*(b \circledS c)$. Also since $b \circledS c \in A(b,c)$, we have $c \leq b*(b \circledS c)$. Then by Proposition 3.5, we get $c \leq a*(b \circledS c)$. Hence we obtain $b \circledS c \in A(a,c)$ and $a \circledS c \leq b \circledS c$.

(ii) By Proposition 2.4. (iii), we have $a*b \leq (b*c)*(a*c)$. Hence we see that $a*c \in A(a*b, b*c)$ and $(a*b) \circledS (b*c) \leq a*c$. ∎

Theorem 3.7. *Let* $(X;*,1)$ *be a self-distributive and commutative c-BE-algebra. Then* $(X;\circledS)$ *is a commutative monoid.*

Proof. By Proposition 3.4 (ii) and (iii), we need only to show that $(X;\circledS)$ is associative. Say $(a\circledS b)\circledS c = u$. Then, since $u \in A(a\circledS b, c)$ and $A(a\circledS b, c) = A(c, a\circledS b)$, we know that

$$a\circledS b \leq c * u \tag{3.1}$$

and

$$c \leq (a\circledS b) * u. \tag{3.2}$$

Hence using the equation (3.1), we have, by Proposition 2.4 (i) and (BE4),

$$b * (a\circledS b) \leq b * (c * u) = c * (b * u). \tag{3.3}$$

Since $a \leq b * (a\circledS b)$, using the equation (3.3) and Proposition 3.5, we obtain

$$a \leq c * (b * u). \tag{3.4}$$

From the equation (3.4), we have $b * u \in A(a, c)$ and $a\circledS c \leq b * u$. So we see that $u \in A(a\circledS c, b)$, that is,

$$(a\circledS c)\circledS b \leq (a\circledS b)\circledS c = u. \tag{3.5}$$

Since the equation (3.5) is true for all $a, b, c \in X$, the following inequality is true:

$$(a\circledS b)\circledS c \leq (a\circledS c)\circledS b. \tag{3.6}$$

Hence by Proposition 2.6, using the equation (3.5) and (3.6), we get

$$(a\circledS b)\circledS c = (a\circledS c)\circledS b. \tag{3.7}$$

Then we obtain $(a\circledS b)\circledS c = (b\circledS a)\circledS c = (b\circledS c)\circledS a = a\circledS(b\circledS c)$. ■

Proposition 3.8. *If* $(X;*,1)$ *is a self-distributive and commutative c-BE-algebra and* $X \neq \{1\}$, *then* $(X;\circledS)$ *has no group structure.*

Proof. Let $1 \neq a \in X$. Hence we have $a \leq 1$. If there exists an element $b \in X$ such that $a\circledS b = b\circledS a = 1$, then since $1 = a\circledS b \leq a \leq 1$, we have $a = 1$.This is a contradiction. ■

Proposition 3.9. *Let* $(X;*,1)$ *be a self-distributive and commutative c-BE-algebra. Then* $a \leq b$ *implies* $a \circledS b = a$.

Proof. (i) Let $a \leq b$. Hence we have $a * b = 1$. Then we get

$$\begin{aligned} a * (a \circledS b) &= 1 * (a * (a \circledS b)) \\ &= (a * b) * (a * (a \circledS b)) \\ &= a * (b * (a \circledS b)), \qquad \text{by self-distributivity property} \\ &= 1 \end{aligned}$$

since $a \circledS b \in A(a,b)$. Hence $a \leq b * (a \circledS b)$. Then we have $a \leq a \circledS b$. Also we know that $a \circledS b \leq a$. Hence we obtain $a \circledS b = a$ by Proposition 2.6. ■

Now, in a c-BE-algebra, define the set

$$B(a,b) = \{x \in X : x \circledS a \leq b\} \tag{3.8}$$

Theorem 3.10. *Let* $(X;*,1)$ *is a self-distributive c-BE-algebra. Then the set* $B(a,b)$ *in equation (3.8) has the greatest element and it is* $a * b$.

Proof. Since $a * b \leq a * b$, we have $b \in A(a * b, a)$. Hence we get $(a * b) \circledS a \leq b$. So, it is seen that $a * b \in B(a,b)$. If $c \in B(a,b)$, we write $c \circledS a \leq b$. By Proposition 2.4 (i), we have $a * (c \circledS a) \leq a * b$. Since $c \circledS a \in A(c,a)$, we have $c \leq a * (c \circledS a)$. Then we obtain $c \leq a * b$, by Proposition 3.5. Hence $a * b$ is the greatest element of $B(a,b)$. ■

Proposition 3.11. *Let* $(X;*,1)$ *be a self-distributive and commutative c-BE-algebra. Then*

(i) $a \circledS b \leq a * b \leq (a \circledS c) * (c \circledS b)$,
(ii) $(a * b) \circledS a = a \circledS b$,
(iii) $(a \circledS b) * c = a * (b * c)$,
(iv) $a * (b \circledS c) = (a * b) \circledS (a * c)$,
(v) $a \circledS b$ *is the greatest lower bound of the set* $\{a, b\}$.

Proof. (i) Using Proposition 3.4 (iv) and Proposition 3.6 (i), we have $c \circledS (a \circledS (a * b)) \leq c \circledS b$. We get $(c \circledS a) \circledS (a * b)) \leq c \circledS b$ or by Proposition 3.4 (iii), $(a * b) \circledS (c \circledS a) \leq c \circledS b$. Hence since $a * b \in B(c \circledS a, c \circledS b)$, we obtain $a * b \leq (a \circledS c) * (c \circledS b)$. Also it is known that $a \circledS b \leq b \leq a * b$. By Proposition 3.5, we get $a \circledS b \leq a * b \leq (a \circledS c) * (c \circledS b)$.

(ii) Since $a * b \in B(a,b)$, we have $(a * b) \circledS a \leq b$. Using Proposition 3.4 (i), commutativity and associativity of the operation $\circledS$, we get $(a*b) \circledS (a \circledS a) \leq a \circledS b$. By Proposition 3.9, we see that $a \circledS a = a$ since $a \leq a$. Hence $(a * b) \circledS a \leq a \circledS b$.

Secondly, since $b \leq a * b$, by commutativity of the operation Ⓢ and Proposition 3.6 (i), we have $aⓈb \leq (a * b)Ⓢa$. So we obtain $(a * b)Ⓢa = aⓈb$ by Proposition 2.6.

(iii) $aⓈb \in A(a, b)$ implies $a \leq b * (aⓈb)$. Also from Proposition 2.4 (iii), we have $b*(aⓈb) \leq ((aⓈb)*c)*(b*c)$. So we get $a \leq ((aⓈb)*c)*(b*c)$ by Proposition 3.5. Then we have $(aⓈb) * c \leq a * (b * c))$. Secondly, using Proposition 2.2 (ii), Proposition 2.4 (iii) and (BE4), since

$$\begin{aligned} b &\leq (b * c) * c \\ &\leq (a * (b * c)) * (a * c) \\ &= a * ((a * (b * c)) * c), \end{aligned}$$

we have $b \leq a * ((a * (b * c)) * c)$ or $a \leq b * ((a * (b * c)) * c)$. Then we obtain $aⓈb \leq (a * (b * c)) * c$ or $a * (b * c) \leq (aⓈb) * c$. Consequently we see that $a * (b * c) = (aⓈb) * c$.

(iv) By (i), we have $a * c \leq (aⓈb) * (bⓈc)$ or $aⓈb \leq (a * c) * (bⓈc)$. Then we get $a * (aⓈb) \leq a * ((a * c) * (bⓈc))$ by Proposition 2.4 (i). We can write $a * b \leq (aⓈa) * (aⓈb) \leq (a * c) * (a * (bⓈc))$ by (i). Hence since $a * (bⓈc) \in A(a * b, a * c)$, we have

$$(a * b)Ⓢ(a * c) \leq a * (bⓈc). \tag{3.9}$$

Secondly, since $bⓈc \leq b$, we have $a * (bⓈc) \leq a * b$ by Proposition 2.4 (i). Hence we get

$$(a * (bⓈc))Ⓢ(a * c) \leq (a * b)Ⓢ(a * c). \tag{3.10}$$

Also since $bⓈc \leq c$, we have $a * (bⓈc) \leq a * c$ and so we get $(a * (bⓈc))Ⓢ(a * (bⓈc)) \leq (a * (bⓈc))Ⓢ(a * c)$, that is

$$a * (bⓈc) \leq (a * (bⓈc))Ⓢ(a * c). \tag{3.11}$$

Hence from the equation (3.10) and (3.11) and by Proposition 3.5, we obtain

$$a * (bⓈc) \leq (a * (bⓈc))Ⓢ(a * c). \tag{3.12}$$

The equations (3.9) and (3.12) show that $a * (bⓈc) = (a * (bⓈc))Ⓢ(a * c)$ by Proposition 2.6.

(v) Since $aⓈb \leq a$ and $aⓈb \leq b$, $aⓈb$ is the lower bound of the set $\{a, b\}$. Let c be another lower bound of the set $\{a, b\}$. Then we know that $c * a = 1$ and $c * b = 1$. So since $c * (aⓈb) = (c * a)Ⓢ(c * b) = 1Ⓢ1 = 1$, we have $c \leq aⓈb$. ■

Remark 3.1. Let $(X;*,1)$ be a BE-algebra. In [21], the binary operation "$+$" on X was defined as the following: for any $a,b \in X$,

$$a + b = (a * b) * b.$$

Also the author proved that if $(X;*,1)$ is a commutative BE-algebra, then $(X;+)$ is a semilattice. By Proposition 3.11 (v), we proved that a self-distributive and commutative c-BE-algebra X is a semilattice under the operation "Ⓢ". In a self-distributive and commutative c-BE-algebra, since $a \le a+b$ by Proposition 2.2 (ii) and using Proposition 3.9, we see that $a Ⓢ (a+b) = a$. Also, since $a \le b$ implies $a + b = b$ and since $a Ⓢ b \le a$, we have $(a Ⓢ b) + a = a$. Therefore any self-distributive and commutative c-BE-algebra is a lattice with respect to operations "Ⓢ" and "$+$".

Now we provide characterizations of ideals in a self-distributive c-BE-algebra.

Corollary 3.12 [2]**.** *Let $(X;*,1)$ be a self-distributive BE-algebra. A nonempty subset I of X is an ideal of X if and only if $A(u,v) \subseteq I$ for all $u,v \in I$.*

Theorem 3.13. *Let $(X;*,1)$ be a self-distributive c-BE-algebra. A nonempty subset I of X is an ideal of X if and only if it satisfies the following conditions:*

(i) $\forall a \in I, \forall x \in X,\ a \le x \Longrightarrow x \in I$,

(ii) $\forall a,b \in I, \exists c \in I,\ c \le a$ *and* $c \le b$.

Proof. Let I be an ideal of X. (i) follows from the Corollary 2.8. Let $a,b \in I$. From Corollary 3.12, we have $A(a,b) \subseteq I$. Then we get $a Ⓢ b \in I$. If we take $a Ⓢ b = c$, then we have $c \le a$ and $c \le b$ by Proposition 3.4 (i) which proves (ii). Conversely, let I be a non-empty subset of X satisfying (i) and (ii). Since for $a \in I, a \le 1$ by (BE2), we have $1 \in I$ by (i). For any $a,b,c \in X$, let $b \in I$ and $a*(b*c) \in I$. By (ii), there exists $d \in I$ such that $d \le b$ and $d \le a*(b*c)$. Then using (BE3), (BE4), and self-distributivity, we have

$$1 = d*(a*(b*c)) = d*(b*(a*c)) = (d*b)*(d*(a*c)) = d*(a*c).$$

Hence, we get $d \le a*c$. By (i), it is obtained $a*c \in I$. So I is an ideal of X by Corollary 2.9. ∎

Theorem 3.14. *Let $(X;*,1)$ be a self-distributive c-BE-algebra. A non-empty subset I of X is an ideal of X if and only if it satisfies the following conditions:*

(i) $\forall a \in I, \forall x \in X,\ a \le x \Longrightarrow x \in I$,

(ii) $\forall a,b \in I,\ a Ⓢ b \in I$.

Proof. The necessity is given in the proof of Theorem 3.13. Conversely, since for $a \in I, a \leq 1$ by (BE2), we have $1 \in I$ by (i). Let I be a non-empty subset of X satisfying (i) and (ii). We know that $x * y \in B(x, y)$ in a self-distributive c-BE-algebra. So $(x * y) \circledS x \leq y$ and hence

$$x \circledS (x * y) \leq y. \tag{3.13}$$

Now let $y \in I$ and $x * (y * z) \in I$. By (ii) and (BE4), we get $y \circledS (x * (y * z)) = y \circledS (y * (x * z)) \in I$. From the equation 3.13, it is clear that $y \circledS (y * (x * z)) \leq x * z$. Hence it is obtained $x * z \in I$ by (i). Consequently, I is an ideal of X by Corollary 2.9. ■

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