

## ON THE INTERSECTION GRAPHS OF IDEALS OF DIRECT PRODUCT OF RINGS

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### Abstract

In this paper we first calculate the number of vertices and edges of the intersection graph of ideals of direct product of rings and fields. Then we study Eulerianity and Hamiltonicity in the intersection graph of ideals of direct product of commutative rings.

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## 1. INTRODUCTION

For graph theory terminology in general we follow [6]. Specifically, let  $G = (V, E)$  be an (undirected) graph with vertex set  $V$  and edge set  $E$ . The *size* of  $G$  is  $|E|$ , the number of edges of  $G$ . A *Hamiltonian graph* is a graph with a spanning cycle, called a *Hamiltonian cycle*. A graph is *Eulerian* if it has a closed trail containing all edges. Let  $F = \{S_i : i \in I\}$  be an arbitrary family of sets. The *intersection graph*  $G(F)$  is the one-dimensional skeleton of the nerves of  $F$ , i.e.,  $G(F)$  is the graph whose vertices are  $S_i$ ,  $i \in I$ , and in which the vertices  $S_i$  and  $S_j$  ( $i, j \in I$ ) are adjacent if and only if  $S_i \neq S_j$  and  $S_i \cap S_j \neq \emptyset$ . It is shown that every simple graph (without loops and multiple edges) is an intersection graph [5].

The study of algebraic structures using properties of graphs has become an exciting research topic in the last few decades, leading to many fascinating results and questions. It is interesting to study the intersection graphs  $G(F)$  when the members of  $F$  have an algebraic structure. Several mathematicians studied such graphs on various algebraic structures. Recently Chakrabarty et al. [2] studied intersection graphs of ideals of rings. The *intersection graph of ideals of a ring*  $R$ , denoted by  $G(R)$ , is the undirected simple graph whose vertices are in a one-to-one correspondence with all nontrivial (nonzero, proper) ideals of  $R$  and two distinct vertices are joined by an edge if and only if the corresponding ideals of  $R$  have a nonzero intersection. For references on graphs related to the ring structures see for example [2, 3, 4, 7].

In this paper, we first calculate the number of vertices and the number of edges of the intersection graph of ideals of rings and fields. Then we study Eulerianity and Hamiltonicity in the intersection graph of ideals of direct product of commutative rings. For a ring  $R$  we denote the number of edges of  $G(R)$  by  $e(G(R))$ . For ring theory terminology in general we follow [1].

## 2. ORDER AND SIZE

**Theorem 1.** *Let  $R_1$  and  $R_2$  be two rings with identity. If  $R_i$  has  $t_i$  ideals for  $i = 1, 2$ , then  $|V(G(R_1 \times R_2))| = t_1 t_2 - 2$  and*

$$(1) \quad e(G(R_1 \times R_2)) = \binom{t_1 t_2 - 2}{2} - 2(x + a)(y + b) - x - y + ab,$$

where  $x = \binom{t_1 - 2}{2} - e(G(R_1))$ ,  $y = \binom{t_2 - 2}{2} - e(G(R_2))$ ,  $a = t_1 - 1$ ,  $b = t_2 - 1$ .

**Proof.** Let  $R = R_1 \times R_2$ . We show that  $e(\overline{G(R)}) = 2(x + a)(y + b) + x + y - ab$ , where  $\overline{G(R)}$  is the graph complement of  $G(R)$  and hence the result follows.

Let  $I$  be a nontrivial ideal of  $R$ . Then  $I = I_1 \times I_2 \notin \{(0_{R_1}, 0_{R_2}), R_1 \times R_2\}$ . Let  $A$  be the set of ideals  $I$  of  $R$  such that  $I = I_1 \times I_2$ , where  $I_1 \notin \{0_{R_1}, R_1\}$  and  $I_2 \notin \{0_{R_2}, R_2\}$ . Let  $B$  be the set of all other nontrivial ideals of  $R$ . Then any ideal of  $R$  in  $B$  is one of the following forms:

$$\{0_{R_1}\} \times R_2, R_1 \times \{0_{R_2}\}, \{0_{R_1}\} \times I_2, I_1 \times \{0_{R_2}\}, R_1 \times I_2, I_1 \times R_2$$

where  $I_1$  and  $I_2$  are nontrivial ideals of  $R_1$  and  $R_2$ , respectively.

Let  $I = I_1 \times I_2$  and  $J = I_3 \times I_4$  be two nontrivial ideals of  $R$  such that  $I \cap J = \{(0_{R_1}, 0_{R_2})\}$ . Then  $I_1 \cap I_3 = \{0_{R_1}\}$  and  $I_2 \cap I_4 = \{0_{R_2}\}$ . So the number of edges of  $\overline{G(R)}$  with both end points corresponding to ideals in  $A$  is given by

$$2e(\overline{G(R_1)})e(\overline{G(R_2)}) = 2xy,$$

where  $x$  and  $y$  are defined as above.

Now the vertices corresponding to the ideals  $\{0_{R_1}\} \times R_2, R_1 \times \{0_{R_2}\}, R_1 \times I_2, I_1 \times R_2$  (for all nontrivial ideals  $I_1$  and  $I_2$  of  $R_1$  and  $R_2$  respectively) are adjacent to any vertex corresponding to an ideal in  $A$  in the graph  $G(R)$ . Thus there are no such edges in  $\overline{G(R)}$ .

Next we note that for each pair of nontrivial ideals  $I_2, I_4$  of  $R_2$  with  $I_2 \cap I_4 = \{0_{R_2}\}$  and for all nontrivial ideals  $I_1, I_3$  of  $R_1$ , we have

$$\{0_{R_1}\} \times I_2 \cap I_3 \times I_4 = \{(0_{R_1}, 0_{R_2})\} = I_1 \times I_2 \cap \{0_{R_1}\} \times I_4.$$

So the number of edges of  $\overline{G(R)}$  with one end point corresponding to an ideal in  $A$  and the other end point corresponding to an ideal in  $B$  is given by

$$2(t_1 - 2)e(\overline{G(R_2)}) + 2(t_2 - 2)e(\overline{G(R_1)}) = 2(a - 1)y + 2(b - 1)x,$$

where  $a$  and  $b$  are defined as above.

Finally we calculate the number of edges of  $\overline{G(R)}$  with both end points corresponding to ideals in  $B$ . We denote the degree of the vertex corresponding to an ideal  $T$  in a graph  $X$  by  $d_X(T)$ .

Let  $M$  be the induced subgraph of  $\overline{G(R)}$  generated by the set of vertices corresponding to ideals in  $B$ . Then the following computations are straightforward:

**Fact 1.**  $d_M(\{0_{R_1}\} \times R_2) = 1 + (t_1 - 2) = t_1 - 1$ .

Note that since the vertex corresponding to  $\{0_{R_1}\} \times R_2$  is adjacent to the vertex corresponding to  $R_1 \times \{0_{R_2}\}$  and all the vertices corresponding to  $I_1 \times \{0_{R_2}\}$  (for all nontrivial ideals  $I_1$  of  $R_1$ ) in  $M$ .

**Fact 2.**  $d_M(R_1 \times \{0_{R_2}\}) = t_2 - 1$ .

**Fact 3.**  $d_M(\{0_{R_1}\} \times I_2) = 1 + (t_1 - 2) + d_{\overline{G(R_2)}}(I_2) + d_{\overline{G(R_2)}}(I_2) = t_1 - 1 + 2d_{\overline{G(R_2)}}(I_2)$ .

Note that since the vertex corresponding to  $\{0_{R_1}\} \times I_2$  is adjacent to the vertex corresponding to  $R_1 \times \{0_{R_2}\}$ , all the vertices corresponding to  $I_1 \times \{0_{R_2}\}$  (for all nontrivial ideals  $I_1$  of  $R_1$ ) and all the vertices corresponding to ideals of the forms  $R_1 \times I_4$  and  $\{0_{R_1}\} \times I_4$  (for all nontrivial ideals  $I_4$  of  $R_2$  such that  $I_2 \cap I_4 = \{0_{R_2}\}$ ) in  $M$ .

**Fact 4.**  $d_M(I_1 \times \{0_{R_2}\}) = t_2 - 1 + 2d_{\overline{G(R_1)}}(I_1)$ .

**Fact 5.**  $d_M(R_1 \times I_2) = d_{\overline{G(R_2)}}(I_2)$ .

**Fact 6.**  $d_M(I_1 \times R_2) = d_{\overline{G(R_1)}}(I_1)$ .

Thus the total degree of vertices of  $M$  is

$$(t_1 + t_2 - 2) + (t_2 - 2)(t_1 - 1) + (t_1 - 2)(t_2 - 1) + 3(2x + 2y) = 2(ab + 3x + 3y),$$

as the sum of degrees of all nontrivial ideals of  $R_1$  (respectively,  $R_2$ ) in the graph  $\overline{G(R_1)}$  (respectively,  $\overline{G(R_2)}$ ) is twice the number of edges of  $\overline{G(R_1)}$  (respectively,  $\overline{G(R_2)}$ ). Therefore the number of edges of  $\overline{G(R)}$  with both end points corresponding to ideals in  $B$  is given by

$$ab + 3x + 3y.$$

Hence the total number of edges of  $\overline{G(R)}$  is

$$\begin{aligned} 2xy + 2(a - 1)y + 2(b - 1)x + ab + 3x + 3y &= 2xy + 2ay + 2bx + ab + x + y \\ &= 2(x + a)(y + b) + x + y - ab \end{aligned}$$

as required. ■

**Corollary 2.** Let  $F_1, F_2, \dots, F_n$  be  $n \geq 2$  fields and  $F = F_1 \times F_2 \times \dots \times F_n$ . Then  $|V(G(F))| = 2^n - 2$  and

$$e(G(F)) = \frac{1}{2}(2^{2n} - 3 \cdot 2^n - 3^n + 5).$$

**Proof.** Since the number of ideals of  $F$  is  $2^n$ , we have  $|V(G(F))| = 2^n - 2$ . We prove the result by induction on  $n$ . For  $n = 2$ , there are only 2 nontrivial ideals of  $F$ , namely,  $F_1 \times \{0_{F_2}\}$  and  $\{0_{F_1}\} \times F_2$ . The vertices of  $G(F)$  corresponding to these ideals are non-adjacent and so  $e(G(F)) = 0 = \frac{1}{2}(2^4 - 3 \cdot 2^2 - 3^2 + 5)$ .

Thus the result is true for  $n = 2$ . Suppose the result is true for  $n = k - 1$ . Let  $R_1 = F_1 \times F_2 \times \dots \times F_{k-1}$  and  $R_2 = F_k$ . Then  $F = R_1 \times R_2$ . Now by inductive hypothesis,  $e(G(R_1)) = \frac{1}{2}(2^{2(k-1)} - 3 \cdot 2^{(k-1)} - 3^{(k-1)} + 5)$ . Then straightforward calculations show that  $x = \frac{1}{2}(3^{k-1} - 2^k + 1)$ ,  $y = 0$ ,  $a = 2^{k-1} - 1$ ,  $b = 1$  and  $e(G(F)) = \frac{1}{2}(2^{2k} - 3 \cdot 2^k - 3^k + 5)$ , by (1), as required. ■

### 3. EULERIANITY AND HAMILTONICITY

Let  $R = R_1 \times R_2$ , where  $R_i$  is a commutative ring with identity, and with  $t_i$  ideals for  $i = 1, 2$ . By definition  $G(R_i)$  has  $t_i - 2$  vertices, for  $i = 1, 2$ . Let the ideals of  $R_1$  be  $0 = I_0, I_1, \dots, I_{t_1-2}, I_{t_1-1} = R_1$ , and let the ideals of  $R_2$  be  $0 = J_0, J_1, \dots, J_{t_2-2}, J_{t_2-1} = R_2$ . It is clear that  $I \trianglelefteq R_1 \times R_2$  if and only if  $I = I_1 \times I_2$ , where  $I_1 = \{x_1 : (x_1, x_2) \in I\}$ , and  $I_2 = \{x_2 : (x_1, x_2) \in I\}$ .

For an ideal  $I$  of  $R$  in order to avoid repeating “the vertex in  $G(R)$  corresponded with  $I$ ” we henceforth simply use “the vertex  $I$ ”. It is well known that a connected graph is Eulerian if and only if its vertices all have even degree. We now calculate the degree of a vertex in  $G(R)$ . Let  $I_i \times J_j \trianglelefteq R$ .

If  $i = 0, j \notin \{0, t_2 - 1\}$ , then  $deg_{G(R)}(I_i \times J_j) = t_1 - 1 + t_1 - 1 + t_1 deg_{R_2}(J_j)$ . If  $i = 0, j = t_2 - 1$ , then  $deg_{G(R)}(I_i \times J_j) = t_1(t_2 - 1) - 2$ . If  $i \notin \{0, t_1 - 1\}, j = 0$ , then  $deg_{G(R)}(I_i \times J_j) = t_2 - 1 + t_2 - 1 + t_2 deg_{R_1}(I_i)$ . If  $i = t_1 - 1, j = 0$ , then  $deg_{G(R)}(I_i \times J_j) = t_2(t_1 - 1) - 2$ . If  $i = t_1 - 1, j \notin \{0, t_2 - 1\}$ , then  $deg_{G(R)}(I_i \times J_j) = t_2(t_1 - 1) - 1 + deg_{G(R_2)}(J_j) + 1$ . If  $i \notin \{0, t_1 - 1\}, j = t_2 - 1$ , then  $deg_{G(R)}(I_i \times J_j) = t_1(t_2 - 1) - 1 + deg_{G(R_1)}(I_i) + 1$ . If  $\{i, j\} \cap \{0, t_2 - 1, t_1 - 1\} = \emptyset$ , then

$$\begin{aligned} deg_{G(R)}(I_i \times J_j) &= t_2 - 1 + t_1 - 1 + t_1 - 2 + t_2 - 2 \\ &\quad + (t_2 - 2)deg_{G(R_1)}(I_i) + (t_1 - 2)deg_{G(R_2)}(J_j) \\ &\quad - deg_{G(R_1)}(I_i)deg_{G(R_2)}(J_j). \end{aligned}$$

So the degrees of vertices of  $G(R)$  are listed in the following forms:

- (1):  $2(t_1 - 1) + t_1 deg_{R_2}(J_j)$ , and  $2(t_2 - 1) + t_2 deg_{R_1}(I_i)$ .
- (2):  $t_1(t_2 - 1) - 2$ , and  $t_2(t_1 - 1) - 2$ .
- (3):  $t_2(t_1 - 1) + deg_{G(R_2)}(J_j)$ , and  $t_1(t_2 - 1) + deg_{G(R_1)}(I_i)$ .
- (4):  $2(t_1 - 1) + 2(t_2 - 1) - 2 + (t_2 - 2)deg_{G(R_1)}(I_i) + (t_1 - 2)deg_{G(R_2)}(J_j) - deg_{G(R_1)}(I_i)deg_{G(R_2)}(J_j)$ .

**Theorem 3.** *Let  $G(R_i)$  be connected for  $i = 1, 2$ . If  $G(R_1 \times R_2)$  is Eulerian, then both  $G(R_1)$  and  $G(R_2)$  are Eulerian.*

**Proof.** Let  $G(R_1 \times R_2)$  be Eulerian. Then the degree of any vertex of  $G(R)$  is even. From Form (2) we have that both  $t_1(t_2 - 1)$  and  $t_2(t_1 - 1)$  are even. Then by Form (3),  $\deg_{G(R_1)}(I_i)$  and  $\deg_{G(R_2)}(J_j)$  are even. Then both  $G(R_1)$  and  $G(R_2)$  are Eulerian. ■

The converse of Theorem 3 is not true in general.

**Theorem 4.** *Let  $R_i$  be a ring with  $t_i$  ideals such that  $G(R_i)$  be Eulerian for  $i = 1, 2$ . Then  $G(R_1 \times R_2)$  is Eulerian if and only if  $t_1 + t_2$  is even.*

**Proof.** Let  $G(R_1)$  and  $G(R_2)$  be Eulerian. Then  $\deg_{G(R_1)}(I_i)$  and  $\deg_{G(R_2)}(J_j)$  are even. So Forms (1) and (4) are even. If  $t_1 + t_2$  is even, then either both  $t_1$  and  $t_2$  are even, or both are odd. In each case Forms (2) and (3) are even. Consequently,  $G(R)$  is Eulerian. For the converse suppose that  $G(R)$  is Eulerian. Then Forms (2) and (3) are even. Since  $\deg_{G(R_1)}(I_i)$  and  $\deg_{G(R_2)}(J_j)$  are even, both  $t_1(t_2 - 1)$  and  $t_2(t_1 - 1)$  are even. This implies that  $t_1 + t_2$  is even. ■

We next study Hamiltonicity of the intersection graph of ideals of direct product of commutative rings. If  $C$  is a cycle in a graph with vertex set  $V(C) = \{v_1, v_2, \dots, v_n\}$  and edge set  $\{v_i v_{i+1} : i = 1, 2, \dots, n - 1\} \cup \{v_1 v_n\}$ , then we refer to this cycle as  $C : v_1 - v_2 - \dots - v_n - v_1$ . We begin with the following.

**Proposition 5.** *If  $G(R_1)$  or  $G(R_2)$  is Hamiltonian, then  $G(R)$  is Hamiltonian.*

**Proof.** Let  $G(R_2)$  be Hamiltonian. It is obvious that  $t_2 \geq 5$ . Without loss of generality assume that the Hamiltonian cycle of  $G(R_2)$  is  $J_1 - J_2 - \dots - J_{t_2-2} - J_1$ . Consider the following cycles in  $G(R)$ :

$$C_0 : 0 \times J_1 - 0 \times J_2 - \dots - 0 \times J_{t_2-2} - 0 \times R_2 - 0 \times J_1,$$

$$C_{t_1-1} : R_1 \times 0 - R_1 \times J_1 - R_1 \times J_2 - \dots - R_1 \times J_{t_2-2} - R_1 \times 0,$$

and for  $1 \leq i \leq t_1 - 2$ ,

$$C_i : I_i \times 0 - I_i \times J_1 - \dots - I_i \times J_{t_2-2} - I_i \times R_2 - I_i \times 0.$$

From the above cycles we obtain a Hamiltonian cycle as follows. Let  $0 \leq r \leq t_1 - 2$ .

If  $r$  is even, then we remove the edges  $(I_r \times J_1)(I_r \times J_2)$ ,  $(I_{r+1} \times J_1)(I_{r+1} \times J_2)$ , and add the edges  $(I_r \times J_1)(I_{r+1} \times J_1)$ ,  $(I_r \times J_2)(I_{r+1} \times J_2)$ .

If  $r$  is odd, then we remove the edges  $(I_r \times J_2)(I_r \times J_3)$ ,  $(I_{r+1} \times J_2)(I_{r+1} \times J_3)$ , and add the edges  $(I_r \times J_2)(I_{r+1} \times J_2)$ ,  $(I_r \times J_3)(I_{r+1} \times J_3)$ .

Thus  $G(R)$  is Hamiltonian. ■

Note that the converse of Proposition 5 is not true in general. As an example let  $F_i$  is a field for  $i = 1, 2, 3$ . Then  $F_1 \times 0 \times 0 - F_1 \times F_2 \times 0 - 0 \times F_2 \times 0 - 0 \times F_2 \times F_3 - 0 \times 0 \times F_3 - F_1 \times 0 \times F_3 - F_1 \times 0 \times 0$  is a Hamiltonian cycle in  $G(F_1 \times F_2 \times F_3)$ . But  $G(S)$  is not Hamiltonian for  $S \in \{F_i, F_i \times F_j : i, j = 1, 2, 3, i \neq j\}$ .

**Lemma 6.** *If  $\min\{t_1, t_2\} = 3$ , then  $G(R)$  is Hamiltonian.*

**Proof.** Let  $\min\{t_1, t_2\} = 3$ . Without loss of generality assume that  $t_2 = \min\{t_1, t_2\}$ . If  $t_1 = 3$ , then  $I_1 \times 0 - I_1 \times J_1 - 0 \times J_1 - 0 \times R_2 - I_1 \times R_2 - R_1 \times J_1 - R_1 \times 0 - I_1 \times 0$  is a Hamiltonian cycle in  $G(R)$ . So we suppose that  $t_1 \geq 4$ . Then,

$$\begin{array}{r}
0 \times J_1 - 0 \times R_2 - \\
I_1 \times J_1 - I_1 \times 0 - I_1 \times R_2 - \\
I_2 \times J_1 - I_2 \times 0 - I_2 \times R_2 - \\
I_3 \times J_1 - I_3 \times 0 - I_3 \times R_2 - \\
\cdots \\
I_i \times J_1 - I_i \times 0 - I_i \times R_2 - \\
\cdots \\
I_{t_1-3} \times J_1 - I_{t_1-3} \times 0 - I_{t_1-3} \times R_2 - \\
I_{t_1-2} \times J_1 - R_1 \times 0 - I_{t_1-2} \times 0 - I_{t_1-2} \times R_2 - R_1 \times J_1 - 0 \times J_1
\end{array}$$

is a Hamiltonian cycle. ■

**Proposition 7.** *If  $\min\{t_1, t_2\} \geq 4$ , then  $G(R)$  is Hamiltonian.*

**Proof.** Let  $4 \leq t_2 \leq t_1$ . For  $i = 1, 2, \dots$  or  $\dots, t_2 - 3$  let  $C_i$  be the following cycle.

$$I_i \times 0 - I_i \times J_1 - \cdots - I_i \times J_i - 0 \times J_i - I_{i-1} \times R_2 - I_i \times J_{i+1} - \cdots - I_i \times J_{t_2-2} - I_i \times 0.$$

Let  $C_{t_2-2}$  be the following cycle.

$$I_{t_2-2} \times 0 - I_{t_2-2} \times J_1 - \cdots - I_{t_2-2} \times J_{t_2-2} - 0 \times J_{t_2-2} - I_{t_2-3} \times R_2 - I_{t_2-2} \times R_2 - I_{t_2-2} \times 0.$$

For  $t_2 - 1 \leq i \leq t_1 - 2$ , let  $C_i$  be the cycle

$$I_i \times 0 - I_i \times J_1 - \cdots - I_i \times J_{t_2-2} - I_i \times R_2 - I_i \times 0.$$

Let  $C_{t_1-1}$  be the cycle  $R_1 \times 0 - R_1 \times J_1 - \cdots - R_1 \times J_{t_2-2} - R_1 \times 0$ .

We produce a Hamiltonian cycle from above cycles as follows. We remove the edges  $(0 \times J_1)(0 \times R_2)$  and  $(I_1 \times R_2)(I_2 \times J_3)$ , and add the edges  $(0 \times J_1)(I_1 \times R_2)$  and  $(0 \times R_2)(I_2 \times J_3)$  to obtain a cycle  $C'_2$  from  $C_1$  and  $C_2$ . We remove the edges  $(0 \times J_2)(I_1 \times R_2)$  and  $(I_2 \times R_2)(I_3 \times J_3)$  and add the edges  $(0 \times J_2)(I_2 \times R_2)$  and  $(I_1 \times R_2)(I_3 \times J_3)$  to obtain a cycle  $C'_3$  from  $C'_2$  and  $C_3$ . In general from  $C'_i$  and  $C_{i+1}$  we obtain a cycle  $C'_{i+1}$  by removing the edges  $(0 \times J_i)(I_{i-1} \times R_2)$  and  $(I_i \times R_2)(I_{i+1} \times J_{i+2})$ , and add the edges  $(0 \times J_i)(I_i \times R_2)$  and  $(I_{i-1} \times R_2)(I_{i+1} \times J_{i+2})$ . We proceed this process to obtain the cycle  $C'_{t_2-2}$ .

Now we remove the edges  $(I_{t_2-2} \times J_1)(I_{t_2-2} \times J_2)$  and  $(I_{t_2-1} \times J_1)(I_{t_2-1} \times J_2)$  from  $C'_{t_2-2}$  and  $C_{t_2-1}$  and add the edges  $(I_{t_2-2} \times J_1)(I_{t_2-1} \times J_1)$  and  $(I_{t_2-2} \times J_2)(I_{t_2-1} \times J_2)$ , to obtain a cycle  $C'_{t_2-1}$ . We remove  $(I_{t_2-1} \times J_2)(I_{t_2-1} \times J_3)$  and  $(I_{t_2} \times J_2)(I_{t_2} \times J_3)$  from  $C'_{t_2-1}$  and  $C_{t_2}$  and add the edges  $(I_{t_2-1} \times J_2)(I_{t_2} \times J_2)$  and  $(I_{t_2-1} \times J_3)(I_{t_2} \times J_3)$ , obtain a cycle  $C'_{t_2}$ .

We remove  $(I_{t_2} \times J_1)(I_{t_2} \times J_2)$  and  $(I_{t_2+1} \times J_1)(I_{t_2+1} \times J_2)$  from  $C'_{t_2}$  and  $C_{t_2+1}$  and add the edges  $(I_{t_2} \times J_1)(I_{t_2+1} \times J_1)$  and  $(I_{t_2} \times J_2)(I_{t_2+1} \times J_2)$ , obtain a cycle  $C'_{t_2+1}$ .

We proceed this process to obtain  $C'_{t_1-2}$ . Lastly, we remove  $(I_{t_1-2} \times 0)(I_{t_1-2} \times J_1)$  and  $(R_2 \times 0)(R_2 \times J_1)$  from  $C'_{t_1-2}$ , and  $C_{t_1-1}$ , and add  $(I_{t_1-2} \times 0)(R_2 \times 0)$  and  $(I_{t_1-2} \times J_1)(R_2 \times J_1)$  to obtain  $C'_{t_1-2}$ . Then  $C'_{t_1-2}$  is a Hamiltonian cycle. ■

**Corollary 8.** *If  $\min\{t_1, t_2\} \geq 3$ , then  $G(R)$  is Hamiltonian.*

**Lemma 9.** *If  $R_i$  is a commutative ring with identity for  $i = 1, 2, 3$ , then  $G(R_1 \times R_2 \times R_3)$  is Hamiltonian.*

**Proof.** Let the number of ideals of  $R_i$  be equal to  $t_i$  for  $i = 1, 2, 3$ . Since the graph of direct product of three fields is Hamiltonian, we assume that  $t_i \geq 3$  for some  $i$ . Then Corollary 8 implies that  $G(R_1 \times R_2 \times R_3)$  is Hamiltonian. ■

By Lemma 9, we have the following corollary.

**Corollary 10.** *For two commutative rings  $R_1$  and  $R_2$  with identity, if  $R_1$  or  $R_2$  is direct product of two rings, then  $G(R)$  is Hamiltonian.*

In the rest of the paper we consider  $t_1 = 2$ . By Corollary 8 it remains to study Hamiltonicity in  $G(R)$  when  $R_1$  is a field (i.e.,  $t_1 = 2$ ). The following has a straightforward proof.

**Observation 11.** *If  $t_2 \in \{2, 3\}$ , then  $G(R)$  is not Hamiltonian.*

So henceforth we suppose that  $t_2 \geq 4$ . Also by Corollary 10, we assume  $R_2$  is not direct product of two rings. Since  $R_2$  is an Artinian ring, henceforth  $(R_2, m)$  is local. (We recall that a ring is *local* if it has only one maximal ideal. We also refer  $(R, M)$  as a local ring  $R$  with the unique maximal ideal  $M$ ).

**Lemma 12.** *If  $R_2$  has a unique minimal ideal, then  $G(R)$  is Hamiltonian.*

**Proof.** First suppose that  $t_2 = 4$ . Let  $J_1$  be the unique minimal ideal of  $R_2$ . Let  $J_2$  be the other proper non-trivial ideal. Since  $R_2$  is not direct product of two rings,  $J_2$  is the unique maximal ideal of  $R_2$ . Now  $0 \times J_1 - 0 \times R_2 - 0 \times J_2 - R_1 \times J_2 - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$  is a Hamiltonian cycle. We next suppose that  $t_2 \geq 5$ . Notice that  $G(R_2)$  is a complete graph and so is Hamiltonian. Then the result follows by Proposition 5. ■



Since for any  $n \geq 2$ , the ring  $\mathbb{Z}_{p^n}$  has a unique minimal ideal, a consequence of Corollaries 8, 10, Observation 11, and Lemma 12 we obtain the following.

**Proposition 13.** *For  $k \geq 2$ ,  $\mathbb{Z}_{p_1^{n_1}} \times \mathbb{Z}_{p_2^{n_2}} \times \cdots \times \mathbb{Z}_{p_k^{n_k}}$  is not Hamiltonian if and only if  $k = 2$  and  $n_1 + n_2 \leq 3$ .*

Since for a prime  $p$ ,  $\mathbb{Z}_{p^n}$  is not Hamiltonian if and only if  $n \leq 3$ , as a consequence of Proposition 13 we obtain the following.

**Corollary 14** (Theorem 5.2 of [2]). *All the graphs of  $\mathbb{Z}_n$  are Hamiltonian except when  $n$  is one of the following forms:  $p^2, p^3, pq, p^2q$ , where  $p, q$  are distinct primes.*

**Lemma 15.** *If  $F$  is an infinite field and  $V$  is a  $n$ -dimensional vector space over  $F$ , then  $V$  has infinite subspaces.*

**Lemma 16.** *If  $F$  is a finite field of order  $q$  and  $V$  is a  $n$ -dimensional vector space over  $F$ , then  $V$  has  $\frac{q^n-1}{q-1} = q^{n-1} + q^{n-2} + \cdots + 1$ , 1-dimensional subspaces.*

**Corollary 17.** *Let  $(R_2, M)$  is a local ring such that  $\frac{R_2}{M}$  is finite. If  $I_1, \dots, I_t$  are minimal ideals of  $R_2$  for  $t \geq 2$  such that for any  $i$ ,  $I_i \cong \frac{R_2}{M}$ , and  $I_1 + \cdots + I_t = I_1 \oplus I_2 \oplus \cdots \oplus I_t = J$ , then  $J$  contains  $|\frac{R_2}{M}|^{t-1} + |\frac{R_2}{M}|^{t-2} + \cdots + 1$  minimal ideals of  $R_2$ .*

**Proof.** Notice that  $R_2$  is a Noetherian ring and  $MI_i = 0$  for  $i = 1, 2, \dots, t$ . Thus  $J$  is an  $\frac{R_2}{M}$ -module (vector space) which a subset  $T$  is an  $\frac{R_2}{M}$ -submodule of  $J$  if and only if  $T$  is an  $R_2$ -submodule. ■

**Theorem 18.** *For  $t_2 = 4, 5, \dots, 9$ ,  $G(R)$  is Hamiltonian.*

**Proof.** For  $t_2 = 4$  the result follows from the proof of Lemma 12.

If  $R_2$  has a unique minimal ideal then by Lemma 12,  $G(R_2)$  is complete and so is Hamiltonian. Then by Proposition 5,  $G(R)$  is Hamiltonian. Thus we assume that  $R_2$  has at least two minimal ideals.

If  $t_2 = 5$ , then by Corollary 17,  $R_2$  has at least six ideals, a contradiction. Thus  $t_2 \neq 5$ .

- $t_2 = 6$ . Let  $J_1, J_2$  be two minimal ideals of  $R_2$ . By Corollary 17,  $J_1 + J_2$  contains at least three minimal ideals of  $R_2$ . Without loss of generality assume that  $J_1 + J_2$  contains three minimal ideals  $J_1, J_2$  and  $J_3$ . Since  $t_2 = 6$ , all ideals of  $R_2$  are  $0, J_1, J_2, J_3, J_1 + J_2 = M, R$ . Now,  $0 \times J_1 - 0 \times R_2 - 0 \times J_2 - R_1 \times J_2 - R_1 \times J_3 - 0 \times J_3 - 0 \times (J_1 + J_2) - R_1 \times (J_1 + J_2) - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$  is a Hamiltonian cycle.

- $t_2 = 7$ . Let  $J_1, J_2$  be two distinct minimal ideals of  $R_2$ . By Corollary 17,  $J_1 + J_2$  contains at least three minimal ideals of  $R_2$ . Without loss of generality assume

that  $J_1 + J_2$  contains three minimal ideals  $J_1, J_2$  and  $J_3$ . Let  $J$  be an ideal of  $R_2$  different from  $J_1, J_2, J_3$  and  $J_1 + J_2$ . If  $J \cap (J_1 + J_2) = 0$ , then  $R_2$  has more than 7 ideals, a contradiction. So  $J \cap (J_1 + J_2) \neq 0$ . Now  $0 \times J_1 - 0 \times (J_1 + J_2) - 0 \times J_2 - R_1 \times J_2 - R_1 \times J_3 - 0 \times J_3 - 0 \times R_2 - 0 \times J - R_1 \times J - R_1 \times (J_1 + J_2) - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$ , is a Hamiltonian cycle.

•  $t_2 = 8$ . Let  $J_1, J_2$  be two distinct minimal ideals of  $R_2$ . Since  $R_2$  is local,  $J_1 \cong J_2 (\cong \frac{R_2}{M})$ . Then  $J_1 + J_2 = J_1 \oplus J_2$ . If  $R_2$  has another minimal ideal  $K$  such that  $J_1 + J_2 + K = J_1 \oplus J_2 \oplus K$ , then by Corollary 17,  $J_1 + J_2 + K$  contains at least 7 minimal ideals contradicting that  $t_2 = 8$ . We deduce that  $J_1 + J_2$  contains all minimal ideals of  $R_2$ . Further,  $|\frac{R_2}{M}| \leq 4$ . We discuss the possibilities for  $|\frac{R_2}{M}|$ .

If  $|\frac{R_2}{M}| = 2$ , then  $R_2$  has three minimal ideals  $J_1, J_2, J_3$ . Also note that  $M \neq J_1 + J_2$ , since  $t_2 = 8$ . Let  $J$  be a proper non-trivial ideal different from  $J_1, J_2, J_3, J_1 + J_2, M$ . Since  $J$  is not minimal we may assume that  $J_1 \subseteq J$ . Now  $0 \times J_1 - 0 \times (J_1 + J_2) - R_1 \times (J_1 + J_2) - R_1 \times J_2 - 0 \times J_2 - 0 \times M - R_1 \times M - R_1 \times J_3 - 0 \times J_3 - 0 \times R_2 - 0 \times J - R_1 \times J - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$  is a Hamiltonian cycle.

If  $|\frac{R_2}{M}| = 3$ , then  $R_2$  has four minimal ideals  $J_1, J_2, J_3, J_4$ . Since  $t_2 = 8$ , we have  $M \neq J_1 + J_2$  and so  $0, J_1, J_2, J_3, J_4, J_1 + J_2, M, R_2$  are all ideals of  $R_2$ . Now  $0 \times J_1 - 0 \times (J_1 + J_2) - 0 \times J_2 - R_1 \times J_2 - R_1 \times J_3 - 0 \times J_3 - 0 \times R_2 - 0 \times M - 0 \times J_4 - R_1 \times J_4 - R_1 \times (J_1 + J_2) - R_1 \times M - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$  is a Hamiltonian cycle.

If  $|\frac{R_2}{M}| = 4$ , then  $R_2$  has five minimal ideals  $J_1, J_2, J_3, J_4, J_5$ . So  $0, J_1, J_2, J_3, J_4, J_5, J_1 + J_2, R_2$  are all ideals of  $R_2$ . Now  $0 \times J_1 - 0 \times (J_1 + J_2) - 0 \times J_2 - R_1 \times J_2 - R_1 \times (J_1 + J_2) - 0 \times J_5 - R_1 \times J_5 - R_1 \times J_4 - 0 \times J_4 - 0 \times R_2 - 0 \times J_3 - R_1 \times J_3 - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$  is a Hamiltonian cycle.

•  $t_2 = 9$ . Let  $J_1, J_2$  be two distinct minimal ideals of  $R_2$ . Since  $R$  is local,  $J_1 \cong J_2 (\cong \frac{R_2}{M})$ . Let  $|\frac{R_2}{M}| = q$ . Then by Corollary 17  $q \in \{2, 3, 4, 5, 7\}$ , since  $\frac{R_2}{M}$  is a field. Consequently by Corollary 17  $R_2$  has  $q + 1$  minimal ideals. It is obvious that  $q \neq 7$ . If  $q = 5$ , then the ideals of  $R_2$  are  $0, J_1, J_2, J_3, J_4, J_5, J_6, J_1 + J_2, R_2$ , where  $J_i$  is minimal and  $J_i \subseteq J_1 + J_2$  for  $i = 1, 2, 3, 4, 5, 6$ . Then  $R_1 \times 0 - R_1 \times J_2 - 0 \times J_2 - 0 \times R_2 - 0 \times J_1 - R_1 \times J_1 - R_1 \times J_3 - 0 \times J_3 - 0 \times (J_1 + J_2) - 0 \times J_4 - R_1 \times J_4 - R_1 \times J_5 - 0 \times J_5 - R_1 \times (J_1 + J_2) - 0 \times J_6 - R_1 \times J_6 - R_1 \times 0$  is a Hamiltonian cycle. If  $q = 4$ , then the ideals of  $R_2$  are  $0, J_1, J_2, J_3, J_4, J_5, J_1 + J_2, M, R_2$ , where  $J_i$  is minimal for  $i = 1, 2, 3, 4, 5$ . Then  $0 \times J_1 - 0 \times (J_1 + J_2) - 0 \times J_2 - R_1 \times J_2 - R_1 \times J_3 - 0 \times J_3 - 0 \times R_2 - 0 \times J_4 - R_1 \times J_4 - R_1 \times J_5 - 0 \times J_5 - 0 \times M - R_1 \times M - R_1 \times (J_1 + J_2) - R_1 \times 0 - R_1 \times J_1 - 0 \times J_1$ , is a Hamiltonian cycle. If  $q = 3$ ,  $J_1 + J_2$  contains four minimal ideals  $J_1, J_2, J_3, J_4$ . Let  $J$  be another proper non-trivial ideal. This time  $0 \times J_1 - 0 \times (J_1 + J_2) - 0 \times J_2 - 0 \times M - 0 \times J_3 - 0 \times R_2 - 0 \times J - R_1 \times J - R_1 \times M - R_1 \times 0 - R_1 \times (J_1 + J_2) - 0 \times J_4 - R_1 \times J_4 - R_1 \times J_3 - R_1 \times J_2 -$

$R_1 \times J_1 - 0 \times J_1$  is a Hamiltonian cycle. If  $q = 2$ , then by Corollary 17,  $J_1 + J_2$  contains three minimal ideals of  $J_1, J_2, J_3$  of  $R_2$ . Also  $R_2$  has no minimal ideal. Suppose that the ideals of  $R_2$  are  $0, J_1, J_2, J_1 + J_2, J_3, K_1, K_2, K_3, R_2$ . Without loss of generality assume that  $J_1 \subseteq K_1$ . Note that  $K_i \cap (J_1 + J_2) \neq 0$ . Then  $0 \times J_1 - 0 \times K_1 - 0 \times (J_1 + J_2) - 0 \times J_3 - 0 \times R_2 - 0 \times J_2 - R_1 \times J_2 - R_1 \times K_2 - 0 \times K_2 - 0 \times K_3 - R_1 \times K_3 - R_1 \times 0 - R_1 \times K_1 - R_1 \times (J_1 + J_2) - R_1 \times J_3 - R_1 \times J_1 - 0 \times J_1$  is a Hamiltonian cycle. ■

We close with the following problem.

**Problem 19.** Is  $G(R)$  Hamiltonian for  $t_2 \geq 10$ ?

#### REFERENCES

- [1] M.F. Atiyah and I.G. Macdonald, *Introduction to Commutative Algebra* (Addison-Wesley Publishing Co, Reading, Mass.-London-Don Mills Ont, 1969).
- [2] I. Chakrabarty, Sh. Ghosh, T.K. Mukherjee and M.K. Sen, *Intersection graphs of ideals of rings*, *Discrete Math.* **309** (2009) 5381–5392.  
doi:10.1016/j.disc.2008.11.034
- [3] B. Csikny and G. Pollk, *The graph of subgroups of a finite group*, *Czechoslovak Math. J.* **19** (1969) 241–247.
- [4] R.P. Grimaldi, *Graphs from rings*, *Congr. Numer.* **71** (1990) 95–103.
- [5] E. Szpilrajn-Marczewski, *Sur deux propriétés des classes d'ensembles*, *Fund. Math.* **33** (1945) 303–307.
- [6] D.B. West, *Introduction To Graph Theory* (Prentice-Hall of India Pvt. Ltd, 2003).
- [7] B. Zelinka, *Intersection graphs of finite abelian groups*, *Czechoslovak Math. J.* **25** (2) (1975) 171–174.

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