

FILTERS OF LATTICES WITH RESPECT TO A CONGRUENCE

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Abstract

Some properties of filters on a lattice L are studied with respect to a congruence on L . The notion of a θ -filter of L is introduced and these filters are then characterized in terms of classes of θ . For distributive L , an isomorphism between the lattice of θ -filters of L and the lattice of filters of L/θ is obtained.

Keywords: congruence, filter, closure operator, θ -filter, congruence lattice.

2010 Mathematics Subject Classification: 06D99.

1. INTRODUCTION

Ideals and filters in lattices were investigated by Tarski, Moisil and others, many of whose results are found in Birkhoff's *Lattice Theory* [2]. In 1980, T.S. Blyth [3] discussed ideals and filters of pseudo-complemented semilattices. In [5], G. Grätzer and E.T. Schmidt examined the properties of lattice congruences and dealt with minimal congruence relations in distributive lattices.

In this paper, two mappings are introduced, one from the lattice of filters of a lattice into the lattice of filters of its congruence lattice and the other from the lattice of filters of the congruence lattice into the lattice of filters of the given lattice. Later it is shown that their composition is a closure operator on the lattice of filters. The concept of θ -filters is introduced in lattices with respect to a congruence θ and there are studied some of their properties. The θ -filters are also characterized in terms of congruence classes. Equivalent conditions are derived for every filter of a lattice to become a θ -filter. Finally, an isomorphism is obtained between the lattice of θ -filters of a lattice and the lattice of filters of its congruence lattice.

The reader is referred to [2] for notions and notations. However, some of the preliminary definitions and results are presented for the ready reference of the reader. Throughout the rest of this note, L stands for a bounded lattice unless otherwise mentioned.

2. PRELIMINARIES

In this section, we present certain definitions and important results taken mostly from [1, 2] and [4], those will be required in the paper.

Definition [2]. An algebra (L, \wedge, \vee) of type $(2, 2)$ is called a lattice if for all $x, y, z \in L$, it satisfies the following properties:

- (1) $x \wedge x = x, x \vee x = x,$
- (2) $x \wedge y = y \wedge x, x \vee y = y \vee x,$
- (3) $(x \wedge y) \wedge z = x \wedge (y \wedge z), (x \vee y) \vee z = x \vee (y \vee z),$
- (4) $(x \wedge y) \vee x = x, (x \vee y) \wedge x = x.$

Definition [1]. A lattice L is called distributive if for all $x, y, z \in L$ it satisfies either of the following properties:

- (1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z),$
- (2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$

The least element of a lattice is denoted by 0 and the greatest element by 1. A lattice L with both 0 and 1 is called a bounded lattice.

Definition [2]. Let (L, \wedge, \vee) be a lattice. A partial ordering relation \leq is defined on L by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Definition [2]. A non-empty subset F of a lattice L is called a filter of L if $a \wedge b \in F$ and $a \vee x \in F$ whenever $a, b \in F$ and $x \in L$. A filter F is called proper if $F \neq L$. Let $\mathcal{F}(L)$ denote the set of all filters of the lattice L .

Definition [2]. Let (L, \wedge, \vee) be a lattice. For any $x \in L$, the set $[a] = \{x \in L \mid a \leq x\}$ is a filter which is called the principal filter generated by a .

Definition [2]. Let (L, \wedge, \vee) be a lattice. A proper filter P of L is called a prime filter if for any $a, b \in L$, $a \vee b \in P$ implies $a \in P$ or $b \in P$.

A non-empty set I of a lattice L is called an ideal if $a \vee b \in I$ and $a \wedge x \in I$ whenever $a, b \in I$ and $x \in L$. According to M.H. Stone's celebrated theorem for prime filters, if F is a filter and I is an ideal of a distributive lattice L such that $F \cap I = \emptyset$, then there exists a prime filter P such that $F \subseteq P$ and $I \cap P = \emptyset$.

Definition [4]. A binary relation θ defined on L is called a congruence on L if it satisfies the following conditions:

- (1) θ is an equivalence relation on L .
- (2) $(a, b), (c, d) \in \theta$ implies $(a \wedge c, b \wedge d), (a \vee c, b \vee d) \in \theta$.

For any $a \in L$, the equivalence class of the element a with respect to the congruence θ is defined as $[a]_\theta = \{x \in L \mid (a, x) \in \theta\}$.

Theorem 2.1 [2]. *An equivalence relation θ on L is a congruence on L if and only if $(a, b) \in \theta$ implies $(a \wedge c, b \wedge c), (a \vee c, b \vee c) \in \theta$ for any $c \in L$.*

3. θ -FILTERS IN LATTICES

Let us recall from [4] that the set $Con(L)$ of congruences on a lattice is ordered by set inclusion. It is easily seen to be a topped \cap -structure on L^2 . Hence $Con(L)$, when ordered by inclusion, is a complete lattice. The least element $0_{Con(L)}$ and the greatest element $1_{Con(L)}$ are given by $0_{Con(L)} = \{(a, a) \mid a \in L\}$ and $1_{Con(L)} = L^2$.

In the following, we first introduce two mappings.

Definition. Let θ be a congruence on a lattice L . Define mappings $\vec{\theta}$ and $\overleftarrow{\theta}$ as follows:

- (1) For any filter F of L , define $\vec{\theta}(F) := \{ [x]_\theta \mid (x, y) \in \theta \text{ for some } y \in F \}$.
- (2) For any filter \widehat{F} of L/θ , define $\overleftarrow{\theta}(\widehat{F}) := \{x \in L \mid (x, y) \in \theta \text{ for some } [y]_\theta \in \widehat{F}\}$.

In the following lemma, some basic properties of the above two mappings are observed.

Lemma 3.1. *Let θ be a congruence on a lattice L . Then we have the following:*

- (1) *For any filter F of L , $\vec{\theta}(F)$ is a filter of L/θ .*

- (2) For any filter \widehat{F} of L/θ , $\overleftarrow{\theta}(\widehat{F})$ is a filter of L .
- (3) $\overrightarrow{\theta}$ and $\overleftarrow{\theta}$ are isotone.
- (4) For any filter F of L , $x \in F$ implies $[x]_\theta \in \overrightarrow{\theta}(F)$.
- (5) For any filter \widehat{F} of L/θ , $[x]_\theta \in \widehat{F}$ implies $x \in \overleftarrow{\theta}(\widehat{F})$.

Proof. (1) If $x \in F$, then $[x]_\theta \in \overrightarrow{\theta}(F)$. Hence $\overrightarrow{\theta}(F) \neq \emptyset$. Let $[x]_\theta, [y]_\theta \in \overrightarrow{\theta}(F)$. Then $(x, x_1) \in \theta$ and $(y, y_1) \in \theta$ for some $x_1, y_1 \in F$. Hence $(x \wedge y, x_1 \wedge y_1) \in \theta$ and $x_1 \wedge y_1 \in F$. Therefore $[x]_\theta \wedge [y]_\theta = [x \wedge y]_\theta \in \overrightarrow{\theta}(F)$. Again, let $[a]_\theta \in \overrightarrow{\theta}(F)$ and $[x]_\theta \in L/\theta$. Hence $(a, b) \in \theta$ for some $b \in F$. Thus we get $(a \vee x, b \vee x) \in \theta$ and $b \vee x \in F$. Hence we have $[a]_\theta \vee [x]_\theta = [a \vee x]_\theta \in \overrightarrow{\theta}(F)$. Therefore $\overrightarrow{\theta}(F)$ is a filter of L/θ .

(2) Clearly $1 \in \overleftarrow{\theta}(\widehat{F})$. Let $x, y \in \overleftarrow{\theta}(\widehat{F})$. Then $(x, x_1) \in \theta$ and $(y, y_1) \in \theta$ for some $[x_1]_\theta, [y_1]_\theta \in \widehat{F}$. Hence $(x \wedge y, x_1 \wedge y_1) \in \theta$. Since \widehat{F} is a filter, we get $[x_1 \wedge y_1]_\theta = [x_1]_\theta \wedge [y_1]_\theta \in \widehat{F}$. Thus we have $x \wedge y \in \overleftarrow{\theta}(\widehat{F})$. Again, let $a \in \overleftarrow{\theta}(\widehat{F})$ and $x \in L$. Then we get $(a, b) \in \theta$ for some $[b]_\theta \in \widehat{F}$ and hence $(a \vee x, b \vee x) \in \theta$. Since \widehat{F} is a filter, we get $[b \vee x]_\theta = [b]_\theta \vee [x]_\theta \in \widehat{F}$. Hence $a \vee x \in \overleftarrow{\theta}(\widehat{F})$. Therefore $\overleftarrow{\theta}(\widehat{F})$ is a filter of L .

(3) Let F_1, F_2 be two filters of L such that $F_1 \subseteq F_2$. Let $[x]_\theta \in \overrightarrow{\theta}(F_1)$. Then $(x, y) \in \theta$ for some $y \in F_1 \subseteq F_2$. Consequently, we get $[x]_\theta \in \overrightarrow{\theta}(F_2)$. Therefore $\overrightarrow{\theta}(F_1) \subseteq \overrightarrow{\theta}(F_2)$. Again, let $\widehat{F}_1, \widehat{F}_2$ be two filters of L/θ such that $\widehat{F}_1 \subseteq \widehat{F}_2$. Suppose $x \in \overleftarrow{\theta}(\widehat{F}_1)$. Then $(x, y) \in \theta$ for some $[y]_\theta \in \widehat{F}_1 \subseteq \widehat{F}_2$. Hence $x \in \overleftarrow{\theta}(\widehat{F}_2)$. Therefore $\overleftarrow{\theta}(\widehat{F}_1) \subseteq \overleftarrow{\theta}(\widehat{F}_2)$.

(4) For any $x \in F$, we have $(x, x) \in \theta$. Hence we conclude that $[x]_\theta \in \overrightarrow{\theta}(F)$.

(5) For any $[x]_\theta \in \widehat{F}$, we have $(x, x) \in \theta$. Hence we get $x \in \overleftarrow{\theta}(\widehat{F})$. ■

It is known that in a distributive lattice L , the class $\mathcal{F}(L)$ of all filters of L forms a complete distributive lattice with respect to the following operations:

$$F \vee G = \{ x \mid x = i \wedge j \text{ for some } i \in F, j \in G \}$$

$$F \wedge G = F \cap G \quad \text{for all } F, G \in \mathcal{F}(L).$$

Theorem 3.2. For any congruence θ on a distributive lattice L , the mapping $\overrightarrow{\theta}$ defined above is a homomorphism from the lattice $\mathcal{F}(L)$ to the lattice of filters of L/θ .

Proof. Let F, G be two filters of L . By the isotone property of $\overrightarrow{\theta}$, it can be observed that $\overrightarrow{\theta}(F) \vee \overrightarrow{\theta}(G) \subseteq \overrightarrow{\theta}(F \vee G)$. Conversely, let $[x]_\theta \in \overrightarrow{\theta}(F \vee G)$. Then we can write $(x, i \wedge j) \in \theta$ for some $i \in F$ and $j \in G$. Since $i \in F$, we get $[i]_\theta \in \overrightarrow{\theta}(F)$. Similarly, we get $[j]_\theta \in \overrightarrow{\theta}(G)$. Thus $[x]_\theta = [i \wedge j]_\theta = [i]_\theta \wedge [j]_\theta \in$

$\overrightarrow{\theta}(F) \vee \overrightarrow{\theta}(G)$. Therefore we conclude $\overrightarrow{\theta}(F) \vee \overrightarrow{\theta}(G) = \overrightarrow{\theta}(F \vee G)$. It is clear that $\overrightarrow{\theta}(F \cap G) \subseteq \overrightarrow{\theta}(F) \cap \overrightarrow{\theta}(G)$. Conversely, let $[x]_{\theta} \in \overrightarrow{\theta}(F) \cap \overrightarrow{\theta}(G)$. Then $(x, y_1) \in \theta$ and $(x, y_2) \in \theta$ for some $y_1 \in F$ and $y_2 \in G$. Therefore $(x, y_1 \vee y_2) = (x \vee x, y_1 \vee y_2) \in \theta$ and $y_1 \vee y_2 \in F \cap G$. Hence we get that $[x]_{\theta} \in \overrightarrow{\theta}(F \cap G)$. Therefore $\overrightarrow{\theta}(F) \cap \overrightarrow{\theta}(G) = \overrightarrow{\theta}(F \cap G)$. Hence $\overrightarrow{\theta}$ is a homomorphism. ■

In view of the mappings $\overrightarrow{\theta}$ and $\overleftarrow{\theta}$, the following result is obvious and hence the proof is omitted.

Theorem 3.3. *Let θ be a congruence on L . Then for any filter F of L ,*

$$\overleftarrow{\theta}(\overrightarrow{\theta}(F)) = \bigcup_{x \in F} [x]_{\theta}.$$

From the above theorem, the following results are clear.

Lemma 3.4. *Let θ be a congruence on L . For any filter F of L , $\overrightarrow{\theta}(\overleftarrow{\theta}(\overrightarrow{\theta}(F))) = \overrightarrow{\theta}(F)$.*

Now it is clear that the composition $\overleftarrow{\theta} \overrightarrow{\theta}$ is a closure operator.

Proposition 3.5. *The mapping $F \rightarrow \overleftarrow{\theta} \overrightarrow{\theta}(F)$ is a closure operator on $\mathcal{F}(L)$. That is, for any two filters F, G of L ,*

- (a) $F \subseteq \overleftarrow{\theta} \overrightarrow{\theta}(F)$,
- (b) $\overleftarrow{\theta} \overrightarrow{\theta}[\overleftarrow{\theta} \overrightarrow{\theta}(F)] = \overleftarrow{\theta} \overrightarrow{\theta}(F)$,
- (c) $F \subseteq G \Rightarrow \overleftarrow{\theta} \overrightarrow{\theta}(F) \subseteq \overleftarrow{\theta} \overrightarrow{\theta}(G)$.

Moreover, we have the following:

Proposition 3.6. *Let θ be a congruence on a distributive lattice L . Then $\overrightarrow{\theta}$ is residuated map with residual map $\overleftarrow{\theta}$.*

We now introduce the notion of θ -filters of a lattice.

Definition. Let θ be a congruence on a lattice L . A filter F of L is called a θ -filter if $\overleftarrow{\theta} \overrightarrow{\theta}(F) = F$.

Let $\mathcal{F}_{\theta}(L)$ denote the set of all θ -filters of a lattice L . For any congruence θ on a bounded lattice L , it can be easily observed that the filter $\{1\}$ is a θ -filter if and only if $[1]_{\theta} = \{1\}$. From Definition 2.1(2), it can be observed that $\overleftarrow{\theta}(\widehat{F}) = \bigcup_{X \in \widehat{F}} X$ for all filters \widehat{F} of L/θ and hence $\overleftarrow{\theta}(\widehat{F})$ is a θ -filter of L and also $\overrightarrow{\theta}(\overleftarrow{\theta}(\widehat{F})) = \widehat{F}$. Moreover, we have the following:

Lemma 3.7. *Let θ be a congruence on a bounded lattice L . Then the following hold:*

- (1) If F is a θ -filter of L then $[1]_\theta \subseteq F$.
- (2) A θ -filter F of L is proper if and only if $F \cap [0]_\theta = \emptyset$.

The following characterization theorem of θ -filters is a direct consequence of the above observations. Hence the proof is omitted.

Theorem 3.8. *Let θ be a congruence on a lattice L . For any filter F of L , the following conditions are equivalent:*

- (1) F is a θ -filter.
- (2) For any $x, y \in L$, $[x]_\theta = [y]_\theta$ and $x \in F$ imply $y \in F$.
- (3) $F = \bigcup_{x \in F} [x]_\theta$.
- (4) $x \in F$ implies $[x]_\theta \subseteq F$.

In the following, a set of equivalent conditions is obtained in order to characterize the smallest congruence in terms of θ -filters of lattices.

Theorem 3.9. *Let θ be a congruence on a lattice L . Then the following conditions are equivalent:*

- (1) θ is the smallest congruence on L .
- (2) Every filter of L is a θ -filter.
- (3) Every principal filter of L is a θ -filter.

Moreover, if L is distributive, then the above conditions are equivalent to the fact that every prime filter of L is a θ -filter.

Proof. (1) \Rightarrow (2) Follows immediately from (4) of Theorem 2.10.

(2) \Rightarrow (3) It is obvious.

(3) \Rightarrow (1) Assume that every principal filter is a θ -filter of L . Let $x, y \in L$ such that $(x, y) \in \theta$. Then $[x]_\theta = [y]_\theta$. Since $[y]$ is a θ -filter of L , we get $x \in [x]_\theta = [y]_\theta \subseteq [y]$. Thus $y \leq x$. Similarly, we get $x \leq y$. Hence θ is the smallest congruence on L .

To prove the remaining assertion, let us assume that L is distributive. Suppose that every prime filter of L is a θ -filter. Let $(x, y) \in \theta$. Suppose $x \neq y$. Without loss of generality, assume that $(x) \cap (y) = \emptyset$. Since L is distributive, there exists a prime filter P of L such that $(y) \subseteq P$ and $P \cap (x) = \emptyset$. Hence $x \notin P$ and $y \in P$, which is a contradiction to the fact that P is a θ -filter of L . Thus $x = y$. Thus θ is the smallest congruence on L . ■

Remark 3.10. Let F, G be two arbitrary filters of a distributive lattice L . Then for any congruence θ on L , by Theorem 2.4 and Proposition 2.6, it can be easily seen that $\overleftarrow{\theta} \overrightarrow{\theta}(F \cap G) = \overleftarrow{\theta} \overrightarrow{\theta}(F) \cap \overleftarrow{\theta} \overrightarrow{\theta}(G)$. Now consider the following set

$$\overleftarrow{\theta} \overrightarrow{\theta}(F \vee G) = \{a \in L \mid [a]_{\theta} = [x \wedge y]_{\theta} \text{ for some } x \in F, y \in G\}.$$

In the following theorem, it can be easily proved that the above set $\overleftarrow{\theta} \overrightarrow{\theta}(F \vee G)$ is the supremum of $\overleftarrow{\theta} \overrightarrow{\theta}(F)$ and $\overleftarrow{\theta} \overrightarrow{\theta}(G)$ in the poset $(\mathcal{F}_{\theta}(L), \subseteq)$.

Theorem 3.11. *Let θ be a congruence on a distributive lattice L . For any two θ -filters F, G of L , $\overleftarrow{\theta} \overrightarrow{\theta}(F \vee G)$ is the smallest θ -filter of L including both F and G .*

In view of the above Remark 2.12 and Theorem 2.13, we can conclude that the set $\mathcal{F}_{\theta}(L)$ of all θ -filters of a distributive lattice L forms a complete distributive lattice with respect to the following operations:

$$F \wedge G = F \cap G \text{ and } F \sqcup G = \overleftarrow{\theta} \overrightarrow{\theta}(F \vee G)$$

in which the greatest element is L . Since $\overrightarrow{\theta}$ is a bijection from $\mathcal{F}_{\theta}(L)$ to $\mathcal{F}(L/\theta)$ and for $F, G \in \mathcal{F}_{\theta}(L)$, we have $F \subseteq G$ if and only if $\overrightarrow{\theta}(F) \subseteq \overrightarrow{\theta}(G)$, the following theorem is an easy consequence.

Theorem 3.12. *Let θ be a congruence on a distributive lattice L . Then the lattice $\mathcal{F}_{\theta}(L)$ of all θ -filters of L is isomorphic to the lattice of filters of L/θ .*

Acknowledgements

The authors would like to thank the referee for his valuable comments and suggestions to improve this presentation.

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Received 21 September 2014
 First Revision 5 October 2014
 Second Revision 20 October 2014