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CHARACTERIZATIONS OF ORDERED Γ-ABEL-GRASSMANN'S GROUPOIDS

Madad Khan, Venus Amjid

Department of Mathematics
COMSATS Institute of Information Technology
Abbottabad, Pakistan

e-mail: madadmath@yahoo.com, venusmath@yahoo.com

Gul Zaman

Department of Mathematics
University of Malakand, Chakdara, Pakistan

e-mail: gzaman@uom.edu.pk

and

Naveed Yaqoob

Department of Mathematics
Quaid-i-Azam University, Islamabad, Pakistan

e-mail: nayaqoob@ymail.com

Abstract

In this paper, we introduced the concept of ordered Γ-AG-groupoids, Γ-ideals and some classes in ordered Γ-AG-groupoids. We have shown that every Γ-ideal in an ordered Γ-AG**-groupoid S is Γ-prime if and only if it is Γ-idempotent and the set of Γ-ideals of S is Γ-totally ordered under inclusion. We have proved that the set of Γ-ideals of S form a semilattice, also we have investigated some classes of ordered Γ-AG**-groupoid and it has shown that weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular and $(2,2)$-regular ordered Γ-AG**-groupoids coincide. Further we have proved that every intra-regular ordered Γ-AG**-groupoid is regular but the converse is not true in general. Furthermore we have shown that non-associative regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular Γ-AG*-groupoids do not exist.

Keywords: ordered Γ-AG-groupoids, Γ-ideals, regular Γ-AG**-groupoids.

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1. Introduction

The concept of a left almost semigroup (LA-semigroup) [10] was first introduced by Kazim and Naseeruddin in 1972. In [7], the same structure is called a left invertive groupoid. Protić and Stevanović called it an Abel-Grassmann's groupoid (AG-groupoid) [24].

An AG-groupoid is a groupoid S whose elements satisfy the left invertive law $(ab)c = (cb)a$, for all $a, b, c \in S$. In an AG-groupoid, the medial law [10] $(ab)(cd) = (ac)(bd)$ holds for all $a, b, c, d \in S$. An AG-groupoid may or may not contains a left identity. In an AG-groupoid S with left identity, the paramedial law $(ab)(cd) = (dc)(ba)$ holds for all $a, b, c, d \in S$. If an AG-groupoid contains a left identity, then by using medial law, we get $a(bc) = b(ac)$, for all $a, b, c \in S$.

The concept of ordered Γ-semigroups has been studied by many mathematicians, for instance, Chinram *et al.* [1], Hila *et al.* [2, 3, 4, 5, 6], Iampan [8, 9] and Kwon *et al.* [15, 16, 17, 18]. Also see [25].

In this paper, we have introduced the notion of ordered Γ-AG**-groupoids. Here, we have explored all basic ordered Γ-ideals, which includes ordered Γ-ideals (left, right, two-sided) and some classes of ordered Γ-AG-groupoids.

Definition 1. Let S and Γ be two non-empty sets, then S is said to be a Γ-AG-groupoid if there exists a mapping $S \times \Gamma \times S \to S$, written (x, γ, y) as $x\gamma y$, such that S satisfies the left invertive law, that is

$$(x\gamma y)\,\delta z = (z\gamma y)\,\delta x, \text{ for all } x, y, z \in S \text{ and } \gamma, \delta \in \Gamma. \tag{1}$$

Definition 2. A Γ-AG-groupoid S is called a Γ-medial if it satisfies the medial law, that is

$$(x\alpha y)\,\beta\,(s\gamma t) = (x\alpha s)\,\beta\,(y\gamma t)\,, \text{ for all } x, y, s, t \in S \text{ and } \alpha, \beta, \gamma \in \Gamma. \tag{2}$$

Definition 3. A Γ-AG-groupoid S is called a Γ-AG**-groupoid if it satisfy the following law:

$$x\alpha(y\beta z) = y\alpha(x\beta z), \text{ for all } x, y, z \in S \text{ and } \alpha, \beta \in \Gamma. \tag{3}$$

Definition 4. A Γ-AG**-groupoid S is called a Γ-paramedial if it satisfies the paramedial law, that is

$$(x\alpha y)\,\beta\,(s\gamma t) = (t\alpha s)\,\beta\,(y\gamma x)\,, \text{ for all } x, y, s, t \in S \text{ and } \alpha, \beta, \gamma \in \Gamma. \tag{4}$$

If an AG-groupoid (without left identity) satisfies medial law, then it is called an AG**-groupoid [20].

An AG-groupoid has been widely explored in [12, 13, 14, 21] and [24]. An AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup with wide applications in theory of flocks [23].

Definition 5. An AG-groupoid S is called a Γ-AG*-groupoid [20], if the following hold:

(5) $$(a\beta b)\gamma c = b\beta(a\gamma c), \text{ for all } a, b, c \in S \text{ and } \beta, \gamma \in \Gamma.$$

Definition 6. In an AG*-groupoid S, the following law holds (see [24])

(6) $$(x_1\alpha x_2)\beta(x_3\gamma x_4) = (x_{p(1)}\alpha x_{p(2)})\beta(x_{p(3)}\gamma x_{p(4)}) \text{ for all } \alpha, \beta, \gamma \in \Gamma$$

where $\{p(1), p(2), p(3), p(4)\}$ means any permutation on the set $\{1, 2, 3, 4\}$. It is an easy consequence that if $S = S\Gamma S$, then S becomes a commutative Γ-semigroup.

An AG-groupoid may or may not contains a left identity. The left identity of an AG-groupoid allow us to introduce the inverses of elements in an AG-groupoid. If an AG-groupoid contains a left identity, then it is unique [21].

Definition 7. An ordered Γ-AG-groupoid (po-Γ-AG-groupoid) is a structure $(S, \Gamma, \leq)$ in which the following conditions hold:

(i) (S, Γ) is a Γ-AG-groupoid.

(ii) $(S, \leq)$ is a poset.

(iii) For all a, b and $x \in S$, $a \leq b$ implies $a\beta x \leq b\beta x$ and $x\beta a \leq x\beta b$ for all $\beta \in \Gamma$.

Let S be an ordered Γ-AG-groupoid. For $H \subseteq S$, we define

$$(H] = \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For $H = \{a\}$, usually written as $(a]$.

Definition 8. A non-empty subset A of an ordered Γ-AG-groupoid S is called a Γ-left (resp. Γ-right) ideal of S if

(i) $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$), and

(ii) If $a \in A$ and b is in S such that $b \leq a$, then $b \in A$.

Definition 9. A non-empty subset A of an ordered Γ-AG-groupoid S is called a (Γ-two-sided) ideal of S if A is both Γ-left and Γ-right ideal of S.

Definition 10. A Γ-ideal P of an ordered Γ-AG-groupoid S is called Γ-prime if for any two Γ-ideals A and B of S such that $A\Gamma B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

Definition 11. A Γ-ideal I of an ordered Γ-AG-groupoid S is called Γ-completely prime if for any two elements a and b of S and $\beta \in \Gamma$ such that $a\beta b \in I$, then $a \in I$ or $b \in I$.

Definition 12. A Γ-ideal P of an ordered Γ-AG-groupoid S is said to be Γ-semiprime if $I^2 \subseteq P$ implies that $I \subseteq P$, for any Γ-ideal I of S.

Definition 13. A Γ-AG-groupoid S is said to be Γ-fully semiprime if every Γ-ideal of S is Γ-semiprime. An ordered Γ-AG-groupoid S is called Γ-fully prime if every Γ-ideal of S is Γ-prime.

Definition 14. The set of Γ-ideals of an ordered Γ-AG-groupoid S is called Γ-totally ordered under inclusion if for all Γ-ideals A, B of S, either $A \subseteq B$ or $B \subseteq A$ and is denoted by Γ-ideal(S).

2. Ideals in Ordered Γ-AG-groupoid

In this section we developed some results on ideals in ordered AG-groupoid.

Lemma 1. *Let S be an ordered Γ-AG-groupoid, then the following are true:*

(i) $A \subseteq (A]$, *for all* $A \subseteq S$,

(ii) *If* $A \subseteq B \subseteq S$, *then* $(A] \subseteq (B]$,

(iii) $(A]\Gamma(B] \subseteq (A\Gamma B]$ *for all subsets* A, B *of* S,

(iv) $(A] = ((A]]$ *for all* $A \subseteq S$,

(v) *For every* Γ*-left (resp.* Γ*-right) ideal or* Γ*-bi-ideal* T *of* S, $(T] = T$,

(vi) $((A]\Gamma(B]] = (A\Gamma B]$ *for all subsets* A, B *of* S.

Proof. It is same as in [11]. ■

Lemma 2. $(S\Gamma a]$, $(a\Gamma S]$ *and* $(S\Gamma a\Gamma S]$ *are a* Γ*-left, a* Γ*-right and a* Γ*-ideal of an ordered* Γ*-*AG^{**}*-groupoid* S *respectively, for all* a *in* S *such that* $(S] = (S\Gamma S]$.

Proof. Let a be any element of S. Then it has to be shown that $(S\Gamma a]$ is the Γ-left ideal of S. For this consider an element x in $S\Gamma(S\Gamma a]$, then $x = y\gamma z$ for some y in S and z in $(S\Gamma a]$ where $z \leq s\beta a$ for some $s\beta a$ in $S\Gamma a$ and $\gamma, \beta \in \Gamma$. Since $S = S\Gamma S$ so let $y = y_1\delta y_2$ for some $\delta \in \Gamma$ and $y_1, y_2 \in S$, then by using (4) and (1), we have

$$x \leq y\gamma(s\beta a) = (y_1\delta y_2)\gamma(s\beta a) = (a\delta s)\gamma(y_2\beta y_1) = ((y_2\beta y_1)\delta s)\gamma a \subseteq S\Gamma a.$$

Which implies that x is in $(S\Gamma a]$. For the second condition of $(S\Gamma a]$ to be Γ-left ideal let x be any element in $(S\Gamma a]$, then $x \leq s\beta a$ for some $s\beta a$ in $S\Gamma a$. Let y be any other element of S such that $y \leq x \leq s\beta a$, which implies that y is in $(S\Gamma a]$. Hence $(S\Gamma a]$ is the Γ-left ideal of S. It is to be noted that $(a\Gamma S]$ and $(S\Gamma a\Gamma S]$ can be shown Γ-right and Γ-two-sided ideal respectively with an analogy to the proof of $(S\Gamma a]$ to be Γ-left ideal of S. ∎

Proposition 1. *If S is an ordered Γ-AG-groupoid such that $S = S\Gamma S$, then every Γ-right ideal of S is a Γ-ideal.*

Proof. Let I be a Γ-right ideal of an ordered Γ-AG-groupoid S. Let $x \in S\Gamma(I]$ which implies that $x = y\gamma z$ for some $y \in S$ and $z \in (I]$ where $z \leq i$ for some $i \in I$. Since $S = S\Gamma S$ so let $y = y_1\delta y_2$ for some $\delta \in \Gamma$ and $y_1, y_2 \in S$, then by (1), we get

$$x \leq y\gamma i = (y_1\delta y_2)\gamma i = (i\delta y_2)\gamma y_1 \subseteq (I\Gamma S)\Gamma S \subseteq I.$$

Which implies that $x \in (I]$ and the second condition of $(I]$ to be Γ-left ideal holds obviously. Hence I is a Γ-ideal of S. ∎

Remark 1.

(1) If $(S] = (S\Gamma S]$ then every Γ-right ideal is also a Γ-left ideal and $S\Gamma I \subseteq I\Gamma S$.

(2) If I is a Γ-right ideal of S, then $S\Gamma I$ is a Γ-left and $I\Gamma S$ is a Γ-right ideal of S.

Lemma 3. *If I is a Γ-left ideal of an ordered Γ-AG^{**}-groupoid S, then $(a\Gamma I]$ is a Γ-left ideal of S.*

Proof. Let I be a Γ-left ideal of an ordered Γ-AG**-groupoid S. Let $x \in S\Gamma(a\Gamma I]$ which implies that $x = y\beta z$ for some $y \in S$ and $z \in (a\Gamma I]$ where $z \leq a\gamma i$ for some $a\gamma i \in a\Gamma I$ and $\beta, \gamma \in \Gamma$. Then by using (3), we get

$$x \leq y\beta(a\gamma i) = a\beta(y\gamma i) \subseteq a\Gamma(S\Gamma I) \subseteq a\Gamma I.$$

Which implies that $x \in (a\Gamma I]$ and for the second condition of $(a\Gamma I]$ to be Γ-left ideal let x be any element in $(a\Gamma I]$ then $x \leq a\gamma i$ for some $a\gamma i$ in $a\Gamma I$. Let y be any other element of S such that $y \leq x \leq a\gamma i$, which implies that y is in $(a\Gamma I]$. Hence $(a\Gamma I]$ is a Γ-left ideal of S. ∎

Lemma 4. *Intersection of two Γ-ideals of an ordered Γ-AG-groupoid S is a Γ-ideal.*

Proof. Assume that P and Q are any two Γ-ideals of an ordered Γ-AG-groupoid S. Let $x \in S\Gamma(P \cap Q]$, then $x = y\beta z$ for some $\beta \in \Gamma$, $y \in S$ and $z \in (P \cap Q]$ where $z \in (P]$ which implies that $z \leq a$ for some $a \in P$ and also $z \in (Q]$ which implies that $z \leq b$ for some $b \in Q$. Then by using (2), we have

$$x \leq y\beta a \subseteq S\Gamma P \subseteq P \text{ and } x \leq y\beta b \subseteq S\Gamma Q \subseteq Q.$$

This shows that $x \in (P] \cap (Q] = (P \cap Q]$ and the second condition of $(P \cap Q]$ to be Γ-left ideal is obvious. Similarly $(P \cap Q]$ is a Γ-right ideal of S. Hence $(P \cap Q]$ is a Γ-ideal of S. ■

Lemma 5. *If I is a Γ-left ideal of an ordered Γ-AG^{**}-groupoid S, then $(I\Gamma I]$ is a Γ-ideal of S.*

Proof. Let I be a Γ-left ideal of an ordered Γ-AG**-groupoid S. Let $x \in (I\Gamma I]\Gamma S$, then $x = k\alpha s$ for some k in $(I\Gamma I]$ and s in S, where $k \leq i\beta j$ for some $i\beta j \in I\Gamma I$ and $\alpha, \beta \in \Gamma$. Now by using (1), we get

$$x \leq (i\beta j)\alpha s = (s\beta j)\alpha i \subseteq I\Gamma I.$$

Now let $x \in S\Gamma(I\Gamma I]$, then $x = s\alpha k$ for some s in S and k in $(I\Gamma I]$, where $k \leq i\beta j$ for some $i\beta\ j \in I\Gamma I$ and $\alpha, \beta \in \Gamma$. Now by using (3), we get

$$x \leq s\alpha(i\beta j) = i\alpha(s\beta j) \subseteq I\Gamma I.$$

This implies that $x \in (I\Gamma I]$ and for the second condition of $(I\Gamma I]$ to be a Γ-ideal, let x be any element in $(I\Gamma I]$ then $x \leq i\beta j$ for some $i\beta j$ in $I\Gamma I$. Let y be any other element of S such that $y \leq x \leq i\beta j$, which implies that y is in $(I\Gamma I]$. Hence $(I\Gamma I]$ a Γ-ideal of S. ■

Remark 2. *If I is a Γ-left ideal of S then $(I\Gamma I]$ is a Γ-ideal of S.*

Proposition 2. *A proper Γ-ideal $(M]$ of an ordered Γ-AG^{**}-groupoid S is minimal if and only if $(M] = ((a\Gamma a)\Gamma M]$ for all $a \in S$.*

Proof. Let $(M]$ be the minimal Γ-ideal of S, as $(M\Gamma M]$ is a Γ-ideal so $(M] = (M\Gamma M]$. Now let $x \in ((a\Gamma a)\Gamma M]\Gamma S$ then $x = y\alpha z$ for some y in $((a\Gamma a)\Gamma M]$ and z in S, where $y \leq (a\gamma a)\beta m$ for some $(a\gamma a)\beta m$ in $(a\Gamma a)\Gamma M$ and $\alpha, \beta, \gamma \in \Gamma$. Now by using (1) and (4), we have

$$\begin{aligned} x &\leq ((a\gamma a)\beta m)\alpha z = (z\beta m)\alpha(a\gamma a) = (a\beta a)\alpha(m\gamma z) \\ &\subseteq (a\Gamma a)\Gamma(M\Gamma S) \subseteq (a\Gamma a)\Gamma M. \end{aligned}$$

Which implies that $x \in ((a\Gamma a)\Gamma M]$.

Now let $x \in S((a\Gamma a)\Gamma M]$ then $x = s\alpha t$ for some s in S and t in $((a\Gamma a)\Gamma M]$, where $t \leq (a\gamma a)\beta m$ for some $(a\gamma a)\beta m$ in $(a\Gamma a)\Gamma M$ and $\alpha, \beta, \gamma \in \Gamma$, then by using (3), we have

$$x \leq s\alpha((a\gamma a)\beta m) = (a\gamma a)\alpha(s\beta m) \subseteq (a\Gamma a)\Gamma(S\Gamma M) \subseteq (a\Gamma a)\Gamma M.$$

Which implies that $x \in ((a\Gamma a)\Gamma M]$, and for the second condition of $((a\Gamma a)\Gamma M]$ to be a Γ-ideal let x be any element in $((a\Gamma a)\Gamma M]$ then $x \leq (a\gamma a)\beta m$ for some $(a\gamma a)\beta m$ in $(a\Gamma a)\Gamma M$. Let y be any other element of S such that $y \leq x \leq (a\gamma a)\beta m$, which implies that y is in $((a\Gamma a)\Gamma M]$. Hence $((a\Gamma a)\Gamma M]$ a Γ-ideal of S contain in $(M]$ and as $(M]$ is minimal so $(M] = ((a\Gamma a)\Gamma M]$.

Conversely, assume that $(M] = ((a\Gamma a)\Gamma M]$ for all $a \in S$. Let $(A]$ be the minimal Γ-ideal properly contain in $(M]$ containing a, then $(M] = ((a\Gamma a)\Gamma M] \subseteq (A]$, which is a contradiction. Hence $(M]$ is a minimal Γ-ideal. ■

A Γ-ideal I of an ordered Γ-AG-groupoid S is called minimal if and only if it does not contain any Γ-ideal of S other than itself.

Theorem 6. *If I is a minimal Γ-left ideal of an ordered Γ-AG^{**}-groupoid S, then $((a\Gamma a)\Gamma(I\Gamma I)]$ is a minimal Γ-ideal of S.*

Proof. Assume that I is a minimal Γ-left ideal of an ordered Γ-AG**-groupoid S. Now let $x \in ((a\Gamma a)\Gamma(I\Gamma I)]\Gamma S$ then $x = y\alpha z$ for some y in $((a\Gamma a)\Gamma(I\Gamma I)]$ and z in S where $y \leq (a\delta a)\beta(i\gamma j)$ for some i, j in I and $\alpha, \beta, \gamma, \delta \in \Gamma$, then by using (1) and (4), we have

$$x \leq ((a\delta a)\beta(i\gamma j))\alpha z = (z\beta(i\gamma j))\alpha(a\delta a) = (a\beta a)\alpha((i\gamma j)\delta z)$$

$$= (a\delta a)\alpha((z\gamma j)\delta i) \subseteq (a\Gamma a)\Gamma((S\Gamma I)\Gamma I) \subseteq (a\Gamma a)\Gamma(I\Gamma I).$$

Which implies that $x \in ((a\Gamma a)\Gamma(I\Gamma I)]$ and for the second condition of $((a\Gamma a)\Gamma(I\Gamma I)]$ to be a Γ-ideal let x be any element in $((a\Gamma a)\Gamma(I\Gamma I)]$ then $x \leq (a\delta a)\beta(i\gamma j)$ for some $(a\delta a)\beta(i\gamma j)$ in $((a\Gamma a)\Gamma(I\Gamma I)]$. Which shows that $((a\Gamma a)\Gamma(I\Gamma I)]$ is a Γ-right ideal of S. Similarly $((a\Gamma a)\Gamma(I\Gamma I)]$ is a Γ-left ideal so is Γ-ideal. Let H be a non-empty Γ-ideal of S properly contained in $((a\Gamma a)\Gamma(I\Gamma I)]$. Define $H' = \{r \in I : a\psi r \in H\}$. Then $a\psi(s\xi y) = s\psi(a\xi y) \in S\Gamma H \subseteq H$ imply that H' is a Γ-left ideal of S properly contained in I. But this is a contradiction to the minimality of I. Hence $((a\Gamma a)\Gamma(I\Gamma I)]$ is a minimal Γ-ideal of S. ■

Theorem 7. *An ordered Γ-AG^{**}-groupoid S is Γ-fully prime if and only if every Γ-ideal is Γ-idempotent and Γ-ideal(S) is Γ-totally ordered under inclusion.*

Proof. Assume that an ordered Γ-AG**-groupoid S is Γ-fully prime. Let I be the Γ-ideal of S. Then by Lemma 5, $(I\Gamma I]$ becomes a Γ-ideal of S and obviously $I\Gamma I \subseteq I$ and by Lemma 1, $(I\Gamma I] \subseteq (I]$. Now

$$(I\Gamma I] \subseteq (I\Gamma I] \text{ yields } (I] \subseteq (I\Gamma I] \text{ and hence}$$

$(I] = (I\Gamma I]$. Let P, Q be Γ-ideals of S and $P\Gamma Q \subseteq P$, $P\Gamma Q \subseteq Q$ imply that $P\Gamma Q \subseteq P \cap Q$. Since $P \cap Q$ is Γ-prime, so $P \subseteq P \cap Q$ or $Q \subseteq P \cap Q$ which further imply that $P \subseteq Q$ or $Q \subseteq P$. Hence Γ-ideal(S) is Γ-totally ordered under inclusion.

Converse is same as in [19]. ∎

If S is an ordered Γ-AG**-groupoid then the principal Γ-left ideal generated by a is defined by $\langle a \rangle = S\Gamma a = \{s\gamma a : s \in S, \gamma \in \Gamma\}$, where a is any element of S. Let P be a Γ-left ideal of an ordered Γ-AG-groupoid S, P is called Γ-quasi-prime if for Γ-left ideals A, B of S such that $A\Gamma B \subseteq P$, we have $A \subseteq P$ or $B \subseteq P$. P is called Γ-quasi-semiprime if for any Γ-left ideal I of S such that $I\Gamma I \subseteq P$, we have $I \subseteq P$.

Theorem 8. *If S is an ordered Γ-AG^{**}-groupoid, then a Γ-left ideal P of S is Γ-quasi-prime if and only if $a\Gamma(S\Gamma b) \subseteq P$ implies that either $a \in P$ or $b \in P$, where a, $b \in S$.*

Proof. The proof is same as in [19]. ∎

Corollary 1. *If S is an ordered Γ-AG^{**}-groupoid, then a Γ-left ideal P of S is Γ-quasi-semiprime if and only if $a\Gamma(S\Gamma a) \subseteq P$ implies $a \in P$, for all $a \in S$.*

Proposition 3. *A Γ-ideal I of an ordered Γ-AG-groupoid S is Γ-prime if and only if it is Γ-semiprime and Γ-strongly irreducible.*

Proof. The proof is obvious. ∎

Theorem 9. *Let S be an ordered Γ-AG-groupoid and $\{P_i : i \in N\}$ be a family of Γ-prime ideals Γ-totally ordered under inclusion in S. Then $\cap P_i$ is a Γ-prime ideal.*

Proof. The proof is same as in [19]. ∎

Theorem 10. *For each Γ-ideal I there exists a minimal Γ-prime ideal of I in an ordered Γ-AG-groupoid S.*

Proof. The proof is same as in [19]. ∎

Definition 15. An ordered Γ-AG-groupoid $(S, \Gamma, \leq)$ is called regular if $a \in ((a\Gamma S)\Gamma a]$ for every $a \in S$, or

(1) For every $a \in S$ there exist $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq (a\beta x)\gamma a$.

(2) $A \subseteq ((A\Gamma S)\Gamma A]$ for every $A \subseteq S$.

Definition 16. An ordered Γ-AG-groupoid $(S, \Gamma, \leq)$ is called weakly regular if $a \in ((a\Gamma S)\Gamma(a\Gamma S)]$ for every $a \in S$, or

(1) For every $a \in S$ there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (a\beta x)\delta(a\gamma y)$.

(2) $A \subseteq ((A\Gamma S)\Gamma(A\Gamma S)]$ for every $A \subseteq S$.

Definition 17. An ordered Γ-AG-groupoid $(S, \Gamma, \leq)$ is called intra-regular if $a \in ((S\Gamma(a\delta a))\Gamma S]$ for every $a \in S$ and $\delta \in \Gamma$, or

(1) For every $a \in S$ there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (x\beta(a\delta a))\gamma y$.

(2) $A \subseteq ((S\Gamma(A\Gamma A))\Gamma S]$ for every $A \subseteq S$.

Definition 18. An ordered Γ-AG-groupoid $(S, \Gamma, \leq)$ is called right regular if $a \in ((a\delta a)\Gamma S]$ for every $a \in S$ and $\delta \in \Gamma$, or

(1) For every $a \in S$ there exist $x \in S$ and $\beta, \delta \in \Gamma$ such that $a \leq (a\delta a)\beta x$.

(2) $A \subseteq ((A\Gamma A)\Gamma S]$ for every $A \subseteq S$.

Definition 19. An ordered Γ-AG-groupoid $(S, \Gamma, \leq)$ is called left regular if $a \in (S\Gamma(a\delta a)]$ for every $a \in S$ and $\delta \in \Gamma$, or

(1) For every $a \in S$ there exist $x \in S$ and $\beta, \delta \in \Gamma$ such that $a \leq x\beta(a\delta a)$.

(2) $A \subseteq (S\Gamma(A\Gamma A)]$ for every $A \subseteq S$.

Definition 20. An ordered Γ-AG-groupoid $(S, \Gamma, \leq)$ is called left quasi regular if $a \in ((S\Gamma a)\Gamma(S\Gamma a)]$ for every $a \in S$, or

(1) For every $a \in S$ there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (x\beta a)\delta(y\gamma a)$.

(2) $A \subseteq ((S\Gamma A)\Gamma(S\Gamma A)]$ for every $A \subseteq S$.

Definition 21. An ordered Γ-AG-groupoid $(S, \Gamma, \leq)$ is called completely regular if S is regular, left regular and right regular.

Definition 22. An ordered Γ-AG-groupoid $(S, \Gamma, \leq)$ is called $(2, 2)$-regular if $a \in (((a\delta a)\Gamma S)\Gamma(a\delta a)]$ for every $a \in S$ and $\delta \in \Gamma$, or

(1) For every $a \in S$ there exist $x \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq ((a\delta a)\beta x)\gamma(a\delta a)$.

(2) $A \subseteq (((A\Gamma A)\Gamma S)\Gamma(A\Gamma A)]$ for every $A \subseteq S$.

Definition 23. An ordered Γ-AG-groupoid $(S, \Gamma, \leq)$ is called strongly regular if $a \in ((a\Gamma S)\Gamma a]$ and $a\beta x = x\beta a$ for every $a \in S$ and $\beta \in \Gamma$, or

(1) For every $a \in S$ there exist $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq (a\beta x)\gamma a$ and $a\beta x = x\beta a$.

(2) $A \subseteq ((A\Gamma S)\Gamma A]$ for every $A \subseteq S$.

Example 1. Let us consider an ordered AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.

.	1	2	3
1	1	3	3
2	1	2	2
3	1	2	2

Let us define $\Gamma = \{\alpha, \beta, \gamma\}$ as follows.

α	1	2	3
1	1	1	1
2	1	1	1
3	1	1	1

β	1	2	3
1	2	2	2
2	2	2	2
3	2	2	2

γ	1	2	3
1	2	2	2
2	2	2	2
3	2	2	3

Here S is a Γ-AG-groupoid because $(a\beta b)\gamma c = (c\beta b)\gamma a$ for all $a, b, c \in S$ and S is non-associative, because $(1\alpha 2)\beta 3 \neq 1\alpha(2\beta 3)$. We define order $\leq$ as:

$$\leq := \{(1,1), (2,2), (3,3), (2,1), (2,3), (3,1)\}.$$

Clearly $(S, \leq)$ is a poset and for all a, b and $x \in S$, $a \leq b$ implies $a\beta x \leq b\beta x$ and $x\beta a \leq x\beta b$ for some $\beta \in \Gamma$ so S is a ordered Γ-AG-groupoid.

Note that S is a Γ-ideal itself so by Lemma 1, $(S\Gamma S] \subseteq S$.

Lemma 11. *If S is regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular ordered Γ-AG-groupoid, then $S = (S\Gamma S]$.*

Proof. Assume that S is a regular ordered Γ-AG-groupoid, then $(S\Gamma S] \subseteq S$ is obvious. Let for any $a \in S$, $a \in ((a\Gamma S)\Gamma a]$, then there exists $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq (a\beta x)\gamma a$. Now $a \leq (a\beta x)\gamma a \in S\Gamma S$, thus $a \in (S\Gamma S]$. Similarly if S is weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular or strongly regular, then we can show that $S = (S\Gamma S]$. ∎

The converse is not true in general, because in Example 2, $S = (S\Gamma S]$ holds but S is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular, because $1 \in S$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular.

Example 2. Let us consider an ordered Γ-AG-groupoid $S = \{1, 2, 3, 4\}$ in the following Cayley's table.

.	1	2	3	4
1	4	4	2	4
2	4	4	1	4
3	1	2	3	4
4	4	4	4	4

Let us define $\Gamma = \{\alpha, \beta, \gamma\}$ as follows:

α	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	1	1

β	1	2	3	4
1	2	2	2	2
2	2	2	2	2
3	2	2	2	2
4	2	2	2	3

γ	1	2	3	4
1	2	2	2	2
2	2	2	2	2
3	2	2	2	2
4	2	2	2	4

Here S is a Γ-AG-groupoid because $(a\beta b)\gamma c = (c\beta b)\gamma a$ for all $a, b, c \in S$ and S is non-associative, because $(1\alpha 2)\beta 3 \neq 1\alpha(2\beta 3)$. We define the order $\leq$ as:

$$\leq := \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 4), (2, 4), (3, 4)\}.$$

Clearly $(S, \leq)$ is a poset and for all a, b and $x \in S$, $a \leq b$ implies $a\beta x \leq b\beta x$ and $x\beta a \leq x\beta b$ for some $\beta \in \Gamma$ so S is a ordered Γ-AG-groupoid. $A = \{1, 2, 4\}$ is an ideal of S as $A\Gamma S \subseteq A$ and $S\Gamma A \subseteq A$, also for every $1 \in A$ there exists $4 \in S$ such that $4 \leq 1 \in A$ implies that $4 \in A$, similarly for every $4 \in A$ there exists $2 \in S$ such that $2 \leq 4 \in A$ implies that $4 \in A$.

Theorem 12. *If S is an ordered Γ-AG^{**}-groupoid, then S is an intra-regular if and only if for all $a \in (S]$, $a \leq (x\beta a)\delta(a\gamma z)$ holds for some $x, z \in S$ and $\beta, \gamma, \delta \in \Gamma$.*

Proof. Assume that S is an intra-regular ordered Γ-AG**-groupoid. Let $a \in ((S\Gamma(a\gamma a))\Gamma S]$ for any $a \in S$ and $\gamma \in \Gamma$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (x\beta(a\gamma a))\delta y$. Now by using Lemma 11, $y \leq u\gamma v$ for some $u, v \in S$. Thus by using (2), (1) and (4), we have

$$\begin{aligned} a &\leq (x\beta(a\gamma a))\delta y = (a\beta(x\gamma a))\delta y = (y\beta(x\gamma a))\delta a = (y\beta(x\gamma a))\delta((x\beta(a\gamma a))\delta y) \\ &\leq ((u\gamma v)\beta(x\gamma a))\delta((x\beta(a\gamma a))\delta y) = ((a\gamma x)\beta(v\gamma u))\delta((x\beta(a\gamma a))\delta y) \\ &\leq ((a\gamma x)\beta t)\delta((x\beta(a\gamma a))\delta y) = (((x\beta(a\gamma a))\delta y)\beta t)\delta(a\gamma x) \\ &= ((t\delta y)\beta(x\beta(a\gamma a)))\delta(a\gamma x) = (((a\gamma a)\delta x)\beta(y\beta t))\delta(a\gamma x) \end{aligned}$$

$$\leq (((a\gamma a)\delta x)\beta s)\delta(a\gamma x) = ((s\delta x)\beta(a\gamma a))\delta(a\gamma x) = ((a\delta a)\beta(x\gamma s))\delta(a\gamma x)$$
$$\leq ((a\delta a)\beta w)\delta(a\gamma x) = ((w\delta a)\beta a)\delta(a\gamma x) \leq (z\beta a)\delta(a\gamma x)$$
$$= (x\beta a)\delta(a\gamma z),$$

where $v\gamma u \leq t$, $y\beta t \leq s$, $x\gamma s \leq w$ and $w\delta a \leq z$ for some $t, s, w, z \in S$.

Conversely, let for all $a \in (S]$, $a \leq (x\beta a)\delta(a\gamma z)$ holds for some $x, z \in S$ and $\beta, \gamma, \delta \in \Gamma$. Now by using (3), (1), (2) and (4), we have

$$\begin{aligned} a &\leq (x\beta a)\delta(a\gamma z) = a\delta((x\beta a)\gamma z) \leq ((x\beta a)\delta(a\gamma z))\delta((x\beta a)\gamma z) \\ &= (a\delta((x\beta a)\gamma z))\delta((x\beta a)\gamma z) = (((x\beta a)\gamma z)\delta((x\beta a)\gamma z))\delta a \\ &= (((x\beta a)\gamma(x\beta a))\delta(z\gamma z))\delta a = (((a\beta x)\gamma(a\beta x))\delta(z\gamma z))\delta a \\ &= ((a\gamma((a\beta x)\beta x))\delta(z\gamma z))\delta a = (((z\gamma z)\delta((a\beta x)\beta x))\gamma a)\delta a \\ &= (((a\beta x)\delta((z\gamma z)\beta x))\gamma a)\delta a = (((((z\gamma z)\beta x)\beta x)\delta a)\gamma a)\delta a \\ &= ((((x\beta x)\beta(z\gamma z))\delta a)\gamma a)\delta a = ((a\delta a)\gamma((x\beta x)\beta(z\gamma z)))\delta a \\ &= (a\gamma((x\beta x)\beta(z\gamma z)))\delta(a\delta a) \\ &\leq (a\gamma t)\delta(a\delta a), \text{ where } (x\beta x)\beta(z\gamma z) \leq t \text{ for some } t \in S. \end{aligned}$$

Now by using (4) and (1), we have

$$\begin{aligned} a &\leq (a\gamma t)\delta(a\delta a) \leq (((a\gamma t)\delta(a\delta a))\gamma t)\delta(a\delta a) = (((a\gamma a)\delta(t\delta a))\gamma t)\delta(a\delta a) \\ &= (((a\gamma a)\delta(t\delta a))\gamma t)\delta(a\delta a) = ((t\delta(t\delta a))\gamma(a\gamma a))\delta(a\delta a) \\ &\leq (u\gamma(a\gamma a))\delta v, \text{ where } t\delta(t\delta a) \leq u \text{ and } a\delta a \leq v \text{ for some } u, v \in S \\ &\in (S\Gamma(a\gamma a))\Gamma S. \end{aligned}$$

Which implies that $a \in ((S\Gamma(a\gamma a))\Gamma S]$, thus S is intra-regular. ■

Theorem 13. *If S is an ordered Γ-AG^{**}-groupoid, then the following are equivalent.*

(i) *S is weakly regular.*

(ii) *S is intra-regular.*

Proof. (i)$\Longrightarrow$(ii) Assume that S is a weakly regular ordered Γ-AG**-groupoid. Let $a \in ((a\Gamma S)\Gamma(a\Gamma S)]$ for any $a \in S$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$

such that $a \leq (a\beta x)\delta(a\gamma y)$. Now by Lemma 11, let $x \leq s\psi t$ for some $s, t \in S$, $\psi \in \Gamma$ and $t\gamma s \leq u \in S$, then by using (4) and (1), we have

$$\begin{aligned} a &\leq (a\beta x)\delta(a\gamma y) = (y\beta a)\delta(x\gamma a) = ((x\gamma a)\beta a)\delta y \leq (((s\psi t)\gamma a)\beta a)\delta y \\ &= ((a\gamma a)\beta(s\psi t))\delta y = ((t\gamma s)\beta(a\psi a))\delta y = ((t\gamma s)\beta(a\psi a))\delta y \leq (u\beta(a\psi a))\delta y \\ &\in (S\Gamma(a\psi a))\Gamma S. \end{aligned}$$

Which implies that $a \in ((S\Gamma(a\psi a))\Gamma S]$, thus S is intra-regular.

(ii)$\Longrightarrow$(i) is the same as (i)$\Longrightarrow$(ii). ∎

Theorem 14. *If S is an ordered Γ-AG^{**}-groupoid, then the following are equivalent.*

(i) *S is weakly regular.*

(ii) *S is right regular.*

Proof. (i)$\Longrightarrow$(ii) Assume that S is a weakly regular ordered Γ-AG**-groupoid. Let $a \in ((a\Gamma S)\Gamma(a\Gamma S)]$ for any $a \in S$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (a\beta x)\delta(a\gamma y)$ and by using Lemma 11, let $x\gamma y \leq t$ for some $t \in S$. Now by using (2), we have

$$a \leq (a\beta x)\delta(a\gamma y) = (a\beta a)\delta(x\gamma y) \leq (a\beta a)\delta t \in (a\beta a)\Gamma S.$$

Which implies that $a \in ((a\beta a)\Gamma S]$, thus S is right regular.

(ii)$\Longrightarrow$(i) It follows from Lemma 11 and (2). ∎

Theorem 15. *If S is an ordered Γ-AG^{**}-groupoid, then the following are equivalent.*

(i) *S is weakly regular.*

(ii) *S is left regular.*

Proof. (i)$\Longrightarrow$(ii) Assume that S is a weakly regular ordered Γ-AG**-groupoid. Let $a \in ((a\Gamma S)\Gamma(a\Gamma S)]$ for any $a \in S$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (a\beta x)\delta(a\gamma y)$. Now let $y\beta x \leq t$ for some $t \in S$ then by (2) and (4), we have

$$\begin{aligned} a &\leq (a\beta x)\delta(a\gamma y) = (a\beta a)\delta(x\gamma y) = (y\beta x)\delta(a\gamma a) = (y\beta x)\delta(a\gamma a) \\ &\leq t\delta(a\gamma a) \in S\Gamma(a\gamma a). \end{aligned}$$

Which implies that $a \in (S\Gamma(a\gamma a)]$, thus S is left regular.

(ii)$\Longrightarrow$(i) It follows from Lemma 11, (4) and (2). ∎

Lemma 16. *Every weakly regular ordered Γ-AG^{**}-groupoid is regular.*

Proof. Assume that S is a weakly regular ordered Γ-AG**-groupoid. Let $a \in ((a\Gamma S)\Gamma(a\Gamma S)]$ for any $a \in S$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (a\beta x)\delta(a\gamma y)$. Let $x\gamma y \leq t \in S$ and by using (1), (2), (4) and (3), we have

$$\begin{aligned}
a &\leq (a\beta x)\delta(a\gamma y) = ((a\gamma y)\beta x)\delta a = ((x\gamma y)\beta a)\delta a \leq (t\beta a)\delta a \\
&\leq (t\beta((a\beta x)\delta(a\gamma y)))\delta a = (t\beta((a\beta a)\delta(x\gamma y)))\delta a \\
&= (t\beta((y\beta x)\delta(a\gamma a)))\delta a = (t\beta(a\delta((y\beta x)\gamma a)))\delta a \\
&= (a\beta(t\delta((y\beta x)\gamma a)))\delta a \leq (a\beta u)\delta a, \text{ where } t\delta((y\beta x)\gamma a) \leq u \in S \\
&\in (a\Gamma S)\Gamma a.
\end{aligned}$$

Which implies that $a \in ((a\Gamma S)\Gamma a]$, thus S is regular. ■

The converse of above Lemma is not true in general, as can be seen from the following example.

Example 3 [24]**.** Let us consider an AG-groupoid $S = \{1, 2, 3, 4\}$ in the following Cayley's table.

.	1	2	3	4
1	2	2	4	4
2	2	2	2	2
3	1	2	3	4
4	1	2	1	2

Let us define $\Gamma = \{\alpha, \beta, \gamma\}$ as follows:

α	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	1
4	1	1	1	1

β	1	2	3	4
1	2	2	2	2
2	2	2	2	2
3	2	2	4	4
4	2	2	2	2

γ	1	2	3	4
1	2	2	2	2
2	2	2	2	2
3	2	2	2	2
4	2	2	3	4

Here S is a Γ-AG-groupoid because $(a\beta b)\gamma c = (c\beta b)\gamma a$ for all $a, b, c \in S$. We define order $\leq$ as:

$$\leq := \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (4, 2)\}.$$

Clearly $(S, \leq)$ is a poset and for all a, b and $x \in S$, $a \leq b$ implies $a\beta x \leq b\beta x$ and $x\beta a \leq x\beta b$ for some $\beta \in \Gamma$ so S is a ordered Γ-AG-groupoid. Also S is regular, because $1 \leq (1\alpha 3)\alpha 1$, $2 \leq (2\beta 1)\gamma 2$, $3 \leq (3\beta 3)\gamma 3$ and $4 \leq (4\gamma 3)\beta 4$, but S is not weakly regular, because $1 \notin ((1\Gamma S)\Gamma(1\Gamma S)]$.

Theorem 17. *If S is an ordered Γ-AG^{**}-groupoid, then the following are equivalent.*

(i) *S is weakly regular.*

(ii) *S is completely regular.*

Proof. (i)$\Longrightarrow$(ii) It follows from Theorems 14, 15 and Lemma 16.
(ii)$\Longrightarrow$(i) It follows from Theorem 15. ∎

Theorem 18. *If S is an ordered Γ-AG^{**}-groupoid, then the following are equivalent.*

(i) *S is weakly regular.*

(ii) *S is left quasi regular.*

Proof. The proof of this Lemma is straight forward. ∎

Theorem 19. *If S is an ordered Γ-AG^{**}-groupoid, then the following are equivalent.*

(i) *S is $(2,2)$-regular.*

(ii) *S is completely regular.*

Proof. (i)$\Longrightarrow$(ii) Assume that S is a $(2,2)$-regular ordered Γ-AG**-groupoid. Let $a \in (((a\delta a)\Gamma S)\Gamma(a\delta a)]$ for any $a \in S$ and $\delta \in$, then there exist $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq ((a\delta a)\beta x)\gamma(a\delta a)$. Now let $(a\delta a)\beta x \leq y \in S$, then we have

$$a \leq ((a\delta a)\beta x)\gamma(a\delta a) \leq y\gamma(a\delta a) \in S\Gamma(a\delta a).$$

Which implies that $a \in (S\Gamma(a\delta a)]$, thus S is left regular. Now by using (4), we have

$$\begin{aligned} a &\leq ((a\delta a)\beta x)\gamma(a\delta a) = (a\beta a)\gamma(x\delta(a\delta a)) \\ &\leq (a\delta a)\gamma z, \text{ where } x\delta(a\delta a) \leq z \in S \text{ and } \delta \in \Gamma \\ &\in (a\delta a)\Gamma S. \end{aligned}$$

Which implies that $a \in ((a\delta a)\Gamma S]$, thus S is right regular. Now let $x \leq u\psi v$ for some $u, v \in S$ and $\psi \in \Gamma$, then by using (4), (1) and (3), we have

$$\begin{aligned} a &\leq ((a\delta a)\beta x)\gamma(a\delta a) = (a\beta a)\gamma(x\delta(a\delta a)) \leq (a\beta a)\gamma((u\psi v)\delta(a\delta a)) \\ &= (a\beta a)\gamma((a\psi a)\delta(v\delta u)) \leq (a\beta a)\gamma((a\psi a)\delta t), \text{ where } v\delta u \leq t \in S \\ &= (((a\psi a)\delta t)\beta a)\gamma a = ((a\delta t)\beta(a\psi a))\gamma a = (a\beta((a\delta t)\psi a))\gamma a \end{aligned}$$

$$\leq (a\beta y)\gamma a, \text{ where } (a\delta t)\psi a \leq y \in S$$
$$\in (a\Gamma S)\Gamma a.$$

Which implies that $a \in ((a\Gamma S)\Gamma a]$, so S is regular. Thus S is completely regular.
(ii)$\Longrightarrow$(i) Assume that S is a completely regular ordered Γ-AG**-groupoid . Let $a \in ((a\Gamma S)\Gamma a]$, $a \in ((a\delta a)\Gamma S]$ and $a \in (S\Gamma(a\delta a)]$ for any $a \in S$, then there exist $x, y, z \in S$ and $\beta, \gamma, \psi, \xi, \delta \in \Gamma$ such that $a \leq (a\beta x)\gamma a$, $a \leq (a\delta a)\psi y$ and $a \leq z\xi(a\delta a)$. Now by using (4), (1) and (3), we have

$$\begin{aligned}
a &\leq (a\beta x)\gamma a \leq (a\beta x)\gamma(z\xi(a\delta a)) = ((a\delta a)\beta z)\gamma(x\xi a) = ((x\xi a)\beta z)\gamma(a\delta a) \\
&\leq ((x\xi((a\delta a)\psi y))\beta z)\gamma(a\delta a) = (((a\delta a)\xi(x\psi y))\beta z)\gamma(a\delta a) \\
&\leq (((a\delta a)\xi t)\beta z)\gamma(a\delta a), \text{ where } x\psi y \leq t \in S \\
&= ((z\xi t)\beta(a\delta a))\gamma(a\delta a) = ((a\xi a)\beta(t\delta z))\gamma(a\delta a) \\
&\leq ((a\xi a)\beta w)\gamma(a\delta a), \text{ where } t\delta z \leq w \in S \\
&= ((a\xi a)\beta w)\gamma(a\delta a) \in ((a\xi a)\Gamma S)\Gamma(a\delta a).
\end{aligned}$$

Which implies that $a \in (((a\xi a)\Gamma S)\Gamma(a\delta a)]$, this shows that S is $(2, 2)$-regular. ∎

Lemma 20. *Every strongly regular ordered Γ-AG**-groupoid is completely regular.*

Proof. Assume that S is a strongly regular ordered Γ-AG**-groupoid, then for any $a \in S$ there exists $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq (a\beta x)\gamma a$ and $a\beta x = x\beta a$. Now by using (1), we have

$$a \leq (a\beta x)\gamma a = (x\beta a)\gamma a = (a\beta a)\gamma x \subseteq (a\beta a)\Gamma S.$$

Which implies that $a \in (a^2\Gamma S]$, this shows that S is right regular and by Theorems 14 and 17, it is clear to see that S is completely regular. ∎

Theorem 21. *In an ordered Γ-AG**-groupoid S, the following are equivalent.*

(i) *S is weakly regular,*

(ii) *S is intra-regular,*

(iii) *S is right regular,*

(iv) *S is left regular,*

(v) *S is left quasi regular,*

(vi) *S is completely regular,*

(vii) *For all $a \in S$, there exist $x, y \in S$ such that $a \leq (x\beta a)\delta(a\gamma z)$ holds for some $x, z \in S$ and $\beta, \gamma, \delta \in \Gamma$,*

(viii) *S is $(2,2)$-regular.*

Proof. (i)$\Longleftrightarrow$(ii) It follows from Theorem 13.
(ii)$\Longleftrightarrow$(iii) It follows from Theorems 13 and 14.
(iii)$\Longleftrightarrow$(iv) It follows from Theorems 14 and 15.
(iv)$\Longleftrightarrow$(v) It follows from Theorems 15 and 18.
(v)$\Longleftrightarrow$(vi) It follows from Theorems 18 and 17.
(vi)$\Longleftrightarrow$(i) It follows from Theorem 17.
(ii)$\Longleftrightarrow$(vii) It follows from Theorem 12.
(vi)$\Longleftrightarrow$(viii) It follows from Theorem 19. ∎

Remark 3. Every intra-regular, right regular, left regular, left quasi regular $(2,2)$-regular and completely regular ordered Γ-AG**-groupoids are regular.

The converse is not true in general, as can be seen from Example 3.

Theorem 22. *Regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular Γ-AG^{*}-groupoids become a Γ-semigroups.*

Proof. It follows from (6) and Lemma 11. ∎

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