

CHARACTERIZATIONS OF ORDERED Γ -ABEL-GRASSMANN'S GROUPOIDS

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Abstract

In this paper, we introduced the concept of ordered Γ -AG-groupoids, Γ -ideals and some classes in ordered Γ -AG-groupoids. We have shown that every Γ -ideal in an ordered Γ -AG^{**}-groupoid S is Γ -prime if and only if it is Γ -idempotent and the set of Γ -ideals of S is Γ -totally ordered under inclusion. We have proved that the set of Γ -ideals of S form a semilattice, also we have investigated some classes of ordered Γ -AG^{**}-groupoid and it has shown that weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular and (2, 2)-regular ordered Γ -AG^{**}-groupoids coincide. Further we have proved that every intra-regular ordered Γ -AG^{**}-groupoid is regular but the converse is not true in general. Furthermore we have shown that non-associative regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, (2, 2)-regular and strongly regular Γ -AG^{*}-groupoids do not exist.

Keywords: ordered Γ -AG-groupoids, Γ -ideals, regular Γ -AG^{**}-groupoids.

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1. INTRODUCTION

The concept of a left almost semigroup (LA-semigroup) [10] was first introduced by Kazim and Naseeruddin in 1972. In [7], the same structure is called a left invertive groupoid. Protić and Stevanović called it an Abel-Grassmann's groupoid (AG-groupoid) [24].

An AG-groupoid is a groupoid S whose elements satisfy the left invertive law $(ab)c = (cb)a$, for all $a, b, c \in S$. In an AG-groupoid, the medial law [10] $(ab)(cd) = (ac)(bd)$ holds for all $a, b, c, d \in S$. An AG-groupoid may or may not contains a left identity. In an AG-groupoid S with left identity, the paramedial law $(ab)(cd) = (dc)(ba)$ holds for all $a, b, c, d \in S$. If an AG-groupoid contains a left identity, then by using medial law, we get $a(bc) = b(ac)$, for all $a, b, c \in S$.

The concept of ordered Γ -semigroups has been studied by many mathematicians, for instance, Chinram *et al.* [1], Hila *et al.* [2, 3, 4, 5, 6], Iampan [8, 9] and Kwon *et al.* [15, 16, 17, 18]. Also see [25].

In this paper, we have introduced the notion of ordered Γ -AG***-groupoids*. Here, we have explored all basic ordered Γ -ideals, which includes ordered Γ -ideals (left, right, two-sided) and some classes of ordered Γ -AG-groupoids.

Definition 1. Let S and Γ be two non-empty sets, then S is said to be a Γ -AG-groupoid if there exists a mapping $S \times \Gamma \times S \rightarrow S$, written (x, γ, y) as $x\gamma y$, such that S satisfies the left invertive law, that is

$$(1) \quad (x\gamma y)\delta z = (z\gamma y)\delta x, \text{ for all } x, y, z \in S \text{ and } \gamma, \delta \in \Gamma.$$

Definition 2. A Γ -AG-groupoid S is called a Γ -medial if it satisfies the medial law, that is

$$(2) \quad (x\alpha y)\beta(s\gamma t) = (x\alpha s)\beta(y\gamma t), \text{ for all } x, y, s, t \in S \text{ and } \alpha, \beta, \gamma \in \Gamma.$$

Definition 3. A Γ -AG-groupoid S is called a Γ -AG***-groupoid* if it satisfy the following law:

$$(3) \quad x\alpha(y\beta z) = y\alpha(x\beta z), \text{ for all } x, y, z \in S \text{ and } \alpha, \beta \in \Gamma.$$

Definition 4. A Γ -AG***-groupoid* S is called a Γ -paramedial if it satisfies the paramedial law, that is

$$(4) \quad (x\alpha y)\beta(s\gamma t) = (t\alpha s)\beta(y\gamma x), \text{ for all } x, y, s, t \in S \text{ and } \alpha, \beta, \gamma \in \Gamma.$$

If an AG-groupoid (without left identity) satisfies medial law, then it is called an AG***-groupoid* [20].

An AG-groupoid has been widely explored in [12, 13, 14, 21] and [24]. An AG-groupoid is a non-associative algebraic structure mid way between a groupoid and a commutative semigroup with wide applications in theory of flocks [23].

Definition 5. An AG-groupoid S is called a Γ -AG*-groupoid [20], if the following hold:

$$(5) \quad (a\beta b)\gamma c = b\beta(a\gamma c), \text{ for all } a, b, c \in S \text{ and } \beta, \gamma \in \Gamma.$$

Definition 6. In an AG*-groupoid S , the following law holds (see [24])

$$(6) \quad (x_1\alpha x_2)\beta(x_3\gamma x_4) = (x_{p(1)}\alpha x_{p(2)})\beta(x_{p(3)}\gamma x_{p(4)}) \text{ for all } \alpha, \beta, \gamma \in \Gamma$$

where $\{p(1), p(2), p(3), p(4)\}$ means any permutation on the set $\{1, 2, 3, 4\}$. It is an easy consequence that if $S = \Gamma S$, then S becomes a commutative Γ -semigroup.

An AG-groupoid may or may not contains a left identity. The left identity of an AG-groupoid allow us to introduce the inverses of elements in an AG-groupoid. If an AG-groupoid contains a left identity, then it is unique [21].

Definition 7. An ordered Γ -AG-groupoid (po- Γ -AG-groupoid) is a structure (S, Γ, \leq) in which the following conditions hold:

- (i) (S, Γ) is a Γ -AG-groupoid.
- (ii) (S, \leq) is a poset.
- (iii) For all a, b and $x \in S$, $a \leq b$ implies $a\beta x \leq b\beta x$ and $x\beta a \leq x\beta b$ for all $\beta \in \Gamma$.

Let S be an ordered Γ -AG-groupoid. For $H \subseteq S$, we define

$$(H] = \{t \in S \mid t \leq h \text{ for some } h \in H\}.$$

For $H = \{a\}$, usually written as $(a]$.

Definition 8. A non-empty subset A of an ordered Γ -AG-groupoid S is called a Γ -left (resp. Γ -right) ideal of S if

- (i) $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$), and
- (ii) If $a \in A$ and b is in S such that $b \leq a$, then $b \in A$.

Definition 9. A non-empty subset A of an ordered Γ -AG-groupoid S is called a (Γ -two-sided) ideal of S if A is both Γ -left and Γ -right ideal of S .

Definition 10. A Γ -ideal P of an ordered Γ -AG-groupoid S is called Γ -prime if for any two Γ -ideals A and B of S such that $A\Gamma B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$.

Definition 11. A Γ -ideal I of an ordered Γ -AG-groupoid S is called Γ -completely prime if for any two elements a and b of S and $\beta \in \Gamma$ such that $a\beta b \in I$, then $a \in I$ or $b \in I$.

Definition 12. A Γ -ideal P of an ordered Γ -AG-groupoid S is said to be Γ -semiprime if $I^2 \subseteq P$ implies that $I \subseteq P$, for any Γ -ideal I of S .

Definition 13. A Γ -AG-groupoid S is said to be Γ -fully semiprime if every Γ -ideal of S is Γ -semiprime. An ordered Γ -AG-groupoid S is called Γ -fully prime if every Γ -ideal of S is Γ -prime.

Definition 14. The set of Γ -ideals of an ordered Γ -AG-groupoid S is called Γ -totally ordered under inclusion if for all Γ -ideals A, B of S , either $A \subseteq B$ or $B \subseteq A$ and is denoted by $\Gamma\text{-ideal}(S)$.

2. IDEALS IN ORDERED Γ -AG-GROUPOID

In this section we developed some results on ideals in ordered AG-groupoid.

Lemma 1. *Let S be an ordered Γ -AG-groupoid, then the following are true:*

- (i) $A \subseteq (A]$, for all $A \subseteq S$,
- (ii) If $A \subseteq B \subseteq S$, then $(A] \subseteq (B]$,
- (iii) $(A]\Gamma(B] \subseteq (A\Gamma B]$ for all subsets A, B of S ,
- (iv) $(A] = ((A])$ for all $A \subseteq S$,
- (v) For every Γ -left (resp. Γ -right) ideal or Γ -bi-ideal T of S , $(T] = T$,
- (vi) $((A]\Gamma(B]) = (A\Gamma B]$ for all subsets A, B of S .

Proof. It is same as in [11]. ■

Lemma 2. $(S\Gamma a]$, $(a\Gamma S]$ and $(S\Gamma a\Gamma S]$ are a Γ -left, a Γ -right and a Γ -ideal of an ordered Γ -AG^{**}-groupoid S respectively, for all a in S such that $(S] = (S\Gamma S]$.

Proof. Let a be any element of S . Then it has to be shown that $(S\Gamma a]$ is the Γ -left ideal of S . For this consider an element x in $S\Gamma(S\Gamma a]$, then $x = y\gamma z$ for some y in S and z in $(S\Gamma a]$ where $z \leq s\beta a$ for some $s\beta a$ in $S\Gamma a$ and $\gamma, \beta \in \Gamma$. Since $S = S\Gamma S$ so let $y = y_1\delta y_2$ for some $\delta \in \Gamma$ and $y_1, y_2 \in S$, then by using (4) and (1), we have

$$x \leq y\gamma(s\beta a) = (y_1\delta y_2)\gamma(s\beta a) = (a\delta s)\gamma(y_2\beta y_1) = ((y_2\beta y_1)\delta s)\gamma a \subseteq S\Gamma a.$$

Which implies that x is in $(S\Gamma a]$. For the second condition of $(S\Gamma a]$ to be Γ -left ideal let x be any element in $(S\Gamma a]$, then $x \leq s\beta a$ for some $s\beta a$ in $S\Gamma a$. Let y be

any other element of S such that $y \leq x \leq s\beta a$, which implies that y is in $(S\Gamma a]$. Hence $(S\Gamma a]$ is the Γ -left ideal of S . It is to be noted that $(a\Gamma S]$ and $(S\Gamma a\Gamma S]$ can be shown Γ -right and Γ -two-sided ideal respectively with an analogy to the proof of $(S\Gamma a]$ to be Γ -left ideal of S . ■

Proposition 1. *If S is an ordered Γ -AG-groupoid such that $S = S\Gamma S$, then every Γ -right ideal of S is a Γ -ideal.*

Proof. Let I be a Γ -right ideal of an ordered Γ -AG-groupoid S . Let $x \in S\Gamma(I]$ which implies that $x = y\gamma z$ for some $y \in S$ and $z \in (I]$ where $z \leq i$ for some $i \in I$. Since $S = S\Gamma S$ so let $y = y_1\delta y_2$ for some $\delta \in \Gamma$ and $y_1, y_2 \in S$, then by (1), we get

$$x \leq y\gamma i = (y_1\delta y_2)\gamma i = (i\delta y_2)\gamma y_1 \subseteq (I\Gamma S)\Gamma S \subseteq I.$$

Which implies that $x \in (I]$ and the second condition of $(I]$ to be Γ -left ideal holds obviously. Hence I is a Γ -ideal of S . ■

Remark 1.

- (1) If $(S] = (S\Gamma S]$ then every Γ -right ideal is also a Γ -left ideal and $S\Gamma I \subseteq I\Gamma S$.
- (2) If I is a Γ -right ideal of S , then $S\Gamma I$ is a Γ -left and $I\Gamma S$ is a Γ -right ideal of S .

Lemma 3. *If I is a Γ -left ideal of an ordered Γ -AG^{**}-groupoid S , then $(a\Gamma I]$ is a Γ -left ideal of S .*

Proof. Let I be a Γ -left ideal of an ordered Γ -AG^{**}-groupoid S . Let $x \in S\Gamma(a\Gamma I]$ which implies that $x = y\beta z$ for some $y \in S$ and $z \in (a\Gamma I]$ where $z \leq a\gamma i$ for some $a\gamma i \in a\Gamma I$ and $\beta, \gamma \in \Gamma$. Then by using (3), we get

$$x \leq y\beta(a\gamma i) = a\beta(y\gamma i) \subseteq a\Gamma(S\Gamma I) \subseteq a\Gamma I.$$

Which implies that $x \in (a\Gamma I]$ and for the second condition of $(a\Gamma I]$ to be Γ -left ideal let x be any element in $(a\Gamma I]$ then $x \leq a\gamma i$ for some $a\gamma i$ in $a\Gamma I$. Let y be any other element of S such that $y \leq x \leq a\gamma i$, which implies that y is in $(a\Gamma I]$. Hence $(a\Gamma I]$ is a Γ -left ideal of S . ■

Lemma 4. *Intersection of two Γ -ideals of an ordered Γ -AG-groupoid S is a Γ -ideal.*

Proof. Assume that P and Q are any two Γ -ideals of an ordered Γ -AG-groupoid S . Let $x \in S\Gamma(P \cap Q]$, then $x = y\beta z$ for some $\beta \in \Gamma$, $y \in S$ and $z \in (P \cap Q]$ where $z \in (P]$ which implies that $z \leq a$ for some $a \in P$ and also $z \in (Q]$ which

implies that $z \leq b$ for some $b \in Q$. Then by using (2), we have

$$x \leq y\beta a \subseteq S\Gamma P \subseteq P \text{ and } x \leq y\beta b \subseteq S\Gamma Q \subseteq Q.$$

This shows that $x \in (P] \cap (Q] = (P \cap Q]$ and the second condition of $(P \cap Q]$ to be Γ -left ideal is obvious. Similarly $(P \cap Q]$ is a Γ -right ideal of S . Hence $(P \cap Q]$ is a Γ -ideal of S . ■

Lemma 5. *If I is a Γ -left ideal of an ordered Γ -AG** $-$ groupoid S , then $(I\Gamma I]$ is a Γ -ideal of S .*

Proof. Let I be a Γ -left ideal of an ordered Γ -AG** $-$ groupoid S . Let $x \in (I\Gamma I]\Gamma S$, then $x = k\alpha s$ for some k in $(I\Gamma I]$ and s in S , where $k \leq i\beta j$ for some $i\beta j \in I\Gamma I$ and $\alpha, \beta \in \Gamma$. Now by using (1), we get

$$x \leq (i\beta j)\alpha s = (s\beta j)\alpha i \subseteq I\Gamma I.$$

Now let $x \in S\Gamma(I\Gamma I]$, then $x = s\alpha k$ for some s in S and k in $(I\Gamma I]$, where $k \leq i\beta j$ for some $i\beta j \in I\Gamma I$ and $\alpha, \beta \in \Gamma$. Now by using (3), we get

$$x \leq s\alpha(i\beta j) = i\alpha(s\beta j) \subseteq I\Gamma I.$$

This implies that $x \in (I\Gamma I]$ and for the second condition of $(I\Gamma I]$ to be a Γ -ideal, let x be any element in $(I\Gamma I]$ then $x \leq i\beta j$ for some $i\beta j$ in $I\Gamma I$. Let y be any other element of S such that $y \leq x \leq i\beta j$, which implies that y is in $(I\Gamma I]$. Hence $(I\Gamma I]$ a Γ -ideal of S . ■

Remark 2. *If I is a Γ -left ideal of S then $(I\Gamma I]$ is a Γ -ideal of S .*

Proposition 2. *A proper Γ -ideal $(M]$ of an ordered Γ -AG** $-$ groupoid S is minimal if and only if $(M] = ((a\Gamma a)\Gamma M]$ for all $a \in S$.*

Proof. Let $(M]$ be the minimal Γ -ideal of S , as $(M\Gamma M]$ is a Γ -ideal so $(M] = (M\Gamma M]$. Now let $x \in ((a\Gamma a)\Gamma M]\Gamma S$ then $x = y\alpha z$ for some y in $((a\Gamma a)\Gamma M]$ and z in S , where $y \leq (a\gamma a)\beta m$ for some $(a\gamma a)\beta m$ in $(a\Gamma a)\Gamma M$ and $\alpha, \beta, \gamma \in \Gamma$. Now by using (1) and (4), we have

$$\begin{aligned} x &\leq ((a\gamma a)\beta m)\alpha z = (z\beta m)\alpha(a\gamma a) = (a\beta a)\alpha(m\gamma z) \\ &\subseteq (a\Gamma a)\Gamma(M\Gamma S) \subseteq (a\Gamma a)\Gamma M. \end{aligned}$$

Which implies that $x \in ((a\Gamma a)\Gamma M]$.

Now let $x \in S((a\Gamma a)\Gamma M]$ then $x = s\alpha t$ for some s in S and t in $((a\Gamma a)\Gamma M]$, where $t \leq (a\gamma a)\beta m$ for some $(a\gamma a)\beta m$ in $(a\Gamma a)\Gamma M$ and $\alpha, \beta, \gamma \in \Gamma$, then by

using (3), we have

$$x \leq s\alpha((a\gamma a)\beta m) = (a\gamma a)\alpha(s\beta m) \subseteq (a\Gamma a)\Gamma(S\Gamma M) \subseteq (a\Gamma a)\Gamma M.$$

Which implies that $x \in ((a\Gamma a)\Gamma M]$, and for the second condition of $((a\Gamma a)\Gamma M]$ to be a Γ -ideal let x be any element in $((a\Gamma a)\Gamma M]$ then $x \leq (a\gamma a)\beta m$ for some $(a\gamma a)\beta m$ in $(a\Gamma a)\Gamma M$. Let y be any other element of S such that $y \leq x \leq (a\gamma a)\beta m$, which implies that y is in $((a\Gamma a)\Gamma M]$. Hence $((a\Gamma a)\Gamma M]$ a Γ -ideal of S contain in $(M]$ and as $(M]$ is minimal so $(M] = ((a\Gamma a)\Gamma M]$.

Conversely, assume that $(M] = ((a\Gamma a)\Gamma M]$ for all $a \in S$. Let $(A]$ be the minimal Γ -ideal properly contain in $(M]$ containing a , then $(M] = ((a\Gamma a)\Gamma M] \subseteq (A]$, which is a contradiction. Hence $(M]$ is a minimal Γ -ideal. ■

A Γ -ideal I of an ordered Γ -AG-groupoid S is called minimal if and only if it does not contain any Γ -ideal of S other than itself.

Theorem 6. *If I is a minimal Γ -left ideal of an ordered Γ -AG^{**}-groupoid S , then $((a\Gamma a)\Gamma(I\Gamma I))$ is a minimal Γ -ideal of S .*

Proof. Assume that I is a minimal Γ -left ideal of an ordered Γ -AG^{**}-groupoid S . Now let $x \in ((a\Gamma a)\Gamma(I\Gamma I))\Gamma S$ then $x = y\alpha z$ for some y in $((a\Gamma a)\Gamma(I\Gamma I))$ and z in S where $y \leq (a\delta a)\beta(i\gamma j)$ for some i, j in I and $\alpha, \beta, \gamma, \delta \in \Gamma$, then by using (1) and (4), we have

$$\begin{aligned} x &\leq ((a\delta a)\beta(i\gamma j))\alpha z = (z\beta(i\gamma j))\alpha(a\delta a) = (a\beta a)\alpha((i\gamma j)\delta z) \\ &= (a\delta a)\alpha((z\gamma j)\delta i) \subseteq (a\Gamma a)\Gamma((S\Gamma I)\Gamma I) \subseteq (a\Gamma a)\Gamma(I\Gamma I). \end{aligned}$$

Which implies that $x \in ((a\Gamma a)\Gamma(I\Gamma I))$ and for the second condition of $((a\Gamma a)\Gamma(I\Gamma I))$ to be a Γ -ideal let x be any element in $((a\Gamma a)\Gamma(I\Gamma I))$ then $x \leq (a\delta a)\beta(i\gamma j)$ for some $(a\delta a)\beta(i\gamma j)$ in $((a\Gamma a)\Gamma(I\Gamma I))$. Which shows that $((a\Gamma a)\Gamma(I\Gamma I))$ is a Γ -right ideal of S . Similarly $((a\Gamma a)\Gamma(I\Gamma I))$ is a Γ -left ideal so is Γ -ideal. Let H be a non-empty Γ -ideal of S properly contained in $((a\Gamma a)\Gamma(I\Gamma I))$. Define $H' = \{r \in I : a\psi r \in H\}$. Then $a\psi(s\xi y) = s\psi(a\xi y) \in S\Gamma H \subseteq H$ imply that H' is a Γ -left ideal of S properly contained in I . But this is a contradiction to the minimality of I . Hence $((a\Gamma a)\Gamma(I\Gamma I))$ is a minimal Γ -ideal of S . ■

Theorem 7. *An ordered Γ -AG^{**}-groupoid S is Γ -fully prime if and only if every Γ -ideal is Γ -idempotent and Γ -ideal(S) is Γ -totally ordered under inclusion.*

Proof. Assume that an ordered Γ -AG^{**}-groupoid S is Γ -fully prime. Let I be the Γ -ideal of S . Then by Lemma 5, $(I\Gamma I)$ becomes a Γ -ideal of S and obviously $I\Gamma I \subseteq I$ and by Lemma 1, $(I\Gamma I) \subseteq (I]$. Now

$$(I\Gamma I) \subseteq (I\Gamma I) \text{ yields } (I] \subseteq (I\Gamma I) \text{ and hence}$$

$(I] = (I\Gamma I]$. Let P, Q be Γ -ideals of S and $P\Gamma Q \subseteq P, P\Gamma Q \subseteq Q$ imply that $P\Gamma Q \subseteq P \cap Q$. Since $P \cap Q$ is Γ -prime, so $P \subseteq P \cap Q$ or $Q \subseteq P \cap Q$ which further imply that $P \subseteq Q$ or $Q \subseteq P$. Hence Γ -ideal(S) is Γ -totally ordered under inclusion.

Converse is same as in [19]. ■

If S is an ordered Γ -AG^{**}-groupoid then the principal Γ -left ideal generated by a is defined by $\langle a \rangle = S\Gamma a = \{s\gamma a : s \in S, \gamma \in \Gamma\}$, where a is any element of S . Let P be a Γ -left ideal of an ordered Γ -AG-groupoid S , P is called Γ -quasi-prime if for Γ -left ideals A, B of S such that $A\Gamma B \subseteq P$, we have $A \subseteq P$ or $B \subseteq P$. P is called Γ -quasi-semiprime if for any Γ -left ideal I of S such that $I\Gamma I \subseteq P$, we have $I \subseteq P$.

Theorem 8. *If S is an ordered Γ -AG^{**}-groupoid, then a Γ -left ideal P of S is Γ -quasi-prime if and only if $a\Gamma(S\Gamma b) \subseteq P$ implies that either $a \in P$ or $b \in P$, where $a, b \in S$.*

Proof. The proof is same as in [19]. ■

Corollary 1. *If S is an ordered Γ -AG^{**}-groupoid, then a Γ -left ideal P of S is Γ -quasi-semiprime if and only if $a\Gamma(S\Gamma a) \subseteq P$ implies $a \in P$, for all $a \in S$.*

Proposition 3. *A Γ -ideal I of an ordered Γ -AG-groupoid S is Γ -prime if and only if it is Γ -semiprime and Γ -strongly irreducible.*

Proof. The proof is obvious. ■

Theorem 9. *Let S be an ordered Γ -AG-groupoid and $\{P_i : i \in N\}$ be a family of Γ -prime ideals Γ -totally ordered under inclusion in S . Then $\cap P_i$ is a Γ -prime ideal.*

Proof. The proof is same as in [19]. ■

Theorem 10. *For each Γ -ideal I there exists a minimal Γ -prime ideal of I in an ordered Γ -AG-groupoid S .*

Proof. The proof is same as in [19]. ■

Definition 15. An ordered Γ -AG-groupoid (S, Γ, \leq) is called regular if $a \in ((a\Gamma S)\Gamma a]$ for every $a \in S$, or

- (1) For every $a \in S$ there exist $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq (a\beta x)\gamma a$.
- (2) $A \subseteq ((A\Gamma S)\Gamma A]$ for every $A \subseteq S$.

Definition 16. An ordered Γ -AG-groupoid (S, Γ, \leq) is called weakly regular if $a \in ((a\Gamma S)\Gamma(a\Gamma S))$ for every $a \in S$, or

- (1) For every $a \in S$ there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (a\beta x)\delta(a\gamma y)$.
- (2) $A \subseteq ((A\Gamma S)\Gamma(A\Gamma S))$ for every $A \subseteq S$.

Definition 17. An ordered Γ -AG-groupoid (S, Γ, \leq) is called intra-regular if $a \in ((S\Gamma(a\delta a))\Gamma S)$ for every $a \in S$ and $\delta \in \Gamma$, or

- (1) For every $a \in S$ there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (x\beta(a\delta a))\gamma y$.
- (2) $A \subseteq ((S\Gamma(A\Gamma A))\Gamma S)$ for every $A \subseteq S$.

Definition 18. An ordered Γ -AG-groupoid (S, Γ, \leq) is called right regular if $a \in ((a\delta a)\Gamma S)$ for every $a \in S$ and $\delta \in \Gamma$, or

- (1) For every $a \in S$ there exist $x \in S$ and $\beta, \delta \in \Gamma$ such that $a \leq (a\delta a)\beta x$.
- (2) $A \subseteq ((A\Gamma A)\Gamma S)$ for every $A \subseteq S$.

Definition 19. An ordered Γ -AG-groupoid (S, Γ, \leq) is called left regular if $a \in (S\Gamma(a\delta a))$ for every $a \in S$ and $\delta \in \Gamma$, or

- (1) For every $a \in S$ there exist $x \in S$ and $\beta, \delta \in \Gamma$ such that $a \leq x\beta(a\delta a)$.
- (2) $A \subseteq (S\Gamma(A\Gamma A))$ for every $A \subseteq S$.

Definition 20. An ordered Γ -AG-groupoid (S, Γ, \leq) is called left quasi regular if $a \in ((S\Gamma a)\Gamma(S\Gamma a))$ for every $a \in S$, or

- (1) For every $a \in S$ there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (x\beta a)\delta(y\gamma a)$.
- (2) $A \subseteq ((S\Gamma A)\Gamma(S\Gamma A))$ for every $A \subseteq S$.

Definition 21. An ordered Γ -AG-groupoid (S, Γ, \leq) is called completely regular if S is regular, left regular and right regular.

Definition 22. An ordered Γ -AG-groupoid (S, Γ, \leq) is called (2, 2)-regular if $a \in (((a\delta a)\Gamma S)\Gamma(a\delta a))$ for every $a \in S$ and $\delta \in \Gamma$, or

- (1) For every $a \in S$ there exist $x \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq ((a\delta a)\beta x)\gamma(a\delta a)$.
- (2) $A \subseteq (((A\Gamma A)\Gamma S)\Gamma(A\Gamma A))$ for every $A \subseteq S$.

Definition 23. An ordered Γ -AG-groupoid (S, Γ, \leq) is called strongly regular if $a \in ((a\Gamma S)\Gamma a)$ and $a\beta x = x\beta a$ for every $a \in S$ and $\beta \in \Gamma$, or

- (1) For every $a \in S$ there exist $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq (a\beta x)\gamma a$ and $a\beta x = x\beta a$.
- (2) $A \subseteq ((A\Gamma S)\Gamma A)$ for every $A \subseteq S$.

Example 1. Let us consider an ordered AG-groupoid $S = \{1, 2, 3\}$ in the following multiplication table.

.	1	2	3
1	1	3	3
2	1	2	2
3	1	2	2

Let us define $\Gamma = \{\alpha, \beta, \gamma\}$ as follows.

α	1	2	3	β	1	2	3	γ	1	2	3
1	1	1	1	1	2	2	2	1	2	2	2
2	1	1	1	2	2	2	2	2	2	2	2
3	1	1	1	3	2	2	2	3	2	2	3

Here S is a Γ -AG-groupoid because $(a\beta b)\gamma c = (c\beta b)\gamma a$ for all $a, b, c \in S$ and S is non-associative, because $(1\alpha 2)\beta 3 \neq 1\alpha(2\beta 3)$. We define order \leq as:

$$\leq := \{(1, 1), (2, 2), (3, 3), (2, 1), (2, 3), (3, 1)\}.$$

Clearly (S, \leq) is a poset and for all a, b and $x \in S$, $a \leq b$ implies $a\beta x \leq b\beta x$ and $x\beta a \leq x\beta b$ for some $\beta \in \Gamma$ so S is a ordered Γ -AG-groupoid.

Note that S is a Γ -ideal itself so by Lemma 1, $(S\Gamma S] \subseteq S$.

Lemma 11. *If S is regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, (2, 2)-regular and strongly regular ordered Γ -AG-groupoid, then $S = (S\Gamma S]$.*

Proof. Assume that S is a regular ordered Γ -AG-groupoid, then $(S\Gamma S] \subseteq S$ is obvious. Let for any $a \in S$, $a \in ((a\Gamma S)\Gamma a)$, then there exists $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq (a\beta x)\gamma a$. Now $a \leq (a\beta x)\gamma a \in S\Gamma S$, thus $a \in (S\Gamma S]$. Similarly if S is weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, (2, 2)-regular or strongly regular, then we can show that $S = (S\Gamma S]$. ■

The converse is not true in general, because in Example 2, $S = (S\Gamma S]$ holds but S is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, (2, 2)-regular and strongly regular, because $1 \in S$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, (2, 2)-regular and strongly regular.

Example 2. Let us consider an ordered Γ -AG-groupoid $S = \{1, 2, 3, 4\}$ in the following Cayley's table.

.	1	2	3	4
1	4	4	2	4
2	4	4	1	4
3	1	2	3	4
4	4	4	4	4

Let us define $\Gamma = \{\alpha, \beta, \gamma\}$ as follows:

α	1	2	3	4	β	1	2	3	4	γ	1	2	3	4
1	1	1	1	1	1	2	2	2	2	1	2	2	2	2
2	1	1	1	1	2	2	2	2	2	2	2	2	2	2
3	1	1	1	1	3	2	2	2	2	3	2	2	2	2
4	1	1	1	1	4	2	2	2	3	4	2	2	2	4

Here S is a Γ -AG-groupoid because $(a\beta b)\gamma c = (c\beta b)\gamma a$ for all $a, b, c \in S$ and S is non-associative, because $(1\alpha 2)\beta 3 \neq 1\alpha(2\beta 3)$. We define the order \leq as:

$$\leq := \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 4), (2, 4), (3, 4)\}.$$

Clearly (S, \leq) is a poset and for all a, b and $x \in S$, $a \leq b$ implies $a\beta x \leq b\beta x$ and $x\beta a \leq x\beta b$ for some $\beta \in \Gamma$ so S is a ordered Γ -AG-groupoid. $A = \{1, 2, 4\}$ is an ideal of S as $A\Gamma S \subseteq A$ and $S\Gamma A \subseteq A$, also for every $1 \in A$ there exists $4 \in S$ such that $4 \leq 1 \in A$ implies that $4 \in A$, similarly for every $4 \in A$ there exists $2 \in S$ such that $2 \leq 4 \in A$ implies that $4 \in A$.

Theorem 12. *If S is an ordered Γ -AG^{**}-groupoid, then S is an intra-regular if and only if for all $a \in (S]$, $a \leq (x\beta a)\delta(a\gamma z)$ holds for some $x, z \in S$ and $\beta, \gamma, \delta \in \Gamma$.*

Proof. Assume that S is an intra-regular ordered Γ -AG^{**}-groupoid. Let $a \in ((S\Gamma(a\gamma a))\Gamma S]$ for any $a \in S$ and $\gamma \in \Gamma$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (x\beta(a\gamma a))\delta y$. Now by using Lemma 11, $y \leq u\gamma v$ for some $u, v \in S$. Thus by using (2), (1) and (4), we have

$$\begin{aligned} a &\leq (x\beta(a\gamma a))\delta y = (a\beta(x\gamma a))\delta y = (y\beta(x\gamma a))\delta a = (y\beta(x\gamma a))\delta((x\beta(a\gamma a))\delta y) \\ &\leq ((u\gamma v)\beta(x\gamma a))\delta((x\beta(a\gamma a))\delta y) = ((a\gamma x)\beta(v\gamma u))\delta((x\beta(a\gamma a))\delta y) \\ &\leq ((a\gamma x)\beta t)\delta((x\beta(a\gamma a))\delta y) = (((x\beta(a\gamma a))\delta y)\beta t)\delta(a\gamma x) \\ &= ((t\delta y)\beta(x\beta(a\gamma a)))\delta(a\gamma x) = (((a\gamma a)\delta x)\beta(y\beta t))\delta(a\gamma x) \end{aligned}$$

$$\begin{aligned}
&\leq (((a\gamma a)\delta x)\beta s)\delta(a\gamma x) = ((s\delta x)\beta(a\gamma a))\delta(a\gamma x) = ((a\delta a)\beta(x\gamma s))\delta(a\gamma x) \\
&\leq ((a\delta a)\beta w)\delta(a\gamma x) = ((w\delta a)\beta a)\delta(a\gamma x) \leq (z\beta a)\delta(a\gamma x) \\
&= (x\beta a)\delta(a\gamma z),
\end{aligned}$$

where $v\gamma u \leq t$, $y\beta t \leq s$, $x\gamma s \leq w$ and $w\delta a \leq z$ for some $t, s, w, z \in S$.

Conversely, let for all $a \in (S]$, $a \leq (x\beta a)\delta(a\gamma z)$ holds for some $x, z \in S$ and $\beta, \gamma, \delta \in \Gamma$. Now by using (3), (1), (2) and (4), we have

$$\begin{aligned}
a &\leq (x\beta a)\delta(a\gamma z) = a\delta((x\beta a)\gamma z) \leq ((x\beta a)\delta(a\gamma z))\delta((x\beta a)\gamma z) \\
&= (a\delta((x\beta a)\gamma z))\delta((x\beta a)\gamma z) = (((x\beta a)\gamma z)\delta((x\beta a)\gamma z))\delta a \\
&= (((x\beta a)\gamma(x\beta a))\delta(z\gamma z))\delta a = (((a\beta x)\gamma(a\beta x))\delta(z\gamma z))\delta a \\
&= ((a\gamma((a\beta x)\beta x))\delta(z\gamma z))\delta a = (((z\gamma z)\delta((a\beta x)\beta x))\gamma a)\delta a \\
&= (((a\beta x)\delta((z\gamma z)\beta x))\gamma a)\delta a = (((((z\gamma z)\beta x)\beta x)\delta a)\gamma a)\delta a \\
&= (((((x\beta x)\beta(z\gamma z))\delta a)\gamma a)\delta a = ((a\delta a)\gamma((x\beta x)\beta(z\gamma z)))\delta a \\
&= (a\gamma((x\beta x)\beta(z\gamma z)))\delta(a\delta a) \\
&\leq (a\gamma t)\delta(a\delta a), \text{ where } (x\beta x)\beta(z\gamma z) \leq t \text{ for some } t \in S.
\end{aligned}$$

Now by using (4) and (1), we have

$$\begin{aligned}
a &\leq (a\gamma t)\delta(a\delta a) \leq (((a\gamma t)\delta(a\delta a))\gamma t)\delta(a\delta a) = (((a\gamma a)\delta(t\delta a))\gamma t)\delta(a\delta a) \\
&= (((a\gamma a)\delta(t\delta a))\gamma t)\delta(a\delta a) = ((t\delta(t\delta a))\gamma(a\gamma a))\delta(a\delta a) \\
&\leq (u\gamma(a\gamma a))\delta v, \text{ where } t\delta(t\delta a) \leq u \text{ and } a\delta a \leq v \text{ for some } u, v \in S \\
&\in (S\Gamma(a\gamma a))\Gamma S.
\end{aligned}$$

Which implies that $a \in ((S\Gamma(a\gamma a))\Gamma S]$, thus S is intra-regular. ■

Theorem 13. *If S is an ordered Γ -AG^{**}-groupoid, then the following are equivalent.*

- (i) S is weakly regular.
- (ii) S is intra-regular.

Proof. (i) \implies (ii) Assume that S is a weakly regular ordered Γ -AG^{**}-groupoid. Let $a \in ((a\Gamma S)\Gamma(a\Gamma S)]$ for any $a \in S$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$

such that $a \leq (a\beta x)\delta(a\gamma y)$. Now by Lemma 11, let $x \leq s\psi t$ for some $s, t \in S$, $\psi \in \Gamma$ and $t\gamma s \leq u \in S$, then by using (4) and (1), we have

$$\begin{aligned} a &\leq (a\beta x)\delta(a\gamma y) = (y\beta a)\delta(x\gamma a) = ((x\gamma a)\beta a)\delta y \leq (((s\psi t)\gamma a)\beta a)\delta y \\ &= ((a\gamma a)\beta(s\psi t))\delta y = ((t\gamma s)\beta(a\psi a))\delta y = ((t\gamma s)\beta(a\psi a))\delta y \leq (u\beta(a\psi a))\delta y \\ &\in (S\Gamma(a\psi a))\Gamma S. \end{aligned}$$

Which implies that $a \in ((S\Gamma(a\psi a))\Gamma S]$, thus S is intra-regular.

(ii) \implies (i) is the same as (i) \implies (ii). ■

Theorem 14. *If S is an ordered Γ -AG^{**}-groupoid, then the following are equivalent.*

- (i) S is weakly regular.
- (ii) S is right regular.

Proof. (i) \implies (ii) Assume that S is a weakly regular ordered Γ -AG^{**}-groupoid. Let $a \in ((a\Gamma S)\Gamma(a\Gamma S))$ for any $a \in S$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (a\beta x)\delta(a\gamma y)$ and by using Lemma 11, let $x\gamma y \leq t$ for some $t \in S$. Now by using (2), we have

$$a \leq (a\beta x)\delta(a\gamma y) = (a\beta a)\delta(x\gamma y) \leq (a\beta a)\delta t \in (a\beta a)\Gamma S.$$

Which implies that $a \in ((a\beta a)\Gamma S]$, thus S is right regular.

(ii) \implies (i) It follows from Lemma 11 and (2). ■

Theorem 15. *If S is an ordered Γ -AG^{**}-groupoid, then the following are equivalent.*

- (i) S is weakly regular.
- (ii) S is left regular.

Proof. (i) \implies (ii) Assume that S is a weakly regular ordered Γ -AG^{**}-groupoid. Let $a \in ((a\Gamma S)\Gamma(a\Gamma S))$ for any $a \in S$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (a\beta x)\delta(a\gamma y)$. Now let $y\beta x \leq t$ for some $t \in S$ then by (2) and (4), we have

$$\begin{aligned} a &\leq (a\beta x)\delta(a\gamma y) = (a\beta a)\delta(x\gamma y) = (y\beta x)\delta(a\gamma a) = (y\beta x)\delta(a\gamma a) \\ &\leq t\delta(a\gamma a) \in S\Gamma(a\gamma a). \end{aligned}$$

Which implies that $a \in (S\Gamma(a\gamma a))$, thus S is left regular.

(ii) \implies (i) It follows from Lemma 11, (4) and (2). ■

Lemma 16. *Every weakly regular ordered Γ -AG^{**}-groupoid is regular.*

Proof. Assume that S is a weakly regular ordered Γ -AG^{**}-groupoid. Let $a \in ((a\Gamma S)\Gamma(a\Gamma S))$ for any $a \in S$, then there exist $x, y \in S$ and $\beta, \gamma, \delta \in \Gamma$ such that $a \leq (a\beta x)\delta(a\gamma y)$. Let $x\gamma y \leq t \in S$ and by using (1), (2), (4) and (3), we have

$$\begin{aligned} a &\leq (a\beta x)\delta(a\gamma y) = ((a\gamma y)\beta x)\delta a = ((x\gamma y)\beta a)\delta a \leq (t\beta a)\delta a \\ &\leq (t\beta((a\beta x)\delta(a\gamma y)))\delta a = (t\beta((a\beta a)\delta(x\gamma y)))\delta a \\ &= (t\beta((y\beta x)\delta(a\gamma a)))\delta a = (t\beta(a\delta((y\beta x)\gamma a)))\delta a \\ &= (a\beta(t\delta((y\beta x)\gamma a)))\delta a \leq (a\beta u)\delta a, \text{ where } t\delta((y\beta x)\gamma a) \leq u \in S \\ &\in (a\Gamma S)\Gamma a. \end{aligned}$$

Which implies that $a \in ((a\Gamma S)\Gamma a)$, thus S is regular. ■

The converse of above Lemma is not true in general, as can be seen from the following example.

Example 3 [24]. Let us consider an AG-groupoid $S = \{1, 2, 3, 4\}$ in the following Cayley's table.

.	1	2	3	4
1	2	2	4	4
2	2	2	2	2
3	1	2	3	4
4	1	2	1	2

Let us define $\Gamma = \{\alpha, \beta, \gamma\}$ as follows:

α	1	2	3	4	β	1	2	3	4	γ	1	2	3	4
1	1	1	1	1	1	2	2	2	2	1	2	2	2	2
2	1	1	1	1	2	2	2	2	2	2	2	2	2	2
3	1	1	1	1	3	2	2	4	4	3	2	2	2	2
4	1	1	1	1	4	2	2	2	2	4	2	2	3	4

Here S is a Γ -AG-groupoid because $(a\beta b)\gamma c = (c\beta b)\gamma a$ for all $a, b, c \in S$. We define order \leq as:

$$\leq := \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (4, 2)\}.$$

Clearly (S, \leq) is a poset and for all a, b and $x \in S$, $a \leq b$ implies $a\beta x \leq b\beta x$ and $x\beta a \leq x\beta b$ for some $\beta \in \Gamma$ so S is a ordered Γ -AG-groupoid. Also S is regular, because $1 \leq (1\alpha 3)\alpha 1$, $2 \leq (2\beta 1)\gamma 2$, $3 \leq (3\beta 3)\gamma 3$ and $4 \leq (4\gamma 3)\beta 4$, but S is not weakly regular, because $1 \notin ((1\Gamma S)\Gamma(1\Gamma S))$.

Theorem 17. *If S is an ordered Γ -AG^{**}-groupoid, then the following are equivalent.*

- (i) S is weakly regular.
- (ii) S is completely regular.

Proof. (i) \implies (ii) It follows from Theorems 14, 15 and Lemma 16.

(ii) \implies (i) It follows from Theorem 15. ■

Theorem 18. *If S is an ordered Γ -AG^{**}-groupoid, then the following are equivalent.*

- (i) S is weakly regular.
- (ii) S is left quasi regular.

Proof. The proof of this Lemma is straight forward. ■

Theorem 19. *If S is an ordered Γ -AG^{**}-groupoid, then the following are equivalent.*

- (i) S is (2, 2)-regular.
- (ii) S is completely regular.

Proof. (i) \implies (ii) Assume that S is a (2, 2)-regular ordered Γ -AG^{**}-groupoid. Let $a \in ((a\delta a)\Gamma S)\Gamma(a\delta a)$ for any $a \in S$ and $\delta \in \Gamma$, then there exist $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq ((a\delta a)\beta x)\gamma(a\delta a)$. Now let $(a\delta a)\beta x \leq y \in S$, then we have

$$a \leq ((a\delta a)\beta x)\gamma(a\delta a) \leq y\gamma(a\delta a) \in S\Gamma(a\delta a).$$

Which implies that $a \in (S\Gamma(a\delta a))$, thus S is left regular. Now by using (4), we have

$$\begin{aligned} a &\leq ((a\delta a)\beta x)\gamma(a\delta a) = (a\beta a)\gamma(x\delta(a\delta a)) \\ &\leq (a\delta a)\gamma z, \text{ where } x\delta(a\delta a) \leq z \in S \text{ and } \delta \in \Gamma \\ &\in (a\delta a)\Gamma S. \end{aligned}$$

Which implies that $a \in ((a\delta a)\Gamma S)$, thus S is right regular. Now let $x \leq u\psi v$ for some $u, v \in S$ and $\psi \in \Gamma$, then by using (4), (1) and (3), we have

$$\begin{aligned} a &\leq ((a\delta a)\beta x)\gamma(a\delta a) = (a\beta a)\gamma(x\delta(a\delta a)) \leq (a\beta a)\gamma((u\psi v)\delta(a\delta a)) \\ &= (a\beta a)\gamma((a\psi a)\delta(v\delta u)) \leq (a\beta a)\gamma((a\psi a)\delta t), \text{ where } v\delta u \leq t \in S \\ &= (((a\psi a)\delta t)\beta a)\gamma a = ((a\delta t)\beta(a\psi a))\gamma a = (a\beta((a\delta t)\psi a))\gamma a \end{aligned}$$

$$\begin{aligned} &\leq (a\beta y)\gamma a, \text{ where } (a\delta t)\psi a \leq y \in S \\ &\in (a\Gamma S)\Gamma a. \end{aligned}$$

Which implies that $a \in ((a\Gamma S)\Gamma a]$, so S is regular. Thus S is completely regular. (ii) \implies (i) Assume that S is a completely regular ordered Γ -AG^{**}-groupoid. Let $a \in ((a\Gamma S)\Gamma a]$, $a \in ((a\delta a)\Gamma S]$ and $a \in (S\Gamma(a\delta a))$ for any $a \in S$, then there exist $x, y, z \in S$ and $\beta, \gamma, \psi, \xi, \delta \in \Gamma$ such that $a \leq (a\beta x)\gamma a$, $a \leq (a\delta a)\psi y$ and $a \leq z\xi(a\delta a)$. Now by using (4), (1) and (3), we have

$$\begin{aligned} a &\leq (a\beta x)\gamma a \leq (a\beta x)\gamma(z\xi(a\delta a)) = ((a\delta a)\beta z)\gamma(x\xi a) = ((x\xi a)\beta z)\gamma(a\delta a) \\ &\leq ((x\xi((a\delta a)\psi y))\beta z)\gamma(a\delta a) = (((a\delta a)\xi(x\psi y))\beta z)\gamma(a\delta a) \\ &\leq (((a\delta a)\xi t)\beta z)\gamma(a\delta a), \text{ where } x\psi y \leq t \in S \\ &= ((z\xi t)\beta(a\delta a))\gamma(a\delta a) = ((a\xi a)\beta(t\delta z))\gamma(a\delta a) \\ &\leq ((a\xi a)\beta w)\gamma(a\delta a), \text{ where } t\delta z \leq w \in S \\ &= ((a\xi a)\beta w)\gamma(a\delta a) \in ((a\xi a)\Gamma S)\Gamma(a\delta a). \end{aligned}$$

Which implies that $a \in (((a\xi a)\Gamma S)\Gamma(a\delta a))$, this shows that S is (2, 2)-regular. ■

Lemma 20. *Every strongly regular ordered Γ -AG^{**}-groupoid is completely regular.*

Proof. Assume that S is a strongly regular ordered Γ -AG^{**}-groupoid, then for any $a \in S$ there exists $x \in S$ and $\beta, \gamma \in \Gamma$ such that $a \leq (a\beta x)\gamma a$ and $a\beta x = x\beta a$. Now by using (1), we have

$$a \leq (a\beta x)\gamma a = (x\beta a)\gamma a = (a\beta a)\gamma x \subseteq (a\beta a)\Gamma S.$$

Which implies that $a \in (a^2\Gamma S]$, this shows that S is right regular and by Theorems 14 and 17, it is clear to see that S is completely regular. ■

Theorem 21. *In an ordered Γ -AG^{**}-groupoid S , the following are equivalent.*

- (i) S is weakly regular,
- (ii) S is intra-regular,
- (iii) S is right regular,
- (iv) S is left regular,
- (v) S is left quasi regular,

- (vi) S is completely regular,
- (vii) For all $a \in S$, there exist $x, y \in S$ such that $a \leq (x\beta a)\delta(a\gamma z)$ holds for some $x, z \in S$ and $\beta, \gamma, \delta \in \Gamma$,
- (viii) S is $(2, 2)$ -regular.

Proof. (i) \iff (ii) It follows from Theorem 13.

(ii) \iff (iii) It follows from Theorems 13 and 14.

(iii) \iff (iv) It follows from Theorems 14 and 15.

(iv) \iff (v) It follows from Theorems 15 and 18.

(v) \iff (vi) It follows from Theorems 18 and 17.

(vi) \iff (i) It follows from Theorem 17.

(ii) \iff (vii) It follows from Theorem 12.

(vi) \iff (viii) It follows from Theorem 19. ■

Remark 3. Every intra-regular, right regular, left regular, left quasi regular $(2, 2)$ -regular and completely regular ordered Γ -AG**-groupoids are regular.

The converse is not true in general, as can be seen from Example 3.

Theorem 22. Regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2, 2)$ -regular and strongly regular Γ -AG*-groupoids become a Γ -semigroups.

Proof. It follows from (6) and Lemma 11. ■

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