Discussiones Mathematicae
General Algebra and Applications 33 (2013) 201–209
doi:10.7151/dmgaa.1206

NIL-EXTENSIONS OF COMPLETELY SIMPLE SEMIRINGS

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Abstract

A semiring S is said to be a quasi completely regular semiring if for any $a \in S$ there exists a positive integer n such that na is completely regular. The present paper is devoted to the study of completely Archimedean semirings. We show that a semiring S is a completely Archimedean semiring if and only if it is a nil-extension of a completely simple semiring. This result extends the crucial structure theorem of completely Archimedean semigroup.

Keywords: ideal extension, nil-extension, bi-ideal, completely Archimedean semirings, completely simple semiring.

2010 Mathematics Subject Classification: 16A78, 20M10, 20M07.

1. Introduction

Nil-extension of completely simple semigroups was first considered by S. Bogdanovic and S. Milic [1] in 1984. In [2], S. Bogdanovic discussed decomposition of completely π-regular semigroups into a semilattice of Archimedean semigroups. Later on, N. Kehayopulu and K. P. Shum [7] also studied the nil-extensions of regular poe-semigroups in 2003. In recent years the study of nil extension of completely regular semigroups has been subject with attraction. It has been proven that a completely Archimedean semigroup is a nil-extension of completely simple semigroups and furthermore it is Archimedean and completely π - regular. Moreover, characterization of nil-extension of a band and retractive nil-extensions of completely simple semigroups were also a matter of interest.

The structure of semirings has been recently studied by many authors, for example, by F. Pastijn, Y.Q. Guo, M.K. Sen, K.P. Shum and others (See [11, 15]). Recently, in paper [15], the study of completely regular semirings have derived prolific results which were analogue properties as completely regular semigroups, and it has also been derived that a completely regular semiring is a b-lattice of completely simple semirings. Many interesting results in completely regular semigroups and inverse semigroups have been extended to semirings by M.K. Sen, S.K. Maity and K.P. Shum in ([8, 14]). An extension of completely regular semirings to quasi completely regular semirings [9] having analogous results has proved a further breakthrough. It extended the idea of GV semigroups as semilattice of completely Archimedean semigroups to semirings, and it has been derived that quasi completely regular semirings can be described as the b-lattice of completely Archimedean semirings. It has also been established that quasi completely regular semirings are idempotent semiring of quasi skew-rings. In recent years, varieties of idempotent semirings and their subvarieties have been studied by F. Pastijin, Y.Q. Guo, X. Zhao and K.P. Shum in [12, 16] and [17].

In this paper, we generalize another result on semigroups to semirings. We show that a semiring is completely Archimedean if and only if it is nil-extension of a completely simple semiring if and only if it is Archimedean and quasi completely regular. The preliminaries and prerequisites we need for this article are discussed in Section 2. In Section 3 we prove some characterization theorems and study few properties of completely Archimedean semirings and finally discuss our main result.

2. Preliminaries

A semiring $(S, +, \cdot)$ is a type (2, 2)- algebra whose semigroup reducts $(S, +)$ and $(S, \cdot)$ are connected by ring like distributivity, that is, $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$ for all $a, b, c \in S$. A semiring $(S, +, \cdot)$ is said to be a b-lattice if $(S, \cdot)$ is a band and $(S, +)$ is a semilattice. A semiring $(S, +, \cdot)$ is said to be a skew-ring if its additive reduct $(S, +)$ is a group, not necessarily an abelian group. A semiring $(S, +, \cdot)$ is called additively regular if for every element $a \in S$ there exists an element $x \in S$ such that $a + x + a = a$. We call a semiring $(S, +, \cdot)$ additively quasi regular if for every element $a \in S$ there exists a positive integer n such that na is additively regular. We define an element a in a semiring $(S, +, \cdot)$ as quasi completely regular [9] if there exists a positive integer n such that na is completely regular, that is, there exists an element $x \in S$ such that

$$\begin{array}{rrcl} \text{(i)} & na + x + na & = & na, \\ \text{(ii)} & na + x & = & x + na, \\ \text{and} \quad \text{(iii)} & na(na + x) & = & na + x. \end{array}$$

In fact, conditions (i) and (ii) follow immediately from definition when the additive reduct $(S, +)$ of the semiring $(S, +, \cdot)$ is a quasi completely regular semigroup.

Condition (iii) is an additional condition which makes the element a in $(S,+,\cdot)$ quasi completely regular. Naturally, a semiring S is said to be a quasi completely regular semiring if every element of S is quasi completely regular.

There are plenty of examples of quasi completely regular semirings, for example, every completely regular semiring is a quasi completely regular semiring. However, the converse of the above statement is not true in general. This follows from the following example:

Example 2.1. We consider the set $S = \{0, 1, 2, \ldots, c\}$ where c be a positive integer. On S we define addition $'+'$ and multiplication $'\cdot'$ as follows:

$$x \oplus y = \min\{x + y, c\}$$
$$x \odot y = \min\{xy, c\}.$$

Then $(S,+,\cdot)$ is a quasi completely regular semiring but not a completely regular semiring.

Throughout this paper, we always let $E^+(S)$ be the set of all additive idempotents of the semiring S. Also we denote the set of all inverse elements of an additively regular element a in a semiring $(S,+,\cdot)$ by $V^+(a)$. As usual, we denote the Green's relations on the semiring $(S,+,\cdot)$ by $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{J}$ and $\mathcal{H}$ and correspondingly, the $\mathcal{L}$-relation, $\mathcal{R}$-relation, $\mathcal{D}$-relation, $\mathcal{J}$-relation and $\mathcal{H}$-relation on $(S,+)$ are denoted by $\mathcal{L}^+$, $\mathcal{R}^+$, $\mathcal{D}^+$, $\mathcal{J}^+$ and $\mathcal{H}^+$, respectively. In fact, the relations $\mathcal{L}^+$, $\mathcal{R}^+$, $\mathcal{D}^+$, $\mathcal{J}^+$ and $\mathcal{H}^+$ are all congruence relations on the multiplicative reduct $(S,\cdot)$. Thus if any one of these happens to be a congruence on $(S,+)$, it will be a congruence on the semiring $(S,+,\cdot)$. For any $a \in S$, we let H_a^+ be the $\mathcal{H}^+$-class in $(S,+)$ containing a. We further denote the Green's relations on a quasi completely regular semigroup as $\mathcal{L}^*$,$\mathcal{R}^*$, $\mathcal{H}^*$, $\mathcal{D}^*$ and $\mathcal{J}^*$. For other notations and terminologies not given in this paper, the reader is referred to Sen, Maity and Shum [15] and also the texts of Howie [6], Golan [4], and Hebisch and Weinert [5].

3. Completely Archimedean semiring

In this section we discuss some characterization theorems and properties of completely Archimedean semirings. We define nil-extension on semirings with the help of bi-ideals.

We recall that a semiring $(S,+,\cdot)$ is a quasi completely regular semiring if for each $a \in S$ there exists a positive integer n and an element $x \in S$ such that the conditions (i), (ii) and (iii) are satisfied.

Every completely regular semiring is a quasi completely regular semiring. However, the converse of the above statement is not true in general. In fact, quasi completely regular semirings are generalizations of completely regular semirings.

Example 3.1 [9]**.** Let $S = \{a, b, c, d, e, f, g, h\}$. On S we define addition $'+'$ and multiplication $'\cdot'$ with the following Cayley tables

$+$	a	b	c	d	e	f	g	h
a	c	e	c	f	g	h	g	h
b	e	d	g	d	f	f	h	h
c	c	g	c	h	g	h	g	h
d	f	d	h	d	f	f	h	h
e	g	f	g	f	h	h	h	h
f	h	f	h	f	h	h	h	h
g	g	h	g	h	h	h	h	h
h	h	h	h	h	h	h	h	h

$\cdot$	a	b	c	d	e	f	g	h
a	c	a	c	c	c	c	c	c
b	a	d	c	d	f	f	h	h
c	c	c	c	c	c	c	c	c
d	c	d	c	d	h	h	h	h
e	c	f	c	h	h	h	h	h
f	c	f	c	h	h	h	h	h
g	c	h	c	h	h	h	h	h
h	c	h	c	h	h	h	h	h

Then $(S, +, \cdot)$ is a quasi completely regular semiring.

Theorem 3.2 [9]**.** *The following statements on a semiring S are equivalent:*

(i) *$a \in S$ is quasi completely regular.*

(ii) *There exists a positive integer n and a unique element $y \in V^+(na)$ such that $na + y = y + na$ and $na(na + y) = na + y$.*

(iii) *There exists a positive integer n such that na lies in a subskew-ring of S.*

The unique element in $V^+(na)$ satisfying condition (ii) of Theorem 3.2 is denoted by $(na)'$.

Theorem 3.3 [9]**.** *Let S be a semiring and let a be an element of S such that na lies in a subskew-ring R of S for some positive integer n. If e is the zero of R, then*

(a) *$e + x = x + e \in R$,*

(b) *$ma \in R$ for any integer $m \geq n$,*

(c) *$ae = ea = e$.*

Lemma 3.4. *If S is a quasi completely regular semiring, then $E^+(S) = \{na + (na)' : a \in S$ and for some $n \in \mathbb{N}\}$ and $e^2 = e$ for all $e \in E^+(S)$.*

Definition 3.5. Let $(S, +, \cdot)$ be an additively quasi regular semiring. We consider the relations $\mathcal{L}^{*+}$, $\mathcal{R}^{*+}$, $\mathcal{J}^{*+}$, $\mathcal{H}^{*+}$ and $\mathcal{D}^{*+}$ defined by

$$a\,\mathcal{L}^{*+}\,b \text{ if and only if } pa\,\mathcal{L}^{+}\,qb\ ,$$
$$a\,\mathcal{R}^{*+}\,b \text{ if and only if } pa\,\mathcal{R}^{+}\,qb\ ,$$
$$a\,\mathcal{J}^{*+}\,b \text{ if and only if } pa\,\mathcal{J}^{+}\,qb\ ,$$
$$\mathcal{H}^{*+} = \mathcal{L}^{*+} \cap \mathcal{R}^{*+} \qquad \text{and} \qquad \mathcal{D}^{*+} = \mathcal{L}^{*+}\, o\, \mathcal{R}^{*+};$$

where p and q are the smallest positive integers such that pa and qb are additively regular.

Definition 3.6. A semiring $(S, +, \cdot)$ is said to be Archimedean if $(S, +)$ is an Archimedean semigroup, i.e., if for any two elements $a, b \in S$, there exists a positive integer n such that $na \in S + b + S$.

Definition 3.7. A quasi completely regular semiring $(S, +, \cdot)$ is said to be completely Archimedean if any two elements of S are $\mathcal{J}^{*+}$- related.

Definition 3.8. A congruence ρ on a semiring S is called a b-lattice congruence (idempotent semiring congruence) if S/ρ is a b-lattice (respectively, an idempotent semiring). A semiring S is called a b-lattice (idempotent semiring) Y of semirings $S_\alpha (\alpha \in Y)$ if S admits a b-lattice congruence (respectively, an idempotent semiring congruence) ρ on S such that $Y = S/\rho$ and each S_α is a ρ-class mapped onto α by the natural homomorphism $\rho^{\#} : S \longrightarrow Y$.

Definition 3.9. Let R be subskew-ring of a semiring S. If for every $a \in S$ there exists a positive integer n such that $na \in R$, then S is said to be a quasi skew-ring.

Theorem 3.10 [9]**.** *Let $(S, +, \cdot)$ be a semiring. Then S is an additively quasi regular semiring with exactly one additive idempotent if and only if S is a quasi skew-ring.*

Definition 3.11 [9]**.** Let $(S, +, \cdot)$ be a quasi completely regular semiring. We define R_e by

$$R_e = \{x \in S : e + x = x = x + e, x + y = e = y + x \text{ for some } y \in S\}.$$

where $e \in E^+(S)$. Then it is easy to verify that R_e is a subskew-ring of S containing e as zero and $R_e = H_e^+$.

We know that a GV semigroup is a special kind of completely π - regular semigroup in which every regular element is completely regular. But in the case of semiring, we proved a very interesting result [9] that in a quasi completely regular semiring every additively regular element is completely regular.

Theorem 3.12 [9]**.** *The following conditions on a semiring $(S, +, \cdot)$ are equivalent.*

(i) *S is a quasi completely regular semiring.*
(ii) *Every $\mathcal{H}^{*+}$- class is a quasi skew-ring.*
(iii) *S is (disjoint) union of quasi skew-rings.*

(iv) *S is a b-lattice of completely Archimedean semirings.*

(v) *S is an idempotent semiring of of quasi skew-rings.*

Theorem 3.13. *Let S be a completely Archimedean semiring. Then for all $e, f \in E^+(S)$ with $e+f = f+e = f$ implies $e = f$.*

Proof. Since S is completely Archimedean and $e, f \in E^+(S)$, it follows that $e = x+f+y$ for some $x, y \in S$. Let $a = e+x+e$. Then $a = e+x+f+y+e+x+e = e+x+e+f+y+e+x+e = a+f+y+a$. Thus, a is an additively regular element in S and hence it is completely regular. Thus, there exists $a' \in S$ such that $a+a'+a = a$, $a+a' = a'+a$ and $a(a+a') = a+a'$. Let $b = e+y+e$. Then $a+f+b = e+x+e+f+e+y+e = e+x+f+y+e = e$ and hence $e = a+f+b = a+a'+a+f+b = a+a'+e = a'+a+e = a'+a$. Therefore, $e = a'+a = a'+e+a = a'+a+f+b+a = e+f+b+a = f+b+a$ and thus $e = f+e$. Hence $e = f$. ■

Theorem 3.14. *Let S be a completely Archimedean semiring. Then the subskew-rings are given by $R_e = e+S+e$ where $e \in E^+(S)$.*

Proof. Let $e \in E^+(S)$ and $u \in R_e$. Then $u = e+u+e \in e+S+e$ and thus $R_e \subseteq e+S+e$. Conversely, let $v \in e+S+e$, i.e., $v = e+a+e$ for some $a \in S$. Since S is quasi completely regular semiring, it follows that $nv \in H_f^+$ for some $n \in \mathbb{N}$ and $f \in E^+(S)$. Then $e+f = e+(nv)+(nv)' = e+n(e+a+e)+(nv)' = (nv)+(nv)' = f$. Similarly, $f+e = f$. Since S is a completely Archimedean semiring so by Theorem 3.13, it follows that $e = f$. Therefore, $nv \in H_e^+$. From this and Theorem 3.3 we have that $(n+1)v \in H_e^+$. Hence, $e = (n+1)v + ((n+1)v)' = v+nv+(nv+v)' = nv+(nv+v)'+v$ and $e+v = e+(e+a+e) = e+a+e = v = v+e$ implies we have $v \in H_e^+$ and therefore $e+S+e \subseteq H_e^+ = R_e$. Consequently, $R_e = e+S+e$. ■

Definition 3.15. ([3, 10]) Let $(S, +, \cdot)$ be a semiring. A nonempty subset I of S is said to be a bi-ideal of S if $a \in I$ and $x \in S$ imply that $a+x,\ x+a,\ ax,\ xa \in I$.

Example 3.16. We consider the semiring $(\mathbb{N}, +, \cdot)$ of all natural numbers with respect to usual addition and multiplication of natural numbers. Then the set $I = \{n : n \geq 5\}$ is a bi-ideal of $\mathbb{N}$.

In example 3.16, the ideal I is a bi-ideal but not a k-deal. Again in any ring R, any proper ideal I of R is a k-ideal which is not a bi-ideal of R.

Example 3.17. Let $M_n(\mathbb{N})$ denote the set of all $n \times n$ matrices with entries from $\mathbb{N}$. Then $M_n(\mathbb{N})$ is a semiring with respect to usual addition and usual multiplication of matrices. Consider the set $I = \{(a_{ij}) \in M_n(\mathbb{N}) : a_{ij} \geq 5$ for all $i, j = 1, 2, \ldots, n\}$. Then I is a bi-ideal of $M_n(\mathbb{N})$ but not a k-ideal of $M_n(\mathbb{N})$.

Definition 3.18. Let I be a bi-ideal of a semiring S. We define a relation ρ_I on S by $a\rho_I b$ if and only if either $a, b \in I$ or $a = b$ where $a, b \in S$. It is easy to verify that ρ_I is a congruence on S. This congruence is said to be Rees congruence on S and the quotient semiring S/ρ_I contains a zero, namely I. This quotient semiring S/ρ_I is said to be the Rees quotient semiring and is denoted by S/I. In this case the semiring S is said to be an ideal extension or simply an extension of I by the semiring S/I. An ideal extension S of a semiring I is a nil-extension of I if for any $a \in S$ there exists a positive integer n such that $na \in I$.

Theorem 3.19. *The following conditions on a semiring are equivalent:*

(i) *S is a completely Archimedean semiring;*

(ii) *S is a nil-extension of a completely simple semiring;*

(iii) *S is Archimedean and quasi completely regular.*

Proof. (i) $\Longrightarrow$ (ii) Let S be a completely Archimedean semiring. Let $K = \bigcup_{e \in E^+(S)} H_e^+$. We first show that K is a bi-ideal of S. For this let $y \in K$ and $x \in S$. Now there exists a positive integer m such that $m(x+y)$ is completely regular and hence $m(x+y), g+(x+y) \in H_g^+$ for some $g \in E^+(S)$. Then $m(x+y)+(m(x+y))' = g$. Again $x \in K$ implies $x \in H_e^+$ for some $e \in E^+(S)$. Now, $e+g = e+m(x+y)+(m(x+y))' = m(x+y)+(m(x+y))' = g$. Also $g+e = m(x+y)+(m(x+y))'+e \in e+S+e = H_e^+$ such that $(g+e)+(g+e) = g+e$. Thus $g+e = e$. Therefore, $x+y = (e+x)+y = (g+e+x)+y = g+(x+y) \in H_g^+ \subseteq K$. Similarly, we can show that $y+x \in K$.

Again, $x \in H_e^+ = e+S+e$ implies $x = e+u+e$ for some $u \in S$. Then $xy = ey+uy+ey = f+uy+f \in f+S+f$ where $f = ey \in E^+(S)$. This implies $xy = f+uy+f \in H_f^+ \subseteq K$. Similarly, $yx \in K$. Hence K is a bi-ideal of S. Also K is a subsemiring of S. Therefore K is a completely regular subsemiring of S such that any two elements of K are $\mathcal{J}^+$-related. Hence K is a completely simple semiring. Clearly, for any $a \in S$ there exists a positive integer n such that $na \in K$. Hence S is a nil-extension of K.

(ii) $\Longrightarrow$ (iii) This is obvious.

(iii) $\Longrightarrow$ (i) Let $a, b \in S$ and let m be the smallest positive integer such that mb is additively regular. We assume $c = mb$. Then by definition, $na \in S+c+S$ which implies $na \in S+mb+S$. Similarly, we can prove that $mb \in S+na+S$. Hence any two elements in S are $\mathcal{J}^{*+}$-related. Consequently, S is a completely Archimedean semiring. ∎

Theorem 3.20. *A semiring S is a quasi skew-ring if and only if S is a nil-extension of a skew-ring.*

Proof. First suppose that S is a quasi skew-ring. Then S has a subskew-ring R such that for every $b \in S$ there is a positive integer n such that $nb \in R$. We show that S is a nil-extension of R. For this we only show that R is a bi-ideal of S. Let $a \in R$ and $x \in S$. Since $x + a \in S$, there exists a positive integer m such that $m(x + a), (x + a) + e \in R$ where e is the zero element of the skew-ring R. Now $x + a = x + (a + e) = (x + a) + e \in R$. Similarly, we can show that $a + x \in R$. Again, there exists a positive integer k such that $kx, x + e \in R$. Then $a(x+e) \in R$. Also $xe = ex = e$. Now, $a(x+e) = ax+ae = ax+e = (a+e)x = ax$. Hence $ax \in R$. Similarly, $xa \in R$. Consequently, R is a bi-ideal of S and hence S is a nil-extension of R.

Conversely, we assume that a semiring S is nil-extension of a skew-ring R. Then clearly, S is additively quasi regular semiring with a unique additive idempotent. Hence by Theorem 3.10, it follows that S is quasi skew-ring. ∎

Corollary 3.21. *A semiring is a quasi skew-ring if and only if it is a completely Archimedean semiring with exactly one additive idempotent.*

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Received 23 April 2013
Revised 2 May 2013