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TRANSFORMATION SEMIGROUPS ASSOCIATED TO Γ-SEMIGROUPS

Dariush Heidari

Department of Mathematics, Najafabad Branch
Islamic Azad University
Najafabad, Isfahan, Iran
e-mail: dheidari82@gmail.com

and

Marzieh Amooshahi

Department of Mathematics, Najafabad Branch
Islamic Azad University
Najafabad, Isfahan, Iran
e-mail: mz.amooshahi@gmail.com

Abstract

The concept of Γ-semigroups is a generalization of semigroups. In this paper, we associate two transformation semigroups to a Γ-semigroup and we call them the left and right transformation semigroups. We prove some relationships between the ideals of a Γ-semigroup and the ideals of its left and right transformation semigroups. Finally, we study some relationships between Green's equivalence relations of a Γ-semigroup and its left (right) transformation semigroup.

Keywords: transformation semigroup, Γ-semigroup, Green's relations.

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1. Introduction

The full transformation semigroup on a set X is the semigroup T_X of all transformations on X (i.e., all mappings from X to itself) under the operation of composition of mappings. Any subsemigroup of T_X is called a transformation semigroup. Transformation semigroups are ubiquitous in semigroup theory because of Cayleys Theorem which states that every semigroup embeds in some transformation semigroup. The study of transformation semigroups has applications in fields as diverse as probability, automata theory, and discrete dynamical systems.

The notion of a Γ-semigroup was introduced by Sen in [10] and [11] that is a generalization of a semigroup. Many classical notions of semigroups have been extended to Γ-semigroups (see, for example, [4, 9, 12, 13] and [14]). Dutta and Adhikari have found operator semigroups of a Γ-semigroup to be a very effective tool in studying Γ-semigroups [5]. Recently, Davvaz et al. introduced the notion of Γ-semihypergroups as a generalization of semigroups, a generalization of a semihypergroups and a generalization of Γ-semigroups [1, 6, 7].

In this paper, we associate two transformation semigroups to a Γ-semigroup that are called the left and right transformation semigroups. Also, we prove some relationships between the ideals of a Γ-semigroup and the ideals of its left and right transformation semigroups. Also, some relationships between Green's equivalence relations of a Γ-semigroup and its left (right) transformation semigroup are studied.

2. Preliminaries

In this section, we recall some preliminary definitions of Γ-semigroups.

Definition [11]**.** Let $S = \{a, b, c, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two non-empty sets. Then, S is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \longrightarrow S$, written (a, γ, b) by $a\gamma b$, such that satisfies the following conditions:

(1) $a\alpha b \in S$ for all $a, b \in S$ and $\alpha \in \Gamma$;

(2) $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

A Γ-semigroup S is called commutative if $x\alpha y = y\alpha x$ for every $x, y \in S$ and $\alpha \in \Gamma$.

Let S be a Γ-semigroup. Then, (S, α) is a semigroup for every $\alpha \in \Gamma$ and we denote this semigroup by S_α.

Let A and B be two subsets of a Γ-semigroup S and $\Delta \subseteq \Gamma$. Then, $A\Delta B$ is defined as follows

$$A\Delta B = \{a\delta b \mid a \in A, b \in B \text{ and } \delta \in \Delta\}.$$

For simplicity, we write $a\Delta B$, $A\Delta b$ and $A\delta B$ instead of $\{a\}\Delta B$, $A\Delta\{b\}$ and $A\{\delta\}B$, respectively.

Let S be an arbitrary semigroup and Γ be a non-empty set. Define a mapping $S \times \Gamma \times S \longrightarrow S$ by $a\alpha b = ab$ for all $a, b \in S$ and $\alpha \in \Gamma$. It is easy to see that S is a Γ-semigroup. Thus, a semigroup can be considered as a Γ-semigroup.

In the following, some examples of Γ-semigroups are presented.

Example 1. Let $S = \{-i, 0, i\}$ and $\Gamma = S$. Then, S is a Γ-semigroup under the multiplication over complex numbers while S is not a semigroup under complex numbers multiplication.

Example 2. Let S be the set of all $m \times n$ matrices with entries from a field F and Γ be a set of $n \times m$ matrices with entries from F. Then, S is a Γ-semigroup with the usual product of matrices.

A non-empty subset T of a Γ-semigroup S is said to be a Γ-subsemigroup of S if $T\Gamma T \subseteq T$.

A non-empty subset I of a Γ-semigroup S is called a left (right) Γ-ideal, "ideal, for short" of S, if $S\Gamma I \subseteq I$ ($I\Gamma S \subseteq I$).

Definition [11]**.** Let S be a Γ-semigroup. A proper ideal P of S is called a prime ideal if for any two ideals I and J of S, $I\Gamma J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$.

Definition [5]**.** Let S be a Γ-semigroup. Define a relation ρ on $S \times \Gamma$ as follows:

$$(x, \alpha)\rho(y, \beta) \Longleftrightarrow x\alpha s = y\beta s, \forall s \in S.$$

Obviously ρ is an equivalence relation. Let $[x, \alpha]$ denotes the equivalence class containing (x, α). Let $L = \{[x, \alpha] \mid x \in S, \alpha \in \Gamma\}$. Then, L is a semigroup where $[x, \alpha][y, \beta] = [x\alpha y, \beta]$, for all $x, y \in S$ and $\alpha, \beta \in \Gamma$. The semigroup L is called the left operator semigroup of S. Similarly, right operator semigroup R of a Γ-semigroup S is defined as $R = \{[\alpha, x] \mid \alpha \in \Gamma, x \in S\}$, where $[\alpha, x][\beta, y] = [\alpha, x\beta y]$, for all $x, y \in S$ and $\alpha, \beta \in \Gamma$.

Let S be a Γ-semigroup, $a \in S$ and $\alpha, \beta \in \Gamma$. Then a is said to be α-idempotent if $a\alpha a = a$. Also, a is said to be (α, β)-regular if there exists $x \in S$ such that $a = a\alpha x\beta a$.

3. MAIN RESULTS

Throughout the paper we assume that S is a Γ-semigroup. In this section, we follow the notations of [3], namely, if $f \in T_X$, then we write

$$f = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$

where, $f(x_i) = y_i$ and i belongs to an index set I.

For every $x \in S$ and $\alpha \in \Gamma$ we associate two mappings $f_{x,\alpha} : S \longrightarrow S$ by $f_{x,\alpha}(s) = x\alpha s$ and $f_{\alpha,x} : S \longrightarrow S$ by $f_{\alpha,x}(s) = s\alpha x$.

Now, consider two subsets $T_r(S)$ and $T_l(S)$ of the transformation semigroup T_S defined by

$$T_l(S) = \{f_{x,\alpha} \mid x \in S, \alpha \in \Gamma\}$$

and

$$T_r(S) = \{f_{\alpha,x} \mid x \in S, \alpha \in \Gamma\}.$$

Theorem 3. *$T_l(S)$ and $T_r(S)$ are subsemigroups of the full transformation semigroup T_S.*

Proof. Suppose that $f_{x,\alpha}$ and $f_{y,\beta}$ are two elements of $T_l(S)$ such that $x, y \in S$ and $\alpha, \beta \in \Gamma$. Then, for every $s \in S$, we have

$$\begin{aligned}(f_{x,\alpha} \circ f_{y,\beta})(s) &= f_{x,\alpha}(f_{y,\beta}(s)) \\ &= f_{x,\alpha}(y\beta s) \\ &= x\alpha(y\beta s) \\ &= (x\alpha y)\beta s \\ &= f_{x\alpha y,\beta}(s).\end{aligned}$$

Thus, $f_{x,\alpha} \circ f_{y,\beta} = f_{x\alpha y,\beta} \in T_l(S)$. So, $T_l(S)$ is closed under composition. Therefore, $T_l(S)$ is a subsemigroup of T_S. Similarly, one can prove that $T_r(S)$ is a subsemigroup of T_S. ■

$T_l(S)$ and $T_r(S)$ are called the left and right transformation semigroups of S, respectively.

It is easy to see that if S is a commutative Γ-semigroup, then $T_l(S) = T_r(S)$.

Example 4. Let $S = \{a, b, c, d\}$ and $\Gamma = \{\alpha, \beta\}$. We define the operations α and β as the following tables

α	a	b	c	d
a	b	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

β	a	b	c	d
a	c	c	c	d
b	c	c	c	d
c	c	c	c	d
d	d	d	d	d

then, S is a commutative Γ-semigroup and we obtain the transformation semigroups of S as follows

$$T_l(S) = \left\{ \begin{pmatrix} a & b & c & d \\ b & b & c & d \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ c & c & c & d \end{pmatrix}, \begin{pmatrix} a & b & c & d \\ d & d & d & d \end{pmatrix} \right\} = T_r(S).$$

For every $A \subseteq S$, we define

$$T_l(A) = \{f_{x,\alpha} \in T_l(S) \mid x\alpha s \in A, \forall s \in S\},$$

$$T_r(A) = \{f_{\alpha,x} \in T_r(S) \mid s\alpha x \in A, \forall s \in S\}.$$

Also, for every $B \subseteq T_l(S)$ and $C \subseteq T_r(S)$ we define

$$S_l(B) = \{x \in S \mid f_{x,\alpha} \in B, \forall \alpha \in \Gamma\},$$

$$S_r(C) = \{x \in S \mid f_{\alpha,x} \in C, \forall \alpha \in \Gamma\}.$$

If there exist elements $e \in S$ and $\delta \in \Gamma$ such that $e\delta x = x$ for every $x \in S$, then S is said to have a left unity. Similarly, if there exist elements $e \in S$ and $\delta \in \Gamma$ such that $x\delta e = x$ for every $x \in S$, then S is said to have a right unity.

Remark 5. If S has a left unity, then $f_{e,\delta}$ is the identity mapping. Also, if S has a right unity, then $f_{\delta,e}$ is the identity mapping.

Lemma 6. *Let A and B be two subsets of S and C and D be two subsets of $T_l(S)$. Then*

(1) $T_l(A \cap B) = T_l(A) \cap T_l(B)$;

(2) $S_l(C \cap D) = S_l(C) \cap S_l(D)$;

(3) $S_l(C)\Gamma S_l(D) \subseteq S_l(C \circ D)$;

(4) *If S has a left unity, then* $T_l(A) \circ T_l(B) \subseteq T_l(A\Gamma B)$.

Proof. The proofs of (1) and (2) are straightforward.

(3) Suppose that C and D are two subsets of $T_l(S)$ and $x \in S_l(C)\Gamma S_l(D)$. Then, there exist $y \in S_l(C)$ and $z \in S_l(D)$ and $\alpha \in \Gamma$ such that $x = y\alpha z$. So, $f_{y,\alpha} \in C$ and for every $\beta \in \Gamma$, $f_{z,\beta} \in D$. Thus, $f_{x,\beta} = f_{y,\alpha} \circ f_{z,\beta} \in C \circ D$ which implies that $x \in S_l(C \circ D)$. Therefore, $S_l(C)\Gamma S_L(D) \subseteq S_l(C \circ D)$.

(4) Suppose that $f_{e,\delta} = id_S$, for some $e \in S$ and $\delta \in \Gamma$. Let A and B be two subsets of S. If $f_{x,\alpha} \in T_l(A) \circ T_l(B)$, then there exist $y, z \in S$ and $\beta, \gamma \in \Gamma$ such that

$$f_{x,\alpha} = f_{y,\beta} \circ f_{z,\gamma} = f_{y\beta z,\gamma}$$

where, $f_{y,\beta} \in T_l(A)$ and $f_{z,\gamma} \in T_l(B)$. So, $y\beta S \subseteq A$ and $z\gamma S \subseteq B$, thus we have

$$\begin{aligned} x\alpha s &= f_{x,\alpha}(s) \\ &= f_{y\beta z,\gamma}(s) \\ &= (y\beta z)\gamma s \\ &= y\beta(z\gamma s) \\ &= y\beta(e\delta(z\gamma s)) \\ &= (y\beta e)\delta(z\gamma s) \in A\Gamma B. \end{aligned}$$

So, $f_{x,\alpha} \in T_l(A\Gamma B)$. Therefore, $T_l(A) \circ T_l(B) \subseteq T_l(A\Gamma B)$. ■

The following examples show that in the parts (3) and (4) of the previous lemma, the equality does not necessarily hold.

Example 7. In Example 3, let $C = \left\{ \begin{pmatrix} a & b & c & d \\ b & b & c & d \end{pmatrix} \right\}$ and $D = \left\{ \begin{pmatrix} a & b & c & d \\ d & d & d & d \end{pmatrix} \right\}$. Then, $S_l(C) = \emptyset$, $S_l(D) = \{d\}$ and $S_l(C \circ D) = \{d\}$. So, we have

$$S_l(C)\Gamma S_l(D) = \emptyset \subsetneq \{d\} = S_l(C \circ D).$$

Example 8. Let S be a closed interval $[0, 1]$ and $\Gamma = \{1, 2, 3\}$. For every $x, y \in S$ and $\gamma \in \Gamma$, we define $x\gamma y = \frac{xy}{\gamma}$. Then, for every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, we have

$$(x\alpha y)\beta z = \frac{xyz}{\alpha\beta} = x\alpha(y\beta z).$$

This means that S is a Γ-semigroup such that $f_{1,1} = id_S$.

Now, consider $A = \{0\}$ and $B = \{1\}$ as two subsets of $S = [0, 1]$. Then, $T_l(A) = \{0\}$, $T_l(B) = \emptyset$ and we have

$$T_l(A) \circ T_l(B) = \emptyset \subsetneq \{0\} = T_l(A\Gamma B).$$

Lemma 9. *The following assertions hold*

(1) *If A is an ideal of S, then $T_l(A)$ is an ideal of $T_l(S)$ and $A \subseteq S_l(T_l(A))$. Moreover, if S has a right unity, then $A = S_l(T_l(A))$;*

(2) *If B is an ideal of $T_l(S)$, then $S_l(B)$ is an ideal of S and $B \subseteq T_l(S_l(B))$. Moreover, if S has a left unity, then $B = T_l(S_l(B))$;*

(3) *If S has a right unity and P is a prime ideal of S, then $T_l(P)$ is a prime ideal of $T_l(S)$;*

(4) *If S has a left unity and Q is a prime ideal of $T_l(S)$, then $S_l(Q)$ is a prime ideal of S.*

Proof. (1) Suppose that A is an ideal of S and $f_{a,\alpha} \in T_l(A)$. Then, for every $x, s \in S$ and $\beta \in \Gamma$, we have $a\alpha(x\beta s) \in A$ so $f_{a,\alpha} \circ f_{x,\beta} = f_{a\alpha x,\beta} \in T_l(A)$. Similarly, we can prove that $f_{x,\beta} \circ f_{a,\alpha} = f_{x\beta a,\alpha} \in T_l(A)$. Therefore, $T_l(A)$ is an ideal of $T_l(S)$.

Now, let $x \in A$. Then, for every $s \in S$ and $\alpha \in \Gamma$ we have $x\alpha s \in A$. So, $f_{x,\alpha} \in T_l(A)$, thus $x \in S_l(T_l(A))$, therefore $A \subseteq S_l(T_l(A))$.

Suppose that S has a right unity so there exist $e \in S$ and $\delta \in \Gamma$ such that $s\delta e = s$, for every $s \in S$. If $x \in S_l(T_l(A))$, then for every $\alpha \in \Gamma$ we have $f_{x,\alpha} \in T_l(A)$. So, $x = x\delta e \in x\Gamma S \subseteq A$, thus $S_l(T_l(A)) \subseteq A$, therefore $A = S_l(T_l(A))$.

(2) Suppose that B is an ideal of $T_l(S)$ and $x \in S_l(B)$. Then, $f_{x,\alpha} \in B$ for every $\alpha \in \Gamma$. So, for all $s \in S$ and $\beta \in \Gamma$ we have $f_{x\alpha s,\beta} = f_{x,\alpha} \circ f_{s,\beta} \in B$, thus $x\alpha s \in S_l(B)$. Similarly, we have $s\alpha x \in S_l(B)$. Therefore, $S_l(B)$ is an ideal of S.

Now, let $f_{x,\alpha} \in B$. Then, for every $s \in S$ and $\beta \in \Gamma$ we have $f_{x\alpha s,\beta} = f_{x,\alpha} \circ f_{s,\beta} \in B$, since B is an ideal of $T_l(S)$. So, $x\alpha s \in S_l(B)$, thus $f_{x,\alpha} \in T_l(S_l(B))$, therefore $B \subseteq T_l(S_l(B))$.

Suppose that S has a left unity, then there exist $e \in S$ and $\delta \in \Gamma$ such that $e\delta s = s$, for every $s \in S$. If $f_{x,\alpha} \in T_l(S_l(B))$, then for every $s \in S$ we have $x\alpha s \in S_l(B)$. So, $f_{x,\alpha} = f_{x,\alpha} \circ f_{e,\delta} = f_{x\alpha e,\delta} \in B$.

(3) Suppose that P is a prime ideal of S and C and D are ideals of $T_l(S)$ such that $C \circ D \subseteq T_l(P)$. So, by part (3) of Lemma 6 and part (1), we have $S_l(C)\Gamma S_l(D) \subseteq S_l(T_l(P)) = P$. Now, since P is prime we conclude that $S_l(C) \subseteq P$ or $S_l(D) \subseteq P$. So, by part (1), we have $C \subseteq T_l(P)$ or $D \subseteq T_l(P)$. Therefore, $T_l(P)$ is a prime ideal of $T_l(S)$.

(4) Suppose that Q is a prime ideal of $T_l(S)$ and I and J are ideals of S such that $I\Gamma J \subseteq S_l(Q)$. So, by part (4) of Lemma 6 and part (2), we have $T_l(I) \circ T_l(J) \subseteq T_l(S_l(Q)) = Q$. Now, since Q is prime we conclude that $T_l(I) \subseteq Q$ or $T_l(J) \subseteq Q$. So, by part (2), we have $I \subseteq S_l(Q)$ or $J \subseteq S_l(Q)$. Therefore, $S_l(Q)$ is a prime ideal of S. ■

Theorem 10. *There is an inclusion preserving bijection between the set of all ideals of S and the set of all ideals of $T_l(S)$.*

Proof. Suppose that $\mathcal{I}(S)$ and $\mathcal{I}(T_l(S))$ are the sets of all ideals of S and $T_l(S)$, respectively. Then, it is easy to see that the mapping

$$T_l : \mathcal{I}(S) \longrightarrow \mathcal{I}(T_l(S))$$
$$I \longmapsto T_l(I)$$

is well-defined and is an inclusion preserving mapping. It remains to show that T_l is monotone. Since S has a left unity, there exist $e \in S$ and $\delta \in \Gamma$ such that $e\delta s = s$, for every $s \in S$. Let I and J be two ideals of S such that $T_l(I) = T_l(J)$. Then for every $a \in S$ we have $a \in I$ if and only if $f_{a,\delta} \in T_l(I)$ if and only if $f_{a,\delta} \in T_l(J)$ if and only if $a \in J$. Hence $I = J$ and so T_l is monotone. ■

In the following theorem, we prove that two semigroups associated to a Γ-semigroup in Definition 2 and Theorem 3 are isomorphic.

Theorem 11. *The left transformation semigroup and the left operator semigroup of S are isomorphic.*

Proof. Suppose that $T_l(S)$ is the left transformation semigroup and L is the left operator semigroup of S. Then, the mapping

$$\varphi : T_l(S) \longrightarrow L$$
$$\varphi(f_{x,\alpha}) = [x, \alpha]$$

is well-defined, as if $f_{x,\alpha} = f_{y,\beta}$ for some $x, y \in S$ and $\alpha, \beta \in \Gamma$, then for every $s \in S$ we have $x\alpha s = f_{x,\alpha}(s) = f_{y,\beta}(s) = y\beta s$ and so $[x, \alpha] = [y, \beta]$. Also, for every $x, y \in S$ and $\alpha, \beta \in \Gamma$ we have

$$\begin{aligned} \varphi(f_{x,\alpha} \circ f_{y,\beta}) &= \varphi(f_{x\alpha y,\beta}) \\ &= [x\alpha y, \beta] \\ &= [x, \alpha][y, \beta] \\ &= \varphi(f_{x,\alpha})\varphi(f_{y,\beta}). \end{aligned}$$

So φ is a homomorphism. Obviously, φ is one-to-one and onto. Therefore, φ is an isomorphism and $T_l(S) \cong L$. ■

Green's equivalence relations for semigroups, first studied by Green [8], have played a fundamental role in the development of semigroup theory (for more details see [3]). Chinram and Siammai have studied Green's relations for Γ-semigroups [2].

Definition. Let A be a semigroup and $a, b \in A$. Then the Green's equivalence relations $\mathcal{L}$ and $\mathcal{R}$ are defined by the following rules:

(1) $(a, b) \in \mathcal{L}$ if and only if $< a >_l = < b >_l$;

(2) $(a, b) \in \mathcal{R}$ if and only if $< a >_r = < b >_r$.

Green's relations of a Γ-semigroup are defined similarly. Let $a \in S$. Then

$$< a >_l = S\Gamma a \cup \{a\} \quad \text{and} \quad < a >_r = a\Gamma S \cup \{a\}.$$

Thus $(a, b) \in \mathcal{L}$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = x\alpha b$ and $b = y\beta a$. Similarly, $(a, b) \in \mathcal{R}$ if and only if $a = b$ or there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $a = b\alpha x$ and $b = a\beta y$.

In the following, we study some relationships between Green's relations of a Γ-semigroup and its left (right) transformation semigroup. The notations $\mathcal{L}_S$ and $\mathcal{R}_S$ are used for Green's relations of a Γ-semigroup S. Similarly, $\mathcal{L}_{T_l}$ and $\mathcal{R}_{T_l}$ are used for Green's relations of the left transformation semigroup of S. Also, $\mathcal{L}_{T_r}$ and $\mathcal{R}_{T_r}$ are used for Green's relations of the right transformation semigroup of S.

Lemma 12. *Let x be an element of S and α and β be two elements of Γ. Then*

(1) *If x is α-idempotent, then $f_{x,\alpha}$ is idempotent;*

(2) *If x is (α, β)-regular, then $f_{x,\alpha}$ is regular.*

(3) *If x is α-idempotent, then $f_{\alpha,x}$ is idempotent.*

(4) *If x is (α, β)-regular, then $f_{\beta,x}$ is regular.*

Proof. (1) Since x is α-idempotent, $f_{x,\alpha} \circ f_{x,\alpha} = f_{x\alpha x,\alpha} = f_{x,\alpha}$. So $f_{x,\alpha}$ is idempotent.

(2) Since x is (α, β)-regular, there exists $y \in S$ such that $x = x\alpha y\beta x$. So $f_{x,\alpha} \circ f_{y,\beta} \circ f_{x,\alpha} = f_{x\alpha y\beta x,\alpha} = f_{x,\alpha}$ and thus $f_{x,\alpha}$ is regular.

(3) The proof is similar to (1).

(4) The proof is similar to (2). ∎

Theorem 13. *Let a and b be two elements of S. Then*

(1) *If $(a, b) \in \mathcal{R}_S$, then $(f_{\alpha,a}, f_{\alpha,b}) \in \mathcal{L}_{T_l}$, for every $\alpha \in \Gamma$;*

(2) *If $(a, b) \in \mathcal{L}_S$, then $(f_{a,\alpha}, f_{b,\alpha}) \in \mathcal{L}_{T_l}$, for every $\alpha \in \Gamma$;*

(3) *Let a be α-idempotent and b be β-idempotent for some $\alpha, \beta \in \Gamma$ such that $(f_{a,\alpha}, f_{b,\beta}) \in \mathcal{R}_{T_l}$. Then $(a, b) \in \mathcal{R}_S$.*

Proof. (1) If $(a, b) \in \mathcal{R}_S$, then there exist $x, y \in S$ and $\gamma, \lambda \in \Gamma$ such that $a = b\gamma x$ and $b = a\lambda y$. Thus for every $s \in S$, we have $s\alpha a = s\alpha b\gamma x$ and $s\alpha b = s\alpha a\lambda y$ which implies that $f_{\alpha,a} = f_{\gamma,x} \circ f_{\alpha,b}$ and $f_{\alpha,b} = f_{\lambda,y} \circ f_{\alpha,a}$. Therefore, $(f_{\alpha,a}, f_{\alpha,b}) \in \mathcal{R}_{T_l}$.

(2) If $(a, b) \in \mathcal{L}_S$, then there exist $x, y \in S$ and $\gamma, \lambda \in \Gamma$ such that $a = x\gamma b$ and $b = y\lambda a$. Thus for every $s \in S$, we have $a\alpha s = x\gamma b\alpha s$ and $b\alpha s = y\lambda a\alpha s$ which implies that $f_{a,\alpha} = f_{x,\gamma} \circ f_{b,\alpha}$ and $f_{b,\alpha} = f_{y,\lambda} \circ f_{a,\alpha}$. Therefore, $(f_{a,\alpha}, f_{b,\alpha}) \in \mathcal{L}_{T_l}$.

(3) Suppose that $(f_{a,\alpha}, f_{b,\beta}) \in \mathcal{R}_{T_l}$. Then there exist $x, y \in S$ and $\gamma, \lambda \in \Gamma$ such that $f_{a,\alpha} = f_{b,\beta} \circ f_{x,\gamma} = f_{b\beta x,\gamma}$ and $f_{b,\beta} = f_{a,\alpha} \circ f_{y,\lambda} = f_{a\alpha y,\lambda}$. Thus

$$a = a\alpha a = f_{a,\alpha}(a) = f_{b\beta x,\gamma}(a) = b\beta(x\gamma a) \in < b >_r$$

and

$$b = b\beta b = f_{b,\beta}(b) = f_{a\alpha y,\lambda}(b) = a\alpha(y\lambda b) \in< a >_r .$$

Therefore, $(a, b) \in \mathcal{R}_S$. ∎

Let ρ be an equivalence relation on a set A and $a \in A$. Then the ρ-class containing a is denoted by $[\rho_A]_a$.

Theorem 14. *Let α be an element of Γ and x be an α-idempotent element of S. Then for every $y \in S$ the following assertions hold*

(1) $y \in [\mathcal{L}_S]_x$ *if and only if* $f_{y,\alpha} \in [\mathcal{L}_{T_l}]_{f_{x,\alpha}}$;

(2) $y \in [\mathcal{R}_S]_x$ *if and only if* $f_{\alpha,y} \in [\mathcal{L}_{T_r}]_{f_{\alpha,x}}$.

Proof. (1) $y \in [\mathcal{L}_S]_x$ if and only if there exist $z, t \in S$ and $\gamma, \lambda \in \Gamma$ such that $x = z\gamma y$ and $y = t\lambda x$ if and only if for every $s \in S$,

$$f_{x,\alpha}(s) = x\alpha s = (z\gamma y)\alpha s = f_{z,\gamma} \circ f_{y,\alpha}(s)$$

and

$$f_{y,\alpha}(s) = y\alpha s = (t\lambda x)\alpha s = f_{t,\lambda} \circ f_{x,\alpha}(s)$$

if and only if

$$f_{x,\alpha} = f_{z,\gamma} \circ f_{y,\alpha} \in< f_{y,\alpha} >_l$$

and

$$f_{y,\alpha} = f_{t,\lambda} \circ f_{x,\alpha} \in< f_{x,\alpha} >_l$$

if and only if $f_{y,\alpha} \in [\mathcal{L}_{T_l}]_{f_{x,\alpha}}$.

(2) $y \in [\mathcal{R}_S]_x$ if and only if there exist $z, t \in S$ and $\gamma, \lambda \in \Gamma$ such that $x = y\gamma z$ and $y = x\lambda t$ if and only if for every $s \in S$,

$$f_{\alpha,x}(s) = s\alpha x = s\alpha(y\gamma z) = f_{\gamma,z} \circ f_{\alpha,y}(s)$$

and

$$f_{\alpha,y}(s) = s\alpha y = s\alpha(x\lambda t) = f_{\lambda,t} \circ f_{\alpha,x}(s)$$

if and only if

$$f_{\alpha,x} = f_{\gamma,z} \circ f_{\alpha,y} \in< f_{\alpha,y} >_l$$

and

$$f_{\alpha,y} = f_{\lambda,t} \circ f_{\alpha,x} \in< f_{\alpha,x} >_l$$

if and only if $f_{\alpha,y} \in [\mathcal{L}_{T_r}]_{f_{\alpha,x}}$. ∎

It is necessary to note that all facts that mentioned and proved for the left transformation semigroup of a Γ-semigroup are hold for the right transformation semigroup of a Γ-semigroup and proved in similar ways.

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