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ALL COMPLETELY REGULAR ELEMENTS IN $Hyp_G(n)$

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Abstract

In Universal Algebra, identities are used to classify algebras into collections, called varieties and hyperidentities are use to classify varieties into collections, called hypervarities. The concept of a hypersubstitution is a tool to study hyperidentities and hypervarieties.

Generalized hypersubstitutions and strong identities generalize the concepts of a hypersubstitution and of a hyperidentity, respectively. The set of all generalized hypersubstitutions forms a monoid. In this paper, we determine the set of all completely regular elements of this monoid of type $\tau = (n)$.

Keywords: generalized hypersubstitution, regular element, completely regular element.

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1. Introduction

Hyperidentities and hypervarieties of a given type τ without nullary operations were first introduced by J. Aczèl [1], V.D. Belousov [2], W.D. Neumann [9] and W. Taylor [10]. The main tool used to study hyperidentities and hypervarieties is the concept of a hypersubstitution. The notion of a hypersubstitution originated by K. Denecke, D. Lau, R. Pöschel and D. Schweigert [4]. In 2000, S. Leeratanavalee and K. Denecke generalized the concepts of a hypersubstitution and a hyperidentity to the concepts of a generalized hypersubstitution and a strong hyperidentity, respectively [8]. The set of all generalized hypersubstitutions together with a binary operation and the identity hypersubstitution forms a monoid. There are several published papers on algebraic properties of this monoid and its submonoids. The present paper will determine the set of all completely regular elements of this monoid of type $\tau = (n)$.

2. Monoid of all generalized hypersubstitutions

In this section, we give the concept of the monoid of all generalized hypersubstitutions.

Let $X := \{x_1, x_2, x_3, \ldots\}$ be a countably infinite set of symbols called *variables*. We often refer to these variables as *letters*, to X as an *alphabet*, and also refer to the set $X_n := \{x_1, x_2, x_3, \ldots, x_n\}$ as an *n-element alphabet*. Let $(f_i)_{i\in I}$ be an indexed set which is disjoint from X. Each f_i is called *n_i-ary operation symbol*, where $n_i \geq 1$ is a natural number. Let τ be a function which assigns to every f_i the number n_i as its *arity*. The function τ, on the values of τ written as $(n_i)_{i\in I}$ is called a *type*. An *n-ary term of type τ*, for simply an n-ary term, is defined inductively as follows:

(i) The variables $x_1, x_2, \ldots, x_n$ are n-ary terms.

(ii) If $t_1, t_2, \ldots, t_{n_i}$ are n-ary terms then $f_i(t_1, t_2, \ldots, t_{n_i})$ is an n-ary term.

The smallest set, which contains $x_1, x_2, \ldots, x_n$ and is closed under finite application of (ii), is denoted by $W_\tau(X_n)$. It is clear that every n-ary term is also an m-ary term for all $m \geq n$. Let $W_\tau(X) := \bigcup_{n=1}^{\infty} W_\tau(X_n)$. It is called the set of all terms of type τ.

Example 1. Let $\tau = (2, 3)$. That means we have one binary operation symbol and one ternary operation symbol, say f and g respectively. These are some examples of ternary terms of type (2, 3): $x_1, x_2, x_3, f(x_3, g(x_1, x_3, x_3)), g(f(x_2, x_3), x_1, g(x_3, x_1, x_2))$.

A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i | \, i \in I\} \to W_\tau(X)$ which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. To define a binary operation on this set, we need the concept of generalized superposition of terms $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ which is defined by the following steps:

(i) If $t = x_j$, $1 \leq j \leq m$, then $S^m(t, t_1, \ldots, t_m) = S^m(x_j, t_1, \ldots, t_m) := t_j$.

(ii) If $t = x_j$, $m < j \in \mathbb{N}$, then $S^m(t, t_1, \ldots, t_m) = S^m(x_j, t_1, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, s_2, \ldots, s_{n_i})$, then $S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

Every generalized hypersubstitution σ can be extended to a mapping $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ defined as follows:

(i) $\widehat{\sigma}[x] := x \in X$,

(ii) $\widehat{\sigma}[f_i(t_1, t_2, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i and supposed that $\widehat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

Example 2. Let $\tau = (2)$, i.e., there is only one binary operation symbol, say f. Let $\sigma \in Hyp_G(2)$ where $\sigma(f) = f(x_2, f(x_3, x_2))$. Then

$$\begin{aligned}
\widehat{\sigma}[f(f(x_1, x_5), x_3)] &= S^2(\sigma(f), \widehat{\sigma}[f(x_1, x_5)], \widehat{\sigma}[x_3]) \\
&= S^2(f(x_2, f(x_3, x_2)), S^2(\sigma(f), \widehat{\sigma}[x_1], \widehat{\sigma}[x_5]), x_3) \\
&= S^2(f(x_2, f(x_3, x_2)), S^2(f(x_2, f(x_3, x_2)), x_1, x_5), x_3) \\
&= S^2(f(x_2, f(x_3, x_2)), f(x_5, f(x_3, x_5)), x_3) \\
&= f(x_3, f(x_3, x_3)).
\end{aligned}$$

We define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings. Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, x_2, \ldots, x_{n_i})$. In [8], S. Leeratanavalee and K. Denecke proved that:

For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have

(i) $S^n(\widehat{\sigma}[t], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]) = \widehat{\sigma}[S^n(t, t_1, \ldots, t_n)]$,

(ii) $(\widehat{\sigma}_1 \circ \sigma_2)\widehat{} = \widehat{\sigma}_1 \circ \widehat{\sigma}_2$.

Then $\underline{Hyp_G(\tau)} = (Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid and the set of all hypersubstitutions of type τ forms a submonoid of $\underline{Hyp_G(\tau)}$.

3. All completely regular elements in $Hyp_G(n)$

To determine the set of all completely regular elements of $\underline{Hyp_G(n)}$, we first introduce some notations which will be used throughout this paper.

For a type $\tau = (n)$ with n-ary operation symbol f and $t \in W_{(n)}(X)$, we denote

$\sigma_t :=$ the generalized hypersubstitution σ of type $\tau = (n)$ which maps f to the term t,

$var(t) :=$ the set of all variables occurring in the term t.

A subterm of t is defined inductively by the following steps.

(i) Every variable $x \in var(t)$ is a subterm of t.

(ii) If $t = f(t_1, \ldots, t_n)$ then $t_1, \ldots, t_n$ and t itself are subterms of t.

We denote the set of all subterms of t by $sub(t)$.

Lemma 3. *Let $\sigma_s, \sigma_t \in Hyp_G(n)$ where $t = f(t_1, \ldots, t_n)$ such that $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m$ and for distinct $j_1, \ldots, j_m \in \{1, \ldots, n\}$ and $var(t) \cap X_n = \{x_{j_1}, \ldots, x_{j_m}\}$. Then $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$ if and only if $s = f(s_1, \ldots, s_n)$ where $s_{j_l} = x_{i_l}$ for all $l \in \{1, \ldots, m\}$.*

Proof. Assume that $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$ and let $s = f(s_1, \ldots, s_n)$. Suppose that, there exists s_{j_l} such that $s_{j_l} \in W_n(X) \setminus \{x_{i_l}\}$ for some $l \in \{1, \ldots, m\}$. Then

$$\begin{aligned}
(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) &= \hat{\sigma_t}[\hat{\sigma_s}[t]] \\
&= \hat{\sigma_t}[S^n(f(s_1, \ldots, s_n), \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n])] \\
&= \hat{\sigma_t}[f(w_1, \ldots, w_n)] \text{ (where } w_i = S^n(s_i, \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n]) \\
&\quad \text{for all } i \in \{1, \ldots, n\}) \\
&= S^n(f(t_1, \ldots, t_n), \hat{\sigma_t}[w_1], \ldots, \hat{\sigma_t}[w_n]) \\
&= f(u_1, \ldots, u_n) \text{ (where } u_i = S^n(t_i, \hat{\sigma_t}[w_1], \ldots, \hat{\sigma_t}[w_n]) \\
&\quad \text{for all } i \in \{1, \ldots, n\}).
\end{aligned}$$

Since $t_{i_l} = x_{j_l}$ for all $l \in \{1, \ldots, m\}$, thus $u_{i_l} = S^n(t_{i_l}, \hat{\sigma_t}[w_1], \ldots, \hat{\sigma_t}[w_n]) = \hat{\sigma_t}[w_{j_l}]$. Since $w_{j_l} = S^n(s_{j_l}, \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n])$ and $s_{j_l} \neq x_{i_l}$, $w_{j_l} \neq \hat{\sigma_s}[t_{i_l}] = x_{j_l}$, we get $u_{i_l} = \hat{\sigma_t}[w_{j_l}] \neq x_{j_l}$, and then $f(u_1, \ldots, u_n) \neq t$. This is a contradiction. Hence $s_{j_l} = x_{i_l}$ for all $l \in \{1, \ldots, m\}$.

Conversely, let $s = f(s_1, \ldots, s_n)$ where $s_{j_l} = x_{i_l}$ for all $l \in \{1, \ldots, m\}$. Then $(\sigma_t \circ_G \sigma_s \circ_G \sigma_t)(f) = \hat{\sigma_t}[f(w_1, \ldots, w_n)]$ where $w_i = S^n(s_i, \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n])$ for all $i \in \{1, \ldots, n\}$. Since $s_{j_l} = x_{i_l}$ for all $l \in \{1, \ldots, m\}$, $w_{i_l} = S^n(x_{i_l}, \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n]) = \hat{\sigma_s}[t_{i_l}] = x_{j_l}$, we get

$$\hat{\sigma_t}[f(w_1, \ldots, w_n)] = S^n(f(t_1, \ldots, t_n), \hat{\sigma_t}[w_1], \ldots, \hat{\sigma_t}[w_n]) = f(t_1, \ldots, t_n) = t.$$

Hence $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$. ■

Definition [6]**.** An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$.

Let $\sigma_t \in Hyp_G(n)$, we denote

$R_1 := \{\sigma_{x_i} | x_i \in X\}$;

$R_2 := \{\sigma_t | var(t) \cap X_n = \emptyset\}$;

$R_3 := \{\sigma_t | t = f(t_1, \ldots, t_n)$ where $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m$ and for distinct $j_1, \ldots, j_m \in \{1, \ldots, n\}$ and $var(t) \cap X_n = \{x_{j_1}, \ldots, x_{j_m}\}\}$.

In 2010, W. Puninagool and S. Leeratanavalee [5] showed that $\bigcup_{i=1}^{3} R_i$ is the set of all regular elements in $Hyp_G(n)$.

Definition [6]**.** For any monoid S, an element $u \in S$ is called *unit* if there exists $u^{-1} \in S$ such that $uu^{-1} = e = u^{-1}u$ where e is the identity element of S, and $U(S)$ denote the set of all unit elements of S.

Let S_n be the set of all permutations of $\{1, 2, \ldots, n\}$. In 2013, A. Boonmee and S. Leeratanavalee [3] showed that $U(Hyp_G(n)) := \{\sigma_t \in Hyp_G(n) | t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ where $\pi \in S_n\}$ is the set of all unit elements in $Hyp_G(n)$. And we see that $U(Hyp_G(n)) \subset R_3$.

Definition [6]**.** An element e of a semigroup S is called *idempotent* if $e^2 = ee = e$, and we denote the set of all idempotents in S by $E(S)$.

Let $\sigma_t \in Hyp_G(n)$, we denote $E := \{\sigma_t | t = f(t_1, \ldots, t_n)$ where $t_{i_1} = x_{i_1}, \ldots, t_{i_m} = x_{i_m}$ for some $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and $var(t) \cap X_n = \{x_{i_1}, \ldots, x_{i_m}\}\} \subset R_3$.

In 2010, W. Puninagool and S. Leeratanavalee [5] showed that $E(Hyp_G(n)) = (R_1 \cup R_2 \cup E)$ is the set of all idempotent elements in $Hyp_G(n)$.

Definition [7]**.** An element a of a semigroup S is called *completely regular element* if there exists $b \in S$ such that $a = aba$ and $ab = ba$.

Proposition 4. *For each $\sigma_t \in U(Hyp_G(n))$, σ_t is a completely regular element in $Hyp_G(n)$.*

Proof. Let $\sigma_t \in U(Hyp_G(n))$. Then there exists $\sigma_{t^{-1}} \in U(Hyp_G(n)) \subseteq Hyp_G(n)$ such that $\sigma_t \circ_G \sigma_{t^{-1}} = \sigma_{id} = \sigma_{t^{-1}} \circ_G \sigma_t$ and $\sigma_t \circ_G \sigma_{t^{-1}} \circ_G \sigma_t = \sigma_t$. ■

Proposition 5. *For each $\sigma_t \in E(Hyp_G(n))$, σ_t is completely regular element in $Hyp_G(n)$.*

Proof. The proof is obvious. ■

Let $\sigma_t \in Hyp_G(n)$, we denote $CR(R_3) := \{\sigma_t | t = f(t_1, \ldots, t_n)$ where $t_{i_1} = x_{\pi(i_1)}, \ldots, t_{i_m} = x_{\pi(i_m)}$ and π is a bijective map on $\{i_1, \ldots, i_m\}$ for some $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and $var(t) \cap X_n = \{x_{\pi(i_1)}, \ldots, x_{\pi(i_m)}\}\}$.

Then we have $(E \cup U(Hyp_G(n))) \subseteq CR(R_3) \subset R_3$.

Proposition 6. *For each $\sigma_t \in CR(R_3)$, σ_t is completely regular element in $Hyp_G(n)$.*

Proof. Let $\sigma_t \in CR(R_3)$. Then $t = f(t_1, \ldots, t_n)$ where $t_{i_1} = x_{\pi(i_1)}, \ldots, t_{i_m} = x_{\pi(i_m)}$ and π is a bijective map on $\{i_1, \ldots, i_m\}$ for some $i_1, \ldots, i_m \in \{1, \ldots, n\}$ and $var(t) \cap X_n = \{x_{\pi(i_1)}, \ldots, x_{\pi(i_m)}\}$. Let $s \in W_{(n)}(X)$ where $s = f(s_1, \ldots, s_n)$ such that $s_{\pi(i_1)} = x_{i_1}, \ldots, s_{\pi(i_m)} = x_{i_m}$. Let $t_k \in sub(t_j)$ and $s_k \in sub(s_j)$ for all $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$ and $k \in \{1, \ldots, n\}$. If $var(t_k) \cap X_n = \emptyset$ then we choose $s_k = t_k$. And, if $t_k = x_{\pi(i_l)}$ and $\pi(i_p) = i_l$ for some $i_p, i_l \in \{i_1, \ldots, i_m\}$ we choose $s_k = x_{i_p}$. By Lemma 3, we have $\sigma_t \circ_G \sigma_s \circ_G \sigma_t = \sigma_t$. Next, we will show that $\sigma_t \circ_G \sigma_s = \sigma_s \circ_G \sigma_t$. Consider

$$(\sigma_t \circ_G \sigma_s)(f) = S^n(f(t_1, \ldots, t_n), \hat{\sigma_t}[s_1], \ldots, \hat{\sigma_t}[s_n]) = f(w_1, \ldots, w_n)$$

where $w_i = S^n(t_i, \hat{\sigma_t}[s_1], \ldots, \hat{\sigma_t}[s_n])$ for all $i \in \{1, \ldots, n\}$. And consider

$$(\sigma_s \circ_G \sigma_t)(f) = S^n(f(s_1, \ldots, s_n), \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n]) = f(u_1, \ldots, u_n)$$

where $u_i = S^n(s_i, \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n])$ for all $i \in \{1, \ldots, n\}$.

Case 1. $i_l \in \{i_1, \ldots, i_m\}$.
Since π is a bijective map on $\{i_1, \ldots, i_m\}$, there exists $i_p \in \{i_1, \ldots, i_m\}$ such that $\pi(i_p) = i_l$. Then

$$u_{i_l} = S^n(s_{i_l}, \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n]) = S^n(x_{i_p}, \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n]) = \hat{\sigma_s}[t_{i_p}] = x_{\pi(i_p)} = x_{i_l}$$

and

$$w_{i_l} = S^n(t_{i_l}, \hat{\sigma_t}[s_1], \ldots, \hat{\sigma_t}[s_n]) = S^n(x_{\pi(i_l)}, \hat{\sigma_t}[s_1], \ldots, \hat{\sigma_t}[s_n]) = \hat{\sigma_t}[s_{\pi(i_l)}] = x_{i_l}.$$

So $u_{i_l} = w_{i_l}$ for all $l \in \{1, \ldots, m\}$.

Case 2. $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$.
Let $t_k \in sub(t_j)$ and $s_k \in sub(s_j)$ for all $k \in \{1, \ldots, n\}$. Then $w_j = S^n(t_j, \hat{\sigma_t}[s_1], \ldots, \hat{\sigma_t}[s_n])$ and $u_j = S^n(s_j, \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n])$. We put $w'_k = S^n(t_k, \hat{\sigma_t}[s_1], \ldots, \hat{\sigma_t}[s_n])$ and $u'_k = S^n(s_k, \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n])$ for all $k \in \{1, \ldots, n\}$. If $var(t_k) \cap X_n = \emptyset$, then $w'_k = t_k$ and $u'_k = s_k = t_k$. If $t_k = x_{\pi(i_l)}$ and $\pi(i_p) = i_l$, then

$$w'_k = S^n(t_k, \hat{\sigma_t}[s_1], \ldots, \hat{\sigma_t}[s_n]) = S^n(x_{\pi(i_l)}, \hat{\sigma_t}[s_1], \ldots, \hat{\sigma_t}[s_n]) = \hat{\sigma_t}[s_{\pi(i_l)}] = x_{i_l}$$

and

$$u'_k = S^n(s_k, \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n]) = S^n(x_{i_p}, \hat{\sigma_s}[t_1], \ldots, \hat{\sigma_s}[t_n]) = \hat{\sigma_s}[t_{i_p}] = x_{\pi(i_p)} = x_{i_l}.$$

So $w_j = u_j$ for all $j \in \{1, \ldots, n\} \setminus \{i_1, \ldots, i_m\}$.

Hence $f(w_1, \ldots, w_n) = f(u_1, \ldots, u_n)$, so $\sigma_t \circ_G \sigma_s = \sigma_s \circ_G \sigma_t$. Therefore σ_t is completely regular element in $Hyp_G(n)$. ■

Lemma 7. *Let $t = f(t_1, \ldots, t_n)$ where $t_{i_1} = x_{j_1}, \ldots, t_{i_m} = x_{j_m}$ for some $i_1, \ldots, i_m$ and for distinct $j_1, \ldots, j_m \in \{1, \ldots, n\}$ and $var(t) \cap X_n = \{x_{j_1}, \ldots, x_{j_m}\}$. If there exists $l \in \{1, \ldots, m\}$ such that $t_{i_l} = x_{j_l}$ where $i_l \notin \{j_1, \ldots, j_m\}$, then $\sigma_t \neq \sigma_s \circ_G \sigma_t^2$ for all $\sigma_s \in Hyp_G(n)$.*

Proof. Assume that the condition holds. Consider

$$(\sigma_t \circ_G \sigma_t)(f) = \hat{\sigma_t}[t] = S^n(f(t_1, \ldots, t_n), \hat{\sigma_t}[t_1], \ldots, \hat{\sigma_t}[t_n]) = f(u_1, \ldots, u_n)$$

where $u_i = S^n(t_i, \hat{\sigma_t}[t_1], \ldots, \hat{\sigma_t}[t_n])$ for all $i \in \{1, \ldots, n\}$. We have $u_i = S^n(t_i, \hat{\sigma_t}[t_1], \ldots, \hat{\sigma_t}[t_n]) \in \{x_{j_1}, \ldots, x_{j_m}\}$ if and only if $t_i = x_{i_k}$ for some $k \in \{1, \ldots, m\}$. Since $i_l \notin \{j_1, \ldots, j_m\}$, $t_i \neq x_{i_l}$ for all $i \in \{1, \ldots, n\}$. So $u_i \neq x_{j_l}$. Hence $\sigma_t^2(f) = f(u_1, \ldots, u_n)$ where $u_i \neq x_{j_l}$ for all $i \in \{1, \ldots, n\}$. Let $\sigma_s \in Hyp_G(n)$. Next, we will show that $\sigma_t \neq \sigma_s \circ_G \sigma_t^2$. If $s = x_i$ where $x_i \in X$, then $(\sigma_s \circ_G \sigma_t^2)(f) = x_j \neq \sigma_t(f)$ for some $x_j \in X$. If $s = f(s_1, \ldots, s_n)$ where $s_1, \ldots, s_n \in W_{(n)}(X)$, then

$$\begin{aligned}
(\sigma_s \circ_G \sigma_t^2)(f) &= \hat{\sigma_s}[f(u_1, \ldots, u_n)] \\
&= S^n(f(s_1, \ldots, s_n), \hat{\sigma_s}[u_1], \ldots, \hat{\sigma_s}[u_n]) \\
&= f(w_1, \ldots, w_n)
\end{aligned}$$

where $w_i = S^n(s_i, \hat{\sigma_s}[u_1], \ldots, \hat{\sigma_s}[u_n])$ for all $i \in \{1, \ldots, n\}$. Since $u_i \neq x_{j_l}$ for all $i \in \{1, \ldots, n\}$, $\hat{\sigma_s}[u_i] \neq x_{j_l}$. So $w_i \neq x_{j_l}$ for all $i \in \{1, \ldots, n\}$. Hence $f(w_1, \ldots, w_n) \neq f(t_1, \ldots, t_n)$, so $\sigma_t \neq \sigma_s \circ_G \sigma_t^2$. ■

Theorem 8. *An element a of a semigroup S is completely regular if and only if $a \in a^2Sa^2$.*

Proof. See [7]. ■

Theorem 9. *Let $CR(Hyp_G(n)) := CR(R_3) \cup R_1 \cup R_2$. Then $CR(Hyp_G(n))$ is the set of all completely regular elements in $Hyp_G(n)$.*

Proof. By Proposition 5 and Proposition 6, every element in $CR(Hyp_G(n))$ is completely regular element. Let σ_t be a regular element where $\sigma_t \notin CR(Hyp_G(n))$. Then $\sigma_t \in R_3 \setminus CR(R_3)$. By Lemma 7, $\sigma_t \neq \sigma_s \circ_G \sigma_t^2$ for all $\sigma_s \in Hyp_G(n)$. Then $\sigma_t \neq (\sigma_t^2 \circ_G \sigma_u) \circ_G \sigma_t^2$ where $\sigma_t^2 \circ_G \sigma_u \in Hyp_G(n)$. By Theorem 8, σ_t is not completely regular element in $Hyp_G(n)$. Therefore $CR(Hyp_G(n))$ is the set of all completely regular elements in $Hyp_G(n)$. ■

Definition [7]**.** An element a of a semigroup S is called *left(right) regular* if $a \in Sa^2$ $(a \in a^2S)$ and a is called *intra-regular* if $a \in Sa^2S$.

Theorem 10. *An element a of a semigroup S is completely regular if and only if a is both left regular and right regular.*

Proof. See [7]. ■

Theorem 11. *Let S be a semigroup and $a \in S$. If a is completely regular, then a is intra-regular.*

Proof. Let a be a completely regular. Then there exists $b \in S$ such that $a = aba$ and $ab = ba$. So $a = aba = a(ab) = aba(ab) = (ab)a^2(b) \in Sa^2S$. ■

Corollary 12. *Let $\sigma_t \in CR(Hyp_G(n))$. Then σ_t is is both left regular and right regular element in $Hyp_G(n)$, and σ_t is intra-regular element in $Hyp_G(n)$.*

Corollary 13. *If $\sigma_t \in R_3 \setminus CR(R_3)$, then σ_t is not left regular element in $Hyp_G(n)$.*

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