

CONGRUENCES ON BANDS OF π -GROUPS

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Abstract

A semigroup S is said to be completely π -regular if for any $a \in S$ there exists a positive integer n such that a^n is completely regular. The present paper is devoted to the study of completely regular semigroup congruences on bands of π -groups.

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1. INTRODUCTION

The study of the structure of semigroups are essentially influenced by the study of the congruences defined on them. We know that the set of all congruences defined on a semigroup S is a partially ordered set with respect to inclusion and relative to this partial order it forms a lattice, the lattice of congruences on S . The study of the lattice of congruences on different types of semigroups such as regular semigroups and eventually regular semigroups led to breakthrough innovations made by T.E. Hall [3], LaTorre [5], S.H. Rao and P. Lakshmi [10]. The congruences that they looked into were group congruences on regular and eventually regular semigroups. In paper [10], S.H. Rao and P. Lakshmi characterized group congruences on eventually regular semigroups in which they used self-conjugate subsemigroups. Further studies were continued by S. Sattayaporn [11] with weakly self-conjugate subsets. Over the years, congruence structures have been an integral part of discussion in mathematics.

In this paper, we study various types of congruences on bands of π -groups. To be more precise, we characterize least completely regular semigroup congruences on bands of π -groups.

2. PRELIMINARIES

A semigroup (S, \cdot) is called regular if for every element $a \in S$ there exists an element $x \in S$ such that $axa = a$. In this case there also exists $y \in S$ such that $aya = a$ and $yay = y$. Such an element y is called an inverse of a . A semigroup (S, \cdot) is said to be π -regular (or power regular) if for every element $a \in S$ there exists a positive integer n such that a^n is regular. An element a in a semigroup (S, \cdot) is said to be completely regular if there exists $x \in S$ such that $a = axa$ and $ax = xa$. We call a semigroup S , a completely regular semigroup if every element of S is completely regular.

An element a in a semigroup (S, \cdot) is said to be completely π -regular if there exists a positive integer n such that a^n is completely regular. Naturally, a semigroup S is said to be completely π -regular if every element of S is completely π -regular.

Lemma 1 [7]. *Let S be a semigroup and let x be an element of S such that x^n belongs to a subgroup G of S for some positive integer n . Then, if e is the identity of G , we have*

- (a) $ex = xe \in G$,
- (b) $x^m \in G$ for any integer $m > n$.

Let a be a completely π -regular element in a semigroup S . Then a^n lies in a subgroup G of S for some positive integer n . The inverse of a^n in G is denoted by $(a^n)^{-1}$. From the above lemma, it follows that for a completely π -regular element a of a semigroup S , all its completely regular powers lie in the same subgroup of S , and let a^0 be the identity of this group and $\bar{a} = (aa^0)^{-1}$. Then clearly, $a^0 = a\bar{a} = \bar{a}a$ and $aa^0 = a^0a$. By a nil-extension of a semigroup we mean any of its ideal extension by a nil-semigroup.

Throughout this paper, we always let $E(S)$ be the set of all idempotents of the semigroup S . Also we denote the set of all inverses of a regular element a in a semigroup S by $V(a)$. For $a \in S$, by “ a^n is a -regular” we mean that n is the smallest positive integer for which a^n is regular.

A semigroup (S, \cdot) is said to be a band if each element of S is an idempotent, i.e., $a^2 = a$ for all $a \in S$. A congruence ρ on a semigroup S is called a band congruence if S/ρ is a band. A semigroup S is called a band B of semigroups S_α ($\alpha \in B$) if S admits a band congruence ρ on S such that $B = S/\rho$ and each S_α is a ρ -class mapped onto α by the natural epimorphism $\rho^\# : S \rightarrow B$. We write $S = (B; S_\alpha)$. For other notations and terminologies not given in this paper, the reader is referred to the texts of Bogdanovic [1] and Howie [4].

3. LEAST COMPLETELY REGULAR SEMIGROUP CONGRUENCES

In this section we characterize the least completely regular semigroup congruences on bands of π -groups. We introduce a relation on π -groups and then extend this relation on bands of π -groups.

Definition 1 [1]. Let S be a semigroup and G be a subgroup of S . If for every $a \in S$ there exists a positive integer n such that $a^n \in G$, then S is said to be a π -group.

Theorem 2 [1]. *Let S be a π -regular semigroup. Then S is a π -group if and only if S has exactly one idempotent.*

Theorem 3 [1]. *A semigroup S is a π -group if and only if S is a nil-extension of a group.*

In order to characterize further the least completely regular semigroup congruence on a band of π -groups, we define the following relation σ .

Definition 2. Let S be a π -group. Then by Theorem 3, S is nil-extension of a group G . We define a relation σ on S as follows. For $a, b \in S$,

$$a \sigma b \text{ if and only if } ab^{m-1}(b^m)^{-1} = e,$$

where e is the identity of G and b^m is b -regular.

Lemma 4. *Let S be a π -group which is nil-extension of a group G . Then the relation σ as defined in Definition 2 is the least group congruence on S such that $S/\sigma \cong G$.*

Proof. Clearly, σ is reflexive. Let $a \sigma b$. Then $ab^{m-1}(b^m)^{-1} = e$, where e is the identity of G and b^m is b -regular.

Let a^n be a -regular. Now, $a^{n-1}(a^n)^{-1}a \in E(S)$. Since S contains exactly one idempotent, it follows that $a^{n-1}(a^n)^{-1}a = e$. Now, $ba^{n-1}(a^n)^{-1} = ba^{n-1}(a^n)^{-1}e = ba^{n-1}(a^n)^{-1}ab^{m-1}(b^m)^{-1} = beb^{m-1}(b^m)^{-1} = ebb^{m-1}(b^m)^{-1} = eb^m(b^m)^{-1} = e$, i.e., $b \sigma a$. Thus, σ is symmetric.

Let $a \sigma b$ and $b \sigma c$ hold. Then, $ab^{m-1}(b^m)^{-1} = e$ and $bc^{k-1}(c^k)^{-1} = e$, where b^m is b -regular and c^k is c -regular.

Now, $ab^{m-1}(b^m)^{-1}bc^{k-1}(c^k)^{-1} = e$ implies $aec^{k-1}(c^k)^{-1} = e$, i.e., $ac^{k-1}(c^k)^{-1} = e$. This implies $a \sigma c$, and hence σ is transitive. Thus, σ is an equivalence relation.

Let $a \sigma b$ and $c \in S$. Then $ab^{m-1}(b^m)^{-1} = e$ and b^m is b -regular.

Let a^n , $(bc)^l$ and c^k be a -regular, (bc) -regular and c -regular, respectively.

Now $c(bc)^{l-1}((bc)^l)^{-1}b = e$ implies $ac(bc)^{l-1}((bc)^l)^{-1}ba^{n-1}(a^n)^{-1} = e$, i.e., $(ac)(bc)^{l-1}((bc)^l)^{-1} = e$, i.e., $(ac)\sigma(bc)$. Similarly, we can prove $(ca)\sigma(cb)$. Consequently, σ is a congruence on S .

Clearly, $a\sigma(ae)$ and $(ae)\sigma$ is regular. Hence $a\sigma$ is regular. Again, $(ae) \in G$ and let x be the inverse of (ae) in G . Then, $(a\sigma)(x\sigma)(a\sigma) = a\sigma$ and $(a\sigma)(x\sigma) = (x\sigma)(a\sigma) = e\sigma$.

Thus, σ is a group congruence. To show σ is the least group congruence on S , let γ be any group congruence on S and let $a\sigma b$. Then $ab^{m-1}(b^m)^{-1} = e$, where b^m is b -regular.

Therefore, $b\gamma(eb) = ab^{m-1}(b^m)^{-1}b = (ae)\gamma a$, i.e., $a\gamma b$. Hence $\sigma \subseteq \gamma$. Thus, σ is the least group congruence on S .

One can easily prove that the mapping $\psi : S/\sigma \rightarrow G$ defined by $\psi(a\sigma) = ae$ is a group isomorphism. ■

Remark. It follows from Theorem 1 [10] that the relation σ on a π -group S defined in Definition 2 is a group congruence if $\{a \in S : ae = e\}$ is substituted for H in Theorem 1 [10].

Using the above lemma, we now characterize the least completely regular semi-group congruence on a band of π -groups.

Definition 3. Let $S = (B; T_\alpha)$ be a band of π -groups, where B is a band and T_α ($\alpha \in B$) is a π -group. Let T_α be the nil-extension of the group G_α and e_α be the identity of G_α for all $\alpha \in B$. For $a \in T_\alpha$ ($\alpha \in B$), where a^n is a -regular, let $(a^n)^{-1}$ denote the inverse of a^n in G_α .

On S we define a relation ρ as follows. For $a, b \in S$, $a\rho b$ if and only if $a, b \in T_\alpha$ for some $\alpha \in B$ and $ab^{m-1}(b^m)^{-1} = e_\alpha$, where b^m is b -regular; i.e., $\rho = \bigcup_{\alpha \in B} \sigma_\alpha$, where σ_α is the least group congruence on T_α for all $\alpha \in B$.

Theorem 5. Let $S = (B; T_\alpha)$ be a band of π -groups. Then the relation ρ as defined in Definition 3 is the least completely regular semigroup congruence on S .

Proof. Clearly, ρ is an equivalence relation on S .

To show ρ is a congruence on S , let $a\rho b$ and $c \in S$. Therefore, $a, b \in T_\alpha$ and $c \in T_\gamma$ for some $\alpha, \gamma \in B$. Now, $a\rho b$ implies $ab^{m-1}(b^m)^{-1} = e_\alpha$, where e_α is the identity of G_α and b^m is b -regular. This implies $ba^{n-1}(a^n)^{-1} = e_\alpha$, where a^n is a -regular.

Let $(bc)^l$ be (bc) -regular. Now, $c(bc)^{l-1}((bc)^l)^{-1}b = e_{\gamma\alpha}$ implies

$$(ac)(bc)^{l-1} \left((bc)^l \right)^{-1} ba^{n-1} (a^n)^{-1} = ae_{\gamma\alpha} a^{n-1} (a^n)^{-1} = e_{\alpha\gamma\alpha},$$

i.e., $(ac)(bc)^{l-1} \left((bc)^l \right)^{-1} e_{\alpha} = e_{\alpha\gamma\alpha},$

i.e., $(ac)(bc)^{l-1} \left((bc)^l \right)^{-1} e_{\alpha} e_{\alpha\gamma} = e_{\alpha\gamma\alpha} e_{\alpha\gamma} = e_{\alpha\gamma},$

i.e., $(ac)(bc)^{l-1} \left((bc)^l \right)^{-1} e_{\alpha\gamma} = e_{\alpha\gamma},$

i.e., $(ac)(bc)^{l-1} \left((bc)^l \right)^{-1} = e_{\alpha\gamma},$

i.e., $ac \rho bc.$

Similarly, we can prove that $ca \rho cb$. Hence, ρ is a congruence on S .

Also, for any $a \in S$, $a \rho (ae_{\alpha})$ (where $a \in T_{\alpha}$) and $ae_{\alpha} \in G_{\alpha}$ is completely regular. This implies $a \rho$ is completely regular. Moreover, it is easy to verify that ρ is the least completely regular semigroup congruence on S . ■

In a semigroup S with nonempty set of idempotents, $E(S)$ is a subsemigroup of S if and only if for all idempotents e, f in S , $(ef)^2 = ef$. However, a semigroup S with nonempty set of idempotents and the property that for any two elements $e, f \in E(S)$, there exists a positive integer n such that $(ef)^n = (ef)^{n+1}$ does not necessarily have $E(S)$ as its subsemigroup. We provide an example of such a semigroup.

Example [6]. Let $S = \{e, f, a, 0\}$. On S we define a multiplication $'\cdot'$ with the following Cayley table:

\cdot	e	f	a	0
e	e	a	a	0
f	0	f	0	0
a	0	a	0	0
0	0	0	0	0

Then (S, \cdot) is a semigroup with $E(S) = \{e, f, 0\}$. Here, $ef = a \notin E(S)$. Hence $E(S)$ is not a subsemigroup of S . But, for any two elements $x, y \in E(S)$, there exists a positive integer n such that $(xy)^n = (xy)^{n+1}$.

Theorem 6. *Let $S = (B; T_{\alpha})$ be a band of π -groups. Then the following two statements are equivalent.*

- (i) For any two elements $e, f \in E(S)$, there exists a positive integer n such that $(ef)^n = (ef)^{n+1}$.
- (ii) S/ρ is an orthogroup, where ρ is the least completely regular semigroup congruence on S as defined in Definition 3.

Proof. Let $S = (B; T_\alpha)$ be a band of π -groups, where B is a band and $T_\alpha (\alpha \in B)$ is a π -group. Furthermore, let T_α be the nil-extension of the group $G_\alpha (\alpha \in B)$.

Suppose S satisfies statement (i) of Theorem 6. Let $e\rho, f\rho \in E(S/\rho)$, where $e, f \in E(S)$. Then there exists a positive integer n such that $(ef)^n = (ef)^{n+1}$, i.e., $(ef)^2(ef)^{n-1}((ef)^n)^{-1} = e_\alpha$, where e_α is the identity of the group G_α containing $(ef)^n$. Therefore, $(ef)^2\rho(ef)$, i.e., $(e\rho)(f\rho) \in E(S/\rho)$. Hence S/ρ is an orthogroup.

Conversely, let us assume that S/ρ is an orthogroup. Let $e, f \in E(S)$ and $ef \in T_\alpha$, where $\alpha \in B$. Let $(ef)^n$ be (ef) -regular.

Clearly, $e\rho, f\rho \in E(S/\rho)$. Since S/ρ is orthodox, $(ef)\rho \in E(S/\rho)$. Thus, we have, $(ef)\rho(ef)\rho = (ef)\rho$, i.e., $(ef)^2\rho(ef)$, i.e., $(ef)^2(ef)^{n-1}((ef)^n)^{-1} = e_\alpha$, i.e., $(ef)^{n+1} = (ef)^n$. Thus, S satisfies statement (i) of Theorem 6. ■

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