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# FUZZY $n$-FOLD INTEGRAL FILTERS IN $BL$-ALGEBRAS

Rajab Ali Borzooei

*Department of Mathematics*
*Shahid Beheshti University*
*Tehran, Iran*

**e-mail:** borzooei@sbu.ac.ir

and

Akbar Paad

*Department of Mathematics*
*Shahid Beheshti University Tehran, Iran*

**e-mail:** Akbar.Paad@gmail.com

**Abstract**

In this paper, we introduce the notion of fuzzy $n$-fold integral filter in $BL$-algebras and we state and prove several properties of fuzzy $n$-fold integral filters. Using a level subset of a fuzzy set in a $BL$-algebra, we give a characterization of fuzzy $n$-fold integral filters. Also, we prove that the homomorphic image and preimage of fuzzy $n$-fold integral filters are also fuzzy $n$-fold integral filters. Finally, we study the relationship among fuzzy $n$-fold obstinate filters, fuzzy $n$-fold integral filters and fuzzy $n$-fold fantastic filters

**Keywords:** $BL$-algebra, fuzzy $n$-fold obstinate filter, $n$-fold obstinate $BL$-algebra, $n$-fold integral $BL$-algebra and fuzzy $n$-fold integra filter.

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## 1. Introduction

$BL$-algebras are the algebraic structure for Hájek basic logic [14] in order to investigate many valued logic by algebraic means. His motivations for introducing $BL$-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Łukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic ($BL$ for short) is proposed as "the most general"many-valued logic with truth values in [0, 1] and $BL$-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0, 1]. The most familiar example of a $BL$-algebra is the unit interval [0, 1] endowed with the structure induced by a continuous t-norm. The concept of an $MV$-algebra is introduced by Chang [3]. Turunen [15] introduced the notion of an implicative filter and *Boolean* filter and proved that these notions are equivalent in $BL$-algebras. *Boolean* filters are an important class of filters, because the quotient $BL$-algebra induced by these filters are *Boolean* algebras. Heveshki and Eslami [7] introduced the notions of $n$-fold implicative filter and $n$-fold positive implicative filter and they prove some relations between these filters and construct quotient algebras via these filters in 2008. Also, Motamed and Borumand Saeid [8] introduced the notion of $n$-fold obstinate filter in 2011. Moreover, Lele [9, 10] studied the notion of fuzzy $n$-fold (positive) implicative filter and fuzzy $n$-fold obstinate filter in $BL$-algebras. In 2012, Borzooei and Paad [1], introduced the notions of $n$-fold integral filter and $n$-fold integral $BL$ algebra and they studied $n$-fold obstinate $BL$ algebras in [2]. Now, in this paper, we define the concepts of fuzzy $n$-fold integral filters and we state and prove several properties of fuzzy $n$-fold integral filters. Using a level subset of a fuzzy set in a $BL$-algebra, we give a characterization of fuzzy $n$-fold integral filters. In the following, we make a link between fuzzy $n$-fold integral filters and fuzzy $(n+1)$-fold integral filters and we show that extension property holds for this class of fuzzy filters. Also, we prove that a $BL$-algebra $L$, is an $n$-fold integral $BL$-algebra if and only if any fuzzy filter of $L$ is a fuzzy $n$-fold integral filter. We prove that the homomorphic image and preimage of fuzzy $n$-fold integral filters are also fuzzy $n$-fold integral filters. Finally, we study the relationship among fuzzy $n$-fold obstinate filters, fuzzy $n$-fold integral filters and fuzzy $n$-fold fantastic filters.

## 2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, see to the references.

**Definition** [14]**.** A $BL$-algebra is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

$(BL1)$ $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

$(BL2)$ $(L, \odot, 1)$ is a commutative monoid,

$(BL3)$ $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,

$(BL4)$ $x \wedge y = x \odot (x \to y)$,

$(BL5)$ $(x \to y) \vee (y \to x) = 1$.

We denote $x^n = \overbrace{x \odot ... \odot x}^{n-times}$, if $n > 0$ and $x^0 = 1$. Also, we denote $(\underbrace{x \to (...(x \to (x}_{n-times} \to y)))...)$ by $x^n \to y$, for all $x, y \in L$.

A $BL$-algebra $L$ is called a Gödel algebra, if $x^2 = x \odot x = x$, for all $x \in L$ and a $BL$-algebra $L$ is called an $MV$-algebra, if $(x^-)^- = x$, for all $x \in L$, where $x^- = x \to 0$.

**Proposition 1** [4, 5, 14]**.** *In any $BL$-algebra the following hold:*

$(BL6)$ *$x \leq y$ if and only if $x \to y = 1$,*

$(BL7)$ *$x^{n+1} \leq x^n$, $\forall n \in \mathbb{N}$,*

$(BL8)$ *$x \leq y$ implies $y \to z \leq x \to z$ and $z \to x \leq z \to y$,*

$(BL9)$ *$0 \leq x$ and $x \odot x^- = 0$,*

$(BL10)$ *$1 \to x = x$ and $x \to 1 = 1$, for all $x, y, z \in L$.*

The following theorems and definitions are from [1, 2, 4, 6, 7, 10, 11, 14, 16] and we refer the reader to them, for more details.

**Definition.** Let $L$ be a $BL$-algebra and $F$ be a non-empty subset of $L$. Then

(i) $F$ is called a *filter* of $L$, if $x \odot y \in F$, for any $x, y \in F$ and if $x \in F$ and $x \leq y$ then $y \in F$, for all $x, y \in L$.

(ii) $F$ is called an *$n$-fold implicative filter* of $L$, if $1 \in F$ and

$$x^n \to (y \to z) \in F \text{ and } x^n \to y \in F \text{ imply } x^n \to z \in F, \text{ for all } x, y, z \in L.$$

(iii) $F$ is called an *$n$-fold positive implicative filter* of $L$, if $1 \in F$ and

$$x \to ((y^n \to z) \to y) \in F \text{ and } x \in F \text{ imply } y \in F, \text{ for all } x, y, z \in L.$$

(iv) $F$ is called an *$n$-fold fantastic filter*, if $1 \in F$ and

$$z \to (y \to x) \in F \text{ and } z \in F \text{ imply } (((x^n \to y) \to y) \to x) \in F,$$
$$\text{for all } x, y, z \in L.$$

(v) A filter $F$ is called an *n-fold obstinate filter*, if whenever $x, y \notin F$, then

$$x^n \to y \in F \text{ and } y^n \to x \in F, \text{ for all } x, y \in L.$$

(vi) A filter $F$ is called an *n-fold integral filter*, if

$$(x^n \odot y^n)^- \in F \text{ implies } (x^n)^- \in F \text{ or } (y^n)^- \in F, \quad \text{for all } x, y \in L.$$

**Note.** 1-fold integral filter is an integral filter.

**Definition.** Let $L$ be a $BL$-algebra. Then

(i) $L$ is called an *n-fold integral $BL$-algebra*, if

$$(x^n \odot y^n) = 0 \text{ implies that } x^n = 0 \text{ or } y^n = 0, \quad \text{for all } x, y \in L$$

(ii) $L$ is called an $n$-fold obstinate $BL$-algebra, if $L$ is an $MV$-algebra and $x^n = 0$, for all $x \in L\backslash\{1\}$.

**Theorem 2.** *Let $F$ be a filter of $BL$-algebra $L$. Then the binary relation $\equiv_F$ on $L$ which is defined by*

$$x \equiv_F y \quad \textit{if and only if } x \to y \in F \quad \textit{and } y \to x \in F$$

*is a congruence relation on $L$. Define $\cdot$, $\rightharpoonup$, $\sqcup$, $\sqcap$ on $\frac{L}{F}$ , the set of all congruence classes of $L$, as follows:*

$$[x] \cdot [y] = [x \odot y], \ [x] \rightharpoonup [y] = [x \to y], \ [x] \sqcup [y] = [x \vee y], \ [x] \sqcap [y] = [x \wedge y].$$

*Then $(\frac{L}{F}, \cdot, \rightharpoonup, \sqcup, \sqcap, [0], [1])$ is a $BL$-algebra which is called quotient $BL$-algebra with respect to $F$.*

**Theorem 3.** *Let $F \subseteq G$, where $F$ be an n-fold integral filter and $G$ be a filter of $L$. Then $G$ is an n-fold integral filter.*

**Theorem 4.** *In any $BL$-algebra $L$, the following conditions are equivalent:*

(i) $\{1\}$ *is an n-fold integral filter,*

(ii) *any filter of $L$ is an n-fold integral filter,*

(iii) *$L$ is an n-fold integral $BL$-algebra.*

**Theorem 5.**

(i) *Let $F$ be a filter of $L$. Then $F$ is an n-fold obstinate filter of $L$ if and only if $F$ is an n-fold integral and n-fold fantastic filter.*

(ii) *Let $F$ be a filter of $L$. Then $F$ is an $n$-fold obstinate filter of $L$ if and only if $\frac{L}{F}$ is an $n$-fold obstinate $BL$-algebra.*

(iii) *Let $F$ be a filter of $L$. Then $F$ is an $n$-fold obstinate filter if and only if*

$$x \notin F \ \ \text{implies that} \ (x^n)^- \in F, \ \text{for all } x \in L.$$

**Definition.** Let $L_1$ and $L_2$ be two $BL$-algebras. Then the map $f : L_1 \to L_2$ is called a $BL$-algebra *homomorphism* if and only if it satisfies the following conditions, for every $x, y \in L_1$:

(i) $f(0) = 0$,

(ii) $f(x \odot y) = f(x) \odot f(y)$,

(iii) $f(x \to y) = f(x) \to f(y)$.

If $f$ is a bijective, then the homomorphism $f$ is called $BL$-algebra *isomorphism*. In this case we write $L_1 \cong L_2$.

In the following, we give some fuzzy algebraic results on $BL$-algebras that come from references [9, 12, 13].

**Definition.** Let $L$ be a $BL$-algebra and $\mu : L \to [0, 1]$ be a fuzzy set on $L$. Then

(i) $\mu$ is called a *fuzzy filter* on $L$, if and only if $\mu(x) \leq \mu(1)$ and $\mu(x \to y) \wedge \mu(x) \leq \mu(y)$, for all $x, y \in L$.

(ii) $\mu$ is called a *fuzzy $n$-fold implicative filter* on $L$, if and only if $\mu(x) \leq \mu(1)$ and

$$\mu(x^n \to (y \to z)) \wedge \mu(x^n \to y) \leq \mu(x^n \to z), \ \text{for all } x, y, z \in L.$$

(iii) $\mu$ is called a *fuzzy $n$-fold positive implicative filter* on $L$, if and only if $\mu(x) \leq \mu(1)$ and

$$\mu(x \to ((y^n \to z) \to y)) \wedge \mu(x) \leq \mu(y), \ \text{for all } x, y, z \in L.$$

(iv) A fuzzy filter $\mu$ is called a *fuzzy $n$-fold obstinate filter* on $L$, if and only if

$$min\{\mu(x^n \to y), \mu(y^n \to x)\} \geq min\{1 - \mu(x), 1 - \mu(y)\}, \ \text{for all } x, y \in L.$$

**Lemma 6.** *Let $L$ be a $BL$-algebra and $\mu$ be a fuzzy filter on $L$. Then the following properties hold:*

(i) *if $x \leq y$, then $\mu(x) \leq \mu(y)$, that is $\mu$ is order-preserving,*

(ii) *if $\mu(x \to y) = \mu(1)$, then $\mu(x) \leq \mu(y)$, for all $x, y \in L$.*

**Definition.** Let $L_1$ and $L_2$ be two $BL$-algebras, $\mu$ a fuzzy subset of $L_1$, $\eta$ a fuzzy subset of $L_2$ and $f : L_1 \to L_2$ a $BL$-algebra homomorphism. The image of $\mu$ under $f$ denoted by $f(\mu)$ is a fuzzy set of $L_2$ defined by:

$$f(\mu)(y) = \sup_{x \in f^{-1}(y)} \mu(x) \text{ if } f^{-1}(y) \neq \emptyset \text{ and } f(\mu)(y) = 0 \text{ if } f^{-1}(y) = \emptyset$$

for all $y \in L_2$.

The preimage of $\eta$ under $f$ denoted by $f^{-1}(\eta)$ is a fuzzy set of $L_1$ defined by: $f^{-1}(\eta)(x) = \eta(f(x))$, for all $x \in L_1$.

Also a fuzzy subset $\mu$ of $X$ has a sup property, if for any nonempty subset $Y$ of $X$, there exists $y_0 \in Y$, such that $\mu(y_0) = \sup_{y \in Y} \mu(y)$.

**Theorem 7.** *Let $L$ be a $BL$-algebra, $\mu$ be a fuzzy set on $L$ and $\mu_t = \{x \in L \mid \mu(x) \geq t\}$, $\forall\ t \in [0,1]$. Then*

(i) *$\mu$ is a fuzzy filter on $L$ if and only if $\forall\ t \in [0,1]$, $\emptyset \neq \mu_t$ is a filter of $L$.*

(ii) *$\mu$ is a fuzzy $n$-fold fantastic filter on $L$ if and only if $\forall\ t \in [0,1]$, $\emptyset \neq \mu_t$ is a $n$-fold fantastic filter of $L$.*

(iii) *$\mu$ is a fuzzy $n$-fold positive implicative filter on $L$ if and only if $\mu$ is a fuzzy filter and $\mu((x^n \to 0) \to x) \leq \mu(x)$, for all $x \in L$.*

(iv) *$\mu$ is a fuzzy $n$-fold obstinate filter on $L$ if and only if $\mu$ is a fuzzy filter and $\mu((x^n)^-) \geq 1 - \mu(x)$, for all $x \in L$.*

**Note.** In the rest of the paper we assume that $L$ is a $BL$-algebra. Unless otherwise is stated.

## 3. Fuzzy $n$-fold integral filters in BL-algebras

**Definition.** Let $\mu$ be a fuzzy filter on $L$. Then $\mu$ is called a *fuzzy $n$-fold integral filter*, if for all $x, y \in L$, it satisfies:

$$\mu((x^n \odot y^n)^-) = \mu((x^n)^-) \vee \mu((y^n)^-)$$

**Example 8.** Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\to$ are defined as the following tables:

Table 1

| $\odot$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | $a$ |
| $b$ | 0 | 0 | $a$ | $b$ |
| 1 | 0 | $a$ | $b$ | 1 |

Table 2

| $\to$ | 0 | $a$ | $b$ | 1 |
|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 |
| $a$ | $b$ | 1 | 1 | 1 |
| $b$ | $a$ | $b$ | 1 | 1 |
| 1 | 0 | $a$ | $b$ | 1 |

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a $BL$-algebra. Now, let the fuzzy set $\mu$ on $L$ is defined by

$$\mu(1) = t_2,\ \mu(0) = \mu(a) = \mu(b) = t_1\ (0 \leq t_1 < t_2 \leq 1).$$

It is easy to check that $\mu$ is a fuzzy filter and it is a fuzzy 3-fold integral filter. But, it is not a fuzzy 2-fold integral filter. Because, $\mu((b^2 \odot b^2)^-) = \mu((a \odot a)^-) = \mu(0^-) = \mu(1) = t_2$ and $\mu((b^2)^-) = \mu(a^-) = \mu(b) = t_1$. Hence, $\mu((b^2 \odot b^2)^-) \neq \mu((b^2)^-) \vee \mu((b^2)^-)$.

**Theorem 9.** *A non empty subset $F$ of $L$ is an $n$-fold integral filter if and only if the characteristic function $\chi_F$ is a fuzzy $n$-fold integral filter on $L$.*

***Proof.*** Let $F$ be an $n$-fold integral filter. Then we show that $\chi_F((x^n \odot y^n)^-) = \chi_F((x^n)^-) \vee \chi_F((y^n)^-)$. If $(x^n \odot y^n)^- \in F$, then $\chi_F((x^n \odot y^n)^-) = 1$ and since $F$ is an $n$-fold integral filter, then $(x^n)^- \in F$ or $(y^n)^- \in F$ and so $\chi_F((x^n)^-) = 1$ or $\chi_F((y^n)^-) = 1$. Hence, $\chi_F((x^n \odot y^n)^-) = \chi_F((x^n)^-) \vee \chi_F((y^n)^-) = 1$. Now, let $(x^n \odot y^n)^- \notin F$. Then $(x^n)^- \notin F$ and $(y^n)^- \notin F$. Indeed by $(BL7)$ and $(BL8)$, $(x^n)^-$and $(y^n)^- \leq (x^n \odot y^n)^-$ and if $(x^n)^- \in F$ or $(y^n)^- \in F$, then $(x^n \odot y^n)^- \in F$ and it is impossible. Hence, $(x^n)^- \notin F$ and $(y^n)^- \notin F$ and so $\chi_F((x^n)^-) = 0$ and $\chi_F((y^n)^-) = 0$. Therefore, $\chi_F((x^n \odot y^n)^-) = \chi_F((x^n)^-) \vee \chi_F((y^n)^-) = 0$.

Conversely, let $\chi_F$ is a fuzzy $n$-fold integral filter on $L$ and $(x^n \odot y^n)^- \in F$. Then $\chi_F((x^n \odot y^n)^-) = \chi_F((x^n)^-) \vee \chi_F((y^n)^-) = 1$ and so $\chi_F((x^n)^-) = 1$ or $\chi_F((y^n)^-) = 1$. Hence, $(x^n)^- \in F$ or $(y^n)^- \in F$. Therefore, $F$ is an $n$-fold integral filter. ■

**Theorem 10.** *Let $\mu$ be a fuzzy filter on $L$. Then $\mu$ is a fuzzy $n$-fold integral filter if and only if for each $t \in [0, 1]$, $\emptyset \neq \mu_t$ is an $n$-fold integral filter.*

***Proof.*** Let $\mu$ be a fuzzy $n$-fold integral filter and $(x^n \odot y^n)^- \in \mu_t$, for $t \in [0, 1]$ and $x, y \in L$. Then $\mu((x^n \odot y^n)^-) \geq t$. Since $\mu((x^n \odot y^n)^-) = \mu((x^n)^-) \vee \mu((y^n)^-)$, then $\mu((x^n)^-) \vee \mu((y^n)^-) \geq t$. Now, by contradiction if $(x^n)^- \notin \mu_t$ and $(y^n)^- \notin \mu_t$, then $\mu((x^n)^-) < t$ and $\mu((y^n)^-) < t$. Hence, $\mu((x^n)^-) \vee \mu((y^n)^-) < t$ and it is a contradiction. Therefore, $(x^n)^- \in \mu_t$ or $(y^n)^- \in \mu_t$ and so $\mu_t$ is an $n$-fold integral filter.

Conversely, Since $\mu$ is a fuzzy filter on $L$, then assume that for each $t \in [0, 1]$, $\emptyset \neq \mu_t$ is an $n$-fold integral filter. Now, we prove that $\mu$ is a fuzzy $n$-fold integral filter. Since by $(BL7)$, $x^n \odot y^n \leq x^n$, then by $(BL8)$, $(x^n)^- \leq (x^n \odot y^n)^-$ and so by Lemma 6, $\mu((x^n)^-) \leq \mu((x^n \odot y^n)^-)$. By similar way we have $\mu((y^n)^-) \leq \mu((x^n \odot y^n)^-)$ and so $\mu((x^n)^-) \vee \mu((y^n)^-) \leq \mu((x^n \odot y^n)^-)$. Now, we show, $\mu((x^n \odot y^n)^-) \leq \mu((x^n)^-) \vee \mu((y^n)^-)$. In the other wise, there exist $a, b \in L$ such that

$$\mu((a^n \odot b^n)^-) > \mu((a^n)^-) \vee \mu((b^n)^-).$$

Let

$$t_0 = \mu((a^n)^-) \vee \mu((b^n)^-) + 1/2\{\mu((a^n \odot b^n)^-) - \mu((a^n)^-) \vee \mu((b^n)^-)\}.$$

Then we have

$$\mu((a^n)^-) \vee \mu((b^n)^-) < t_0 < \mu((a^n \odot b^n)^-)$$

and so $(a^n \odot b^n)^- \in \mu_{t_0}$. Now, since $\mu_{t_0}$ is an $n$-fold integral filter, then $(a^n)^- \in \mu_{t_0}$ or $(b^n)^- \in \mu_{t_0}$. Hence, $\mu((a^n)^-) \geq t_0$ or $\mu((b^n)^-) \geq t_0$ and so $\mu((a^n)^-) \vee \mu((b^n)^-) \geq t_0$, it is a contradiction. Therefore, $\mu((x^n \odot y^n)^-) = \mu((x^n)^-) \vee \mu((y^n)^-)$ and so $\mu$ is a fuzzy $n$-fold integral filter. ∎

In the following theorem, we make a link between fuzzy $n$-fold integral filters and fuzzy $(n+1)$-fold integral filters.

**Theorem 11.** *Let $\mu$ be a fuzzy $n$-fold integral filter. Then $\mu$ is a fuzzy $(n+1)$-fold integral filter.*

***Proof.*** Let $\mu$ be a fuzzy $n$-fold integral filter. Then it is easy to check that $\mu((x^{n+1} \odot y^{n+1})^-) \geq \mu((x^{n+1})^-) \vee \mu((y^{n+1})^-)$, for all $x, y \in L$. Now, since by $(BL7)$, $x^{n+n} \odot y^{n+n} \leq x^{n+1} \odot y^{n+1}$, then by $(BL8)$, $(x^{n+1} \odot y^{n+1})^- \leq (x^{n+n} \odot y^{n+n})^-$ and by Lemma 6, $\mu((x^{n+1} \odot y^{n+1})^-) \leq \mu((x^{n+n} \odot y^{n+n})^-)$. Since $\mu$ is a fuzzy $n$-fold integral filter, then

$$\begin{aligned}
\mu((x^{n+n} \odot y^{n+n})^-) &= \mu((x^{2n} \odot y^{2n})^-) \\
&= \mu((x^2)^n \odot (y^2)^n)^-) \\
&= \mu(((x^2)^n)^-) \vee \mu(((y^2)^n)^-) \\
&= \mu((x^{n+n})^-) \vee \mu((y^{n+n})^-) \\
&= \mu((x^n \odot x^n)^-) \vee \mu((y^n \odot y^n)^-) \\
&= \mu((x^n)^-) \vee \mu((x^n)^-) \vee \mu((y^n)^-) \vee \mu((y^n)^-) \\
&= \mu((x^n)^-) \vee \mu(y^n)^-) \\
&\leq \mu((x^{n+1})^-) \vee \mu((y^{n+1})^-), \text{ by } (BL8) \text{ and Lemma 6.}
\end{aligned}$$

Hence, $\mu((x^{n+1} \odot y^{n+1})^-) = \mu((x^{n+1})^-) \vee \mu((y^{n+1})^-)$ and so $\mu$ is a fuzzy $(n+1)$-fold integral filter. ∎

By mathematical induction, we can prove that every fuzzy $n$-fold integral filter is a fuzzy $(n+k)$-fold integral filter, for any integer $k \geq 0$.

**Note.** Example 8 shows that the converse of Theorem 11 is not correct in general.

**Theorem 12.** (*Extension property for fuzzy n-fold integral filters*) *Let $\mu$ and $\eta$ be two fuzzy filters such that $\mu \subseteq \eta$ and $\mu(1) = \eta(1)$. If $\mu$ is a fuzzy n-fold integral filter, then $\eta$ is a fuzzy n-fold integral filter too.*

***Proof.*** Let $\mu$ be a fuzzy $n$-fold integral filter. Then by Theorem 10, $\emptyset \neq \mu_t$ is an $n$-fold integral filter, for each $t \in [0,1]$ and since $\mu \subseteq \eta$, then $\mu(x) \leq \eta(x)$, for all $x \in L$. Now, if $x \in \mu_t$, then $\mu(x) \geq t$ and so $\eta(x) \geq t$. Hence, $x \in \eta_t$ and $\mu_t \subseteq \eta_t$. If $\emptyset \neq \eta_t$, since $\mu(1) = \eta(1)$ then $\emptyset \neq \mu_t$. Now, by Theorem 3, since $\mu_t$ is an $n$-fold integral filter, then $\eta_t$ is an $n$-fold integral filter, for each $t \in [0,1]$. Hence, by Theorem 10, $\eta$ is a fuzzy $n$-fold integral filter. ■

**Theorem 13.** *Let $\mu$ be a fuzzy set on $L$ defined by*

$$\mu(x) = \begin{cases} 0, & x \neq 1, \\ \alpha, & x = 1. \end{cases}$$

*For fixed $\alpha \in (0,1]$. Then the following are equivalent:*

(i) *$L$ is an n-fold integral $BL$-algebra,*

(ii) *Any fuzzy filter is a fuzzy n-fold integral filter,*

(iii) *$\mu$ is a fuzzy n-fold integral filter.*

***Proof.*** (i)⇒(ii): Let $L$ be an $n$-fold integral $BL$-algebra and $\eta$ be a fuzzy filter on $L$. Then by Theorem 4, every filter of $L$ is an $n$-fold integral filter. Now, since $\eta$ is a fuzzy filter by Theorem 7(i), for each $t \in [0,1]$, $\emptyset \neq \eta_t$ is a filter and so $\eta_t$ is an $n$-fold integral filter of $L$. Theref ore, by Theorem 10, $\eta$ is a fuzzy $n$-fold integral filter on $L$.

(ii)⇒(iii): First, we will prove that $\mu$ is a fuzzy filter. By definition of $\mu$, for any $x \in L$, $\mu(x) \leq \mu(1)$. Now, let $x, y \in L$. We consider two following cases for $y$. If $y = 1$, then

$$\mu(x \to y) \wedge \mu(x) \leq \alpha = \mu(1) = \mu(y).$$

If $y \neq 1$, then we consider two following cases for $x$. If $x = 1$, then by $(BL10)$

$$\mu(x \to y) \wedge \mu(x) = \mu(1 \to y) \wedge \mu(1) = \mu(y) \wedge \mu(1) = \mu(y) \leq \mu(y).$$

If $x \neq 1$, then

$$\mu(x \to y) \wedge \mu(x) = \mu(x \to y) \wedge 0 = 0 \leq \mu(y).$$

Hence, $\mu$ is a fuzzy filter and so by (ii), it is a fuzzy $n$-fold integral filter.

(iii)⇒(i): Since $\mu$ is a fuzzy $n$-fold integral filter, then by Theorem 10, $\mu_\alpha = \{x \in L \mid \mu(x) \geq \alpha\} = \{1\}$ is an $n$-fold integral filter and so by Theorem 4, $L$ is an $n$-fold integral $BL$-algebra. ■

**Corollary 14.** *Let $\mu$ be a fuzzy set on $L$ defined by*

$$\mu(x) = \begin{cases} 0, & x \neq 1, \\ 1, & x = 1. \end{cases}$$

*Then the following are equivalent:*

(i) *$L$ is an integral $BL$-algebra,*

(ii) *Any fuzzy filter is a fuzzy integral filter,*

(iii) *$\mu$ is a fuzzy integral filter.*

***Proof.*** Let $n = 1$ in Theorem 13. Then the proof is clear. ■

**Theorem 15.** *Let $f : L_1 \to L_2$ be a $BL$-algebra homomorphism and $\mu$ be a fuzzy $n$-fold integral filter on $L_2$. Then $f^{-1}(\mu)$ is a fuzzy $n$-fold integral filter on $L_1$.*

***Proof.*** First, we show that $f^{-1}(\mu)$ is a fuzzy filter on $L_1$. Since $f(x) \leq f(1)$, for all $x \in L_1$ and by Lemma 6,

$$f^{-1}(\mu)(x) = \mu(f(x)) \leq \mu(f(1)) = f^{-1}(\mu)(1).$$

Also, since $\mu$ is a fuzzy filter on $L_2$, then for all $x, y \in L_1$,

$$\begin{aligned} f^{-1}(\mu)(x \to y) \wedge f^{-1}(\mu)(x) &= \mu(f(x) \to f(y)) \wedge \mu(f(x)) \\ &\leq \mu(f(y)) \\ &= f^{-1}(\mu)(y). \end{aligned}$$

Then $f^{-1}(\mu)$ is a fuzzy filter on $L_1$. Now, let $\mu$ be a fuzzy $n$-fold integral filter on $L_2$ and $x, y \in L_1$. Then

$$\begin{aligned} f^{-1}(\mu)((x^n \odot y^n)^-) &= \mu(f((x^n \odot y^n)^-)) \\ &= \mu((f(x)^n \odot f(y)^n)^-) \\ &= \mu((f(x)^n)^-) \vee \mu((f(y)^n)^-) \\ &= \mu(f((x^n)^-)) \vee \mu(f((y^n)^-)) \\ &= f^{-1}(\mu)((x^n)^-) \vee f^{-1}(\mu)((y^n)^-). \end{aligned}$$

Therefore, $f^{-1}(\mu)$ is a fuzzy $n$-fold integral filter on $L_1$. ■

**Lemma 16.** *Let $f : L_1 \to L_2$ be a $BL$-algebra isomorphism and $\mu$ be a fuzzy filter on $L_1$. Then $f(\mu)$ is a fuzzy filter on $L_2$.*

***Proof.*** Since $\mu$ is a fuzzy filter on $L_1$, then $\mu(x) \leq \mu(1)$, for all $x \in L_1$. Now, for all $y \in L_2$,

$$f(\mu)(y) = \sup\{\mu(x) \mid x \in f^{-1}(y)\} \leq \sup\{\mu(1) \mid 1 \in f^{-1}(1)\} = f(\mu)(1).$$

Thus, $f(\mu)(y) \leq f(\mu)(1)$, for all $y \in L_2$. Now, suppose that $y_1, y_2 \in L_2$. Since $f$ is a $BL$-algebra isomorphism, then there exist $x_1, x_2 \in L_1$, such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Now,

$$f(\mu)(y_1 \to y_2) = \sup\{\mu(z) \mid z \in f^{-1}(y_1 \to y_2)\}.$$

Also, since $f$ is a $BL$-algebra isomorphism and $z \in f^{-1}(y_1 \to y_2)$, then

$$f(z) = y_1 \to y_2 = f(x_1) \to f(x_2) = f(x_1 \to x_2).$$

And so $z = x_1 \to x_2$. Therefore,

$$\begin{aligned} f(\mu)(y_1 \to y_2) &= \sup\{\mu(x_1 \to x_2) \mid x_1 \to x_2 \in f^{-1}(y_1 \to y_2)\} \\ &= \mu(x_1 \to x_2). \end{aligned}$$

By similar way, we have

$$f(\mu)(y_1) = \sup\{\mu(x_1) \mid x_1 \in f^{-1}(y_1)\} = \mu(x_1)$$
$$f(\mu)(y_2) = \sup\{\mu(x_2) \mid x_1 \in f^{-1}(y_2)\} = \mu(x_2).$$

Moreover, since $\mu$ is a fuzzy filter on $L_1$, then

$$\begin{aligned} f(\mu)(y_1 \to y_2) \wedge f(\mu)(y_1) &= \mu(x_1 \to x_2) \wedge \mu(x_1) \\ &\leq \mu(x_2) \\ &= f(\mu)(y_2). \end{aligned}$$

Therefore, $f(\mu)$ is a fuzzy filter on $L_2$. ■

**Theorem 17.** *Let $f : L_1 \to L_2$ be a $BL$-algebra isomorphism and $\mu$ be a fuzzy $n$-fold integral filter on $L_1$ with sup property. Then $f(\mu)$ is a fuzzy $n$-fold integral filter on $L_2$.*

***Proof.*** By Lemma 16, $f(\mu)$ is a fuzzy filter on $L_2$. Now, we show that $f(\mu)((y_1^n)^-) \vee f(\mu)((y_2^n)^-) \leq f(\mu)((y_1^n \odot y_2^n)^-)$. Also,

$$f(\mu)((y_1^n \odot y_2^n)^-) = \sup_{t \in f^{-1}((y_1^n \odot y_2^n)^-)} \mu(t) \text{ and } f(\mu)((y_1^n)^-) = \sup_{t \in f^{-1}((y_1^n)^-)} \mu(t).$$

Since $f$ is a $BL$-algebra isomorphism and $\mu$ has sup property, then there exist $x_1 \in f^{-1}((y_1^n)^-)$ and $x_3 \in f^{-1}((y_1^n \odot y_2^n)^-)$ such that $\sup_{t\in f^{-1}((y_1^n)^-)} \mu(t) = \mu(x_1)$ and $\sup_{t\in f^{-1}((y_1^n\odot y_2^n)^-)} \mu(t) = \mu(x_3)$. Since $x_1 \in f^{-1}((y_1^n)^-)$, then $f(x_1) = (y_1^n)^- \leq (y_1^n\odot y_2^n)^- = f(x_3)$. Now, since $f^{-1}$ is a $BL$-algebra homomorphism, then $x_1 \leq x_3$ and so by Lemma 6, $\mu(x_1) \leq \mu(x_3)$. Hence, $f(\mu)((y_1^n)^-) \leq f(\mu)((y_1^n \odot y_2^n)^-)$. By similar way, we have $f(\mu)((y_2^n)^-) \leq f(\mu)((y_1^n \odot y_2^n)^-)$ and so $f(\mu)((y_1^n)^-) \vee f(\mu)((y_2^n)^-) \leq f(\mu)((y_1^n \odot y_2^n)^-)$.

Now, we show that $f(\mu)((y_1^n)^-) \vee f(\mu)((y_2^n)^-) \geq f(\mu)((y_1^n \odot y_2^n-)$. Since $f$ is a $BL$-algebra isomorphism, then there exist $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. But, since $(x_1^n)^- \in f^{-1}((y_1^n)^-)$ and $(x_2^n)^- \in f^{-1}((y_1^n)^-)$, then

$$\begin{aligned} f(\mu)((y_1^n)^-) \vee f(\mu)((y_2^n)^-) &= \sup_{t\in f^{-1}((y_1^n)^-)} \mu(t) \vee \sup_{t\in f^{-1}((y_2^n)^-)} \mu(t) \\ &\geq \mu((x_1^n)^-) \vee \mu((x_2^n)^-) \\ &= \mu((x_1^n \odot x_2^n)^-). \end{aligned}$$

By sup property for $\mu$, there exist

$$x_3 \in f^{-1}((y_1^n \odot y_2^n)^-) \text{ such that } \sup_{t\in f^{-1}((y_1^n\odot y_2^n)^-)} \mu(t) = \mu(x_3)$$

and so $f(\mu)((y_1^n \odot y_2^n)^-) = \mu(x_3)$. Now, since $f$ is a $BL$-algebra monomorphism and $f(x_3) = (y_1^n \odot y_2^n)^- = f((x_1^n \odot x_2^n)^-)$, then $x_3 = (x_1^n \odot x_2^n)^-$ and so $\mu(x_3) = \mu((x_1^n \odot x_2^n)^-)$. Hence, $\mu((x_1^n \odot x_2^n)^-) = f(\mu)((y_1^n \odot y_2^n)^-)$ and so $f(\mu)((y_1^n)^-) \vee f(\mu)((y_2^n)^-) \geq f(\mu)((y_1^n \odot y_2^n)^-)$. Therefore, $f(\mu)((y_1^n)^-) \vee f(\mu)((y_2^n)^-) = f(\mu)((y_1^n \odot y_2^n)^-)$ and so $f(\mu)$ is a fuzzy $n$-fold integral filter on $L_2$. ■

## 4. Fuzzy n-fold obstinate filters and fuzzy n-fold integral filters

In this section, we study relationship among fuzzy $n$-fold obstinate filters, fuzzy $n$-fold integral filters and fuzzy $n$-fold fantastic filters.

**Theorem 18.** *Let $\mu$ be a fuzzy filter on $L$. Then $\mu$ is a fuzzy $n$-fold integral filter and fuzzy $n$-fold fantastic filter if and only if $\emptyset \neq \mu_t$ is an $n$-fold obstinate filter, for any $t \in [0,1]$.*

***Proof.*** Let $\mu$ be a fuzzy $n$-fold integral filter and fuzzy $n$-fold fantastic filter. Then by Theorems 7(ii) and 10, $\mu_t$ is a $n$-fold integral filter and $n$-fold fantastic filter of $L$, for any $t \in [0,1]$. Hence, by Theorem 5(i), $\mu_t$ is an $n$-fold obstinate filter, for any $t \in [0,1]$. Conversely, let $\emptyset \neq \mu_t$ is an $n$-fold obstinate filter, for any $t \in [0,1]$. Then by Theorem 5(i), $\mu_t$ is a $n$-fold integral filter and $n$-fold fantastic filter of $L$, for any $t \in [0,1]$. Hence, by Theorem 7(ii) and 10, $\mu$ is a fuzzy $n$-fold integral filter and fuzzy $n$-fold fantastic filter on $L$. ■

**Theorem 19.** *Let $\mu$ be a fuzzy $n$-fold obstinate filter on $L$. Then*

(i) *for any $t \in [0, 0.5]$, $\emptyset \neq \mu_t$ is an $n$-fold obstinate filter of $L$.*

(ii) *for any $t \in (0.5, 1]$, if $\mu_t \neq \emptyset$, then $\mu_t$ either $n$-fold obstinate filter or $\mu_{1-t} = L$.*

***Proof.*** (i) It is hold by Theorem 3.4 of [10].

(ii) Assume that $t \in (0.5,1]$ and $\mu_t \neq \emptyset$. If $\mu_t$ is an $n$-fold obstinate filter, then the proof is complete. Otherwise, suppose that $\mu_t$ is not an $n$-fold obstinate filter. Then by Theorem 5(iii), there exist $a \in L$ such that $a \notin \mu_t$ and $(a^n)^- \notin \mu_t$. Hence, $\mu(a) < t$ and $\mu((a^n)^-) < t$. Now, since $\mu$ is a fuzzy $n$-fold obstinate filter, then $\mu((a^n)^-) \geq 1-\mu(a)$ and so $\mu((a^n)^-) > 1-t$. Hence, $(a^n)^- \in \mu_{1-t}$ and since $t > \mu((a^n)^-) \geq 1-\mu(a)$, then $t > 1-\mu(a)$ and so $\mu(a) > 1-t$. Thus, $a \in \mu_{1-t}$ and so $a^n \in \mu_{1-t}$. Therefore, by $(BL9)$, $0 = (a^n)^- \odot (a^n) \in \mu_{1-t}$ and so $\mu_{1-t} = L$. ■

**Theorem 20.** *Let $L$ be an $n$-fold obstinate $BL$-algebra. Then every fuzzy filter is a fuzzy $n$-fold (positive) implicative filter and fuzzy $n$-fold integral filter.*

***Proof.*** Let $\mu$ be a fuzzy filter on an $n$-fold obstinate $BL$-algebra $L$. Then for all $1 \neq x \in L$, $x^n = 0$ and so

$$\mu((x^n \to 0) \to x) = \mu((0 \to 0) \to x) = \mu(1 \to x) = \mu(x) \leq \mu(x)$$

and for $x = 1$,

$$\mu((1^n \to 0) \to 1) = \mu((1 \to 0) \to 1) = \mu(0 \to 1) = \mu(1) \leq \mu(1).$$

Therefore, by Theorem 7(iii), $\mu$ is a fuzzy $n$-fold positive implicative filter. Also, since

$$\mu((x^n \odot y^n)^-) = \mu((x^n)^-) = \mu((y^n)^-) = \mu(0^-) = \mu(1).$$

Then $\mu((x^n \odot y^n)^-) = \mu((x^n)^-) \vee \mu((y^n)^-)$. Therefore, $\mu$ is a fuzzy $n$-fold integral filter. ■

**Theorem 21.** *Let $\mu$ be a fuzzy $n$-fold obstinate filter on an $n$-fold obstinate $BL$-algebra $L$. Then for all $x \in L$*

(i) $\mu(x) \geq 1 - \mu(1)$,

(ii) $\mu(0) + \mu(1) \geq 1$.

***Proof.*** (i) Let $\mu$ be a fuzzy $n$-fold obstinate filter on $L$. Then by Theorem 7(iv), $\mu((x^n)^-) \geq 1 - \mu(x)$ and since $L$ is an $n$-fold obstinate $BL$-algebra, then $x^n = 0$, for all $x \in L$ and so $\mu(0^-) \geq 1 - \mu(x)$. Therefore, $\mu(x) \geq 1 - \mu(1)$.

(ii) Since $\mu$ be a fuzzy $n$-fold obstinate filter, then by Theorem 7(iv), $\mu((x^n)^-) \geq 1 - \mu(x)$. Now, let $x = 1$. Then $\mu((1^n)^-) \geq 1 - \mu(1)$ and so $\mu(0) \geq 1 - \mu(1)$. Therefore, $\mu(0) + \mu(1) \geq 1$. ■

**Note.** The following example show that there is a fuzzy $n$-fold integral filter and fuzzy $n$-fold fantastic filter such that it is not a fuzzy $n$-fold obstinate filter.

**Example 22.** Let $L$ be $BL$-algebra in Example 8. Now, let the fuzzy set $\mu$ on $L$ is defined by

$$\mu(1) = t_2,\ \mu(b) = \mu(a) = \mu(0) = t_1\ (0 \leq t_1 \leq t_2 < 0.5 \leq 1).$$

It is easy to check that $\mu$ is a fuzzy 3-fold fantastic filter and fuzzy 3-fold integral filter and since $L$ is an 3-fold obstinate $BL$-algebra and $\mu(1) < 0.5$, then by Theorem 21, $\mu$ is not a fuzzy 3-fold Obstinate filter.

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