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FOUR-PART SEMIGROUPS - SEMIGROUPS OF BOOLEAN OPERATIONS

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Abstract

Four-part semigroups form a new class of semigroups which became important when sets of Boolean operations which are closed under the binary superposition operation $f + g := f(g, \ldots, g)$, were studied. In this paper we describe the lattice of all subsemigroups of an arbitrary four-part semigroup, determine regular and idempotent elements, regular and idempotent subsemigroups, homomorphic images, Green's relations, and prove a representation theorem for four-part semigroups.

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1. Introduction

Definition [2]**.** Let

$S_1 = \{a_{11}, a_{12}, \ldots, a_{1n_r}\}$,
$S_2 = \{a_{21}, a_{22}, \ldots, a_{2n_r}\}$,
$S_3 = \{a_{31}, a_{32}, \ldots, a_{3n_s}\}$, where $a^* \in S_3$ is a fixed element,
$S_4 = \{a_{41}, a_{42}, \ldots, a_{4n_s}\}$, where $a^{**} \in S_4$ is a fixed element,

be four non-empty, finite and pairwise disjoint sets and let $S = S_1 \cup S_2 \cup S_3 \cup S_4$. We define a binary operation $*$ on S by

$$a_{ij} * a_{lk} = \begin{cases} a_{lk} & \text{if } a_{ij} \in S_1 \\ a_{tk} & \text{if } a_{ij} \in S_2 \text{ where } t = \begin{cases} 1 & \text{if } l = 2 \\ 2 & \text{if } l = 1 \\ 3 & \text{if } l = 4 \\ 4 & \text{if } l = 3 \end{cases} \\ a^* \in S_3 & \text{if } a_{ij} \in S_3 \\ a^{**} \in S_4 & \text{if } a_{ij} \in S_4. \end{cases}$$

The semigroup $(S; *)$ is said to be a four-part semigroup.

Remark 1.

1. It is easy to see that the binary operation $*$ is well-defined and associative. Therefore $(S; *)$ is a finite semigroup. Since the sets S_1 and S_2 have the same cardinality, as do S_3 and S_4, any four-part semigroup has even cardinality. Four-part semigroups were introduced by R. Butkote ([1]) (see also [2]) to give an abstract description of the semigroup $(O^n(\{0,1\}); +)$ of all n-ary Boolean operations for $n \geq 1$, where $f + g := f(g, \ldots, g)$, $f, g \in O^n(\{0,1\})$ is the n-ary Boolean operation which is defined by $(f+g)(a_1, \ldots, a_n) := f(g(a_1, \ldots, a_n), \ldots, g(a_1, \ldots, a_n))$. The sets S_1, S_2, S_3 and S_4 are then the following collections of Boolean operations : $C_4^n := \{f \in O^n(A) | f(0, \ldots, 0) = 0 \text{ and } f(1, \ldots, 1) = 1\}$, $\neg C_4^n := \{f \in O^n(A) | f(0, \ldots, 0) = 1 \text{ and } f(1, \ldots, 1) = 0\}$ (the notation $\neg C_4^n$ means that each element of this set is the negation of an element of C_4^n), $K_0^n := \{f \in O^n(A) | f(0, \ldots, 0) = 0 \text{ and } f(1, \ldots, 1) = 0\}$ which contains the n-ary constant operation with value 0 and $K_1^n := \{f \in O^n(A) | f(0, \ldots, 0) = 1 \text{ and } f(1, \ldots, 1) = 1\}$. K_1^n contains the n-ary constant operation with value 1. Each element of K_1^n is the negation of some element of K_0^n. Therefore, instead of K_1^n one could also write $\neg K_0^n$. Clearly, $O^n(\{0,1\}) = C_4^n \cup \neg C_4^n \cup K_0^n \cup K_1^n$ is the disjoint union of these sets and it is not difficult to see that $(O^n(\{0,1\}); +)$ is a four-part semigroup

since the operation + satisfies

$$f+g=\begin{cases} g & \text{if } f\in C_4^n \\ \neg g & \text{if } f\in \neg C_4^n \\ c_0^n & \text{if } f\in K_0^n \\ c_1^n & \text{if } f\in \neg K_0^n. \end{cases}$$

Our aim is to determine the semigroup-theoretical properties of four-part semigroups. This can be applied to determine the properties of the semigroup $(O^n(\{0,1\});+)$.

2. To get a semigroup not necessarily all of the sets S_1, S_2, S_3, S_4 have to be non-empty. We analyze all possible cases where at least one of our sets is empty. Clearly, $S_1=\emptyset$ iff $S_2=\emptyset$ and $S_3=\emptyset$ iff $S_4=\emptyset$. Therefore except the case that none of the sets S_1, S_2, S_3, S_4 is the empty set, we have three more cases:

1. $S_1=S_2=\emptyset$, $S_3\neq\emptyset, S_4\neq\emptyset$,
2. $S_3=S_4=\emptyset, S_1\neq\emptyset, S_2\neq\emptyset$,
3. $S_1=S_2=S_3=S_4=\emptyset$.

In the first case we have $S=S_3\cup S_4$ with

$$a_{ij}*a_{lk}=\begin{cases} a^* & \text{if } a_{ij}\in S_3 \\ a^{**} & \text{if } a_{ij}\in S_4 \end{cases}$$

and in the second case we have $S=S_1\cup S_2$ with

$$a_{ij}*a_{lk}=\begin{cases} a_{lk} & \text{if } a_{ij}\in S_1 \\ a_{1k} & \text{if } a_{ij}\in S_2 \text{ and } l=2 \\ a_{2k} & \text{if } a_{ij}\in S_2 \text{ and } l=1. \end{cases}$$

2. Subsemigroups of four-part semigroups

To study subsemigroups of four-part semigroups we define the following kinds of semigroups:

Definition. A semigroup $\mathcal{S}=(S;*)$ is called a constant semigroup if there is an element $b^*\in S$ such that $a*b=b^*$ for any $a,b\in S$, a right-zero constant semigroup if there are two disjoint non-empty sets S_1, S_2 such that $S=S_1\cup S_2$ and there is a fixed element $b^*\in S_2$ such that

$$a*b=\begin{cases} b & \text{if } a\in S_1 \\ b^* & \text{if } a\in S_2, \end{cases}$$

a two-constant semigroup if there are two disjoint non-empty sets S_1, S_2 such that $S = S_1 \cup S_2$ and there are two fixed elements $b^* \in S_1$ and $b^{**} \in S_2$ such that

$$a * b = \begin{cases} b^* & \text{if } a \in S_1 \\ b^{**} & \text{if } a \in S_2, \end{cases}$$

a right-zero two-constant semigroup if there are subsets S_1, S_2, S_3 of S such that $S = \bigcup_{i=1}^{3} S_i$, $S_i \neq \emptyset$, $S_i \cap S_j = \emptyset$ for $i \neq j \in \{1, 2, 3\}$ and there are distinguished elements $b^* \in S_2$ and $b^{**} \in S_3$ such that

$$a * b = \begin{cases} b & \text{if } a \in S_1 \\ b^* & \text{if } a \in S_2 \\ b^{**} & \text{if } a \in S_3, \end{cases}$$

a right-zero φ-semigroup if there is a fixed point free bijective mapping $\varphi : S \to S$ with $\varphi \circ \varphi = id$ and there are two disjoint sets S_1, S_2 of S such that $S = S_1 \cup S_2$ and

$$a * b = \begin{cases} b & \text{if } a \in S_1 \\ \varphi(b) & \text{if } a \in S_2. \end{cases}$$

Lemma 2. *Let $\mathcal{S}$ be a four-part semigroup. Then there is a fixed point free bijective mapping $\varphi : S \to S$ such that $\varphi \circ \varphi = id$, $\varphi(a^*) = a^{**}$, $\varphi(a^{**}) = a^*$, $\varphi(a_{1j}) = a_{2j}$, $\varphi(a_{2j}) = a_{1j}$, $\varphi(a_{3k}) = a_{4k}$ and $\varphi(a_{4k}) = a_{3k}$ for $j = 1, \ldots, n_r$ and $k = 1, \ldots, n_s$.*

Proof. We can define a bijective mapping $\varphi : S \to S$ by definition $\varphi(a_{1j}) = a_{2j}$, $\varphi(a_{2j}) = a_{1j}$, $j = 1, \ldots, n_r$ and $\varphi(a_{3k}) = a_{4k}$ and $\varphi(a_{4k}) = a_{3k}$, $k = 1, \ldots, n_s$ and $\varphi(a^*) = a^{**}$, $\varphi(a^{**}) = a^*$. It is easy to see that φ is a fixed point free bijection satisfying $\varphi \circ \varphi = id$. ■

Lemma 3. *Let S be a four-part semigroup and let $\mathcal{H} \subseteq \mathcal{S}$ be a subsemigroup with $H = H_1 \cup H_2 \cup H_3 \cup H_4$, $H_i \subseteq S_i$, $i = 1, 2, 3, 4$. Then we have*

(i) *If $H_2 \neq \emptyset$, then $H_1 \neq \emptyset$.*

(ii) *If $H_2 \neq \emptyset$, then $H_3 \neq \emptyset$ if and only if $H_4 \neq \emptyset$.*

Proof. (i) Let $H_2 \neq \emptyset$ and $a_{2j} \in H_2 \subseteq S_2$. Then $a_{2j} * a_{2j} = a_{1j} \in S_1 \cap H = H_1$, i.e $H_1 \neq \emptyset$.

(ii) Let $H_2 \neq \emptyset$ and $H_3 \neq \emptyset$ and let $a_{2j} \in H_2$ and $a_{3k} \in H_3$. Then $a_{2j} * a_{3k} = a_{4k} \in S_4 \cap H = H_4$, i.e., $H_4 \neq \emptyset$ and if $a_{4k} \in H_4$, then $a_{2j} * a_{4k} = a_{3k} \in S_3 \cap H = H_3$. ■

Lemma 4. *Let $\mathcal{S}$ be a four-part semigroup and let $\mathcal{H} \subseteq \mathcal{S}$ be a subsemigroup of $\mathcal{S}$.*

(i) *If $H \cap S_2 \neq \emptyset$, then H is a four-part semigroup or a right-zero φ-semigroup.*

(ii) *If $H \cap S_2 = \emptyset$, then $\mathcal{H}$ is a right-zero, a constant, a right-zero constant, a two-constant or a right-zero two-constant semigroup.*

Proof. (i) Because of $H \subseteq S_1 \cup S_2 \cup S_3 \cup S_4$ we can write $H = (S_1 \cap H) \cup (S_2 \cap H) \cup (S_3 \cap H) \cup (S_4 \cap H)$. If $H \cap S_2 \neq \emptyset$, then $H \cap S_1 \neq \emptyset$ by Lemma 3. We consider two cases:

1. $S_3 \cap H = \emptyset$. Then also $S_4 \cap H = \emptyset$ by Lemma 3 and $H = (S_1 \cap H) \cup (S_2 \cap H)$ and with a bijection $\varphi : S_1 \cup S_2 \to S_1 \cup S_2$ defined by $\varphi(a_{1j}) = a_{2j}$ and $\varphi(a_{2j}) = a_{1j}$ for all $j \in \{1, 2, \ldots, n_r\}$ we obtain $\varphi \circ \varphi = id$ on S and

$$a_{ij} * a_{lk} = \begin{cases} a_{lk} & \text{if } a_{ij} \in S_1 \\ \varphi(a_{lk}) & \text{if } a_{ij} \in S_2. \end{cases}$$

   The restriction of φ on H satisfies $\varphi \circ \varphi = id$ on H and

$$a_{ij} * a_{lk} = \begin{cases} a_{lk} & \text{if } a_{ij} \in H_1 \\ \varphi(a_{lk}) & \text{if } a_{ij} \in H_2. \end{cases}$$

   Therefore H is a right-zero φ-semigroup.

2. $S_3 \cap H \neq \emptyset$. Then by Lemma 3 also $S_4 \cap H \neq \emptyset$ and H is the union of four pairwise disjoint non-empty sets. Since $\mathcal{H}$ is a subsemigroup by the definition of $*$, we get $a^*, a^{**} \in H$ and we have

$$a_{ij} * a_{lk} = \begin{cases} a_{lk} & \text{if } a_{ij} \in S_1 \cap H \\ a_{tk} & \text{if } a_{ij} \in S_2 \cap H \quad \text{where } t = \begin{cases} 1 & \text{if } l = 2 \\ 2 & \text{if } l = 1 \\ 3 & \text{if } l = 4 \\ 4 & \text{if } l = 3 \end{cases} \\ a^* \in S_3 & \text{if } a_{ij} \in S_3 \cap H \\ a^{**} \in S_4 & \text{if } a_{ij} \in S_4 \cap H. \end{cases}$$

   This shows that $\mathcal{H}$ is a four-part semigroup.

(ii) If $H \cap S_2 = \emptyset$, then $H = (S_1 \cap H) \cup (S_3 \cap H) \cup (S_4 \cap H)$. Now we have the following cases:

1. $H \cap S_3 = H \cap S_4 = \emptyset$. Then $H = H \cap S_1$ forms a right-zero semigroup.
2. $H \cap S_1 = H \cap S_4 = \emptyset$ or $H \cap S_1 = H \cap S_3 = \emptyset$. Then $\mathcal{H}$ is a constant semigroup.
3. $H \cap S_3 = \emptyset$ or $H \cap S_4 = \emptyset$. Then $\mathcal{H}$ is a right-zero constant semigroup.
4. $H \cap S_1 = \emptyset$. Then $\mathcal{H}$ is a two-constant semigroup.
5. $H \cap S_1 \neq \emptyset$, $H \cap S_3 \neq \emptyset$, $H \cap S_4 \neq \emptyset$. Then $\mathcal{H}$ is a right-zero two-constant semigroup. ∎

Lemma 4 answers to the question which kinds of semigroups can occur as subsemigroups of a four-part semigroup. Now we characterize the different cases.

Proposition 5. *Let $\mathcal{S}$ be a four-part semigroup with constant elements a^* and a^{**}. Then a subset $H \subseteq S$ is the universe of a four-part semigroup with constant elements b^* and b^{**} if and only if*

(i) *$H \cap S_2 \neq \emptyset$, $b^* = a^*$, $b^{**} = a^{**}$ and*

(ii) *H is closed under φ, that is $\varphi(a) \in H$ for all $a \in H$.*

Proof. Assume that the two conditions are satisfied. Then by Lemma 3 and condition (i), $H \cap S_1 \neq \emptyset$ and hence $H \cap S_i \neq \emptyset$ for all $i = 1, 2, 3, 4$ and we define $H_1 := H \cap S_1$, $H_2 := H \cap S_2$, $H_3 := H \cap S_3$, $H_4 := H \cap S_4$. Clearly, $H = H_1 \cup H_2 \cup H_3 \cup H_4$ and $H_i \cap H_j = \emptyset$ for $i \neq j$ and each of these sets is nonempty. We show that H is closed under the multiplication of $\mathcal{S}$ and therefore a subsemigroup. If $a_{1j} \in H_1$ and $b \in H$ is arbitrary, then $a_{1j} * b = b$ since $a_{1j} \in S_1$. If $a_{2j} \in H_2$ and $b \in H$, then $a_{2j} * b = \varphi(b) \in H$ by $\varphi(H) \subseteq H$ and if $a_{4k} \in H_4$, then $a_{2j} * a_{4k} = a_{3k} = \varphi(a_{4k}) \in H$ by $\varphi(H) \subseteq H$. If $a_{3k} \in H_3$, then $a_{3k} * b = a^* \in H$ for any $b \in H$ and if $a_{4k} \in H_4$, then $a_{4k} * b = a^{**} \in H$ for any $b \in H$. This shows that $\mathcal{H}$ is a four-part semigroup.

Assume now conversely, that $\mathcal{H}$ is a four-part subsemigroup of $\mathcal{S}$. We want to show that (i) and (ii) are satisfied. We know that $H = H_1 \cup H_2 \cup H_3 \cup H_4$, $H_i \neq \emptyset$, $H_i \cap H_j = \emptyset$ for $i \neq j$ and $b^* \in H_3$ and $b^{**} \in H_4$ are the constant elements. We need the following

Claim. $H_1 \subseteq S_1, H_2 \subseteq S_2, H_3 \subseteq S_3, H_4 \subseteq S_4$.

Proof of the Claim. Assume that $H_1 \not\subseteq S_1$. Then there is an element $a_{1j} \in H_1$ but $a_{1j} \notin S_1$ and we form $a_{1j} *_H a_{1j} = a_{1j} \in H_1$ using the multiplication $*_H$ of $\mathcal{H}$. But this has to be equal to $a_{1j} *_S a_{1j}$ using the multiplication of $\mathcal{S}$. Since $a_{1j} \notin S_1$, there are the following possibilities for a_{1j}, i.e., $a_{1j} \in H_1 \cap S_2$ or $a_{1j} \in H_1 \cap S_3$ or $a_{1j} \in H_1 \cap S_4$.

1. $a_{1j} \in S_2$, then $a_{1j} *_S a_{1j} = a_{2j} \in S_2$. But $a_{2j} \neq a_{1j}$, a contradiction.

2. $a_{1j} \in S_3$, then $a_{1j} *_H b = b$ for any $b \in H$, but $a_{1j} *_S b = a^* \in S_3$. This gives a contradiction for any $b \in H$.

3. $a_{1j} \in S_4$, then $a_{1j} *_H b = b$ for any $b \in H$, but $a_{1j} *_S b = a^{**} \in S_4$ for any $b \in H$, a contradiction.

These contradictions show that $H_1 \subseteq S_1$.

Assume that $H_2 \not\subseteq S_2$. Then there is an element $a_{2j} \in H_2$, but $a_{2j} \notin S_2$. We form $a_{2j} *_H a_{2j} = a_{1j}$ and this has to be equal to $a_{2j} *_S a_{2j}$. Here we have the following possibilities:

1. $a_{2j} \in S_1$, then $a_{2j} *_S a_{2j} = a_{2j} \neq a_{1j}$.

2. $a_{2j} \in S_3$, then $a_{2j} *_S a_{2j} = a^* \neq a_{1j}$.

3. $a_{2j} \in S_4$, then $a_{2j} *_S a_{2j} = a^{**} \neq a_{1j}$.

This contradiction shows that $H_2 \subseteq S_2$.

Assume that $H_3 \not\subseteq S_3$. Then there is an element $a_{3j} \in H_3$, but $a_{3j} \notin S_3$. If $a_{3j} \in S_1$, then $a_{3j} *_H b = b^*$ for any $b \in H$, but $a_{3j} *_S b = b$, i.e., $b = b^*$ for any $b \in H$, a contradiction. If $a_{3j} \in S_2$, then $a_{3j} *_H b = b^*$ for any $b \in H$, $b^* \in H_3$ but $a_{3j} *_S b = \varphi(b)$, i.e., $\varphi(b) = b^*$ for any b, a contradiction. If $a_{3j} \in S_4$, then $b *_H a_{3j} = a_{4j} = \varphi(a_{3j}) \in S_4$, but $b *_S a_{3j} = \varphi(a_{3j}) \notin S_4$ for any $b \in H_2 \subseteq S_2$.

$H_4 \subseteq S_4$ can be proved in a similar way as $H_3 \subseteq S_3$. This finishes the proof of this claim.

$H_2 \subseteq S_2$ and $H_2 \neq \emptyset$ show that $H \cap S_2 \neq \emptyset$. Moreover, since $b^* \in H_3 \subseteq S_3$ and $b^{**} \in H_4 \subseteq S_4$, we have $b^* = a_{3j} *_H b = a_{3j} *_S b = a^*$ and $b^{**} = a_{4j} *_H b = a_{4j} *_S b = a^{**}$ for any $a_{3j} \in H_3, a_{4j} \in H_4$ and $b \in H$. Since $H_2 \subseteq S_2$, then for every $a \in H$ we have $\phi(a) = a_{2j} *_H a \in H$ for $a_{2j} \in H_2$, that means (ii) is satisfied. This completes the proof. ∎

Proposition 6. *let $\mathcal{S}$ be a four-part semigroup. Then a subset $H \subseteq S$ is the universe of a right-zero two-constant subsemigroup of $\mathcal{S}$ if and only if $H \subseteq S_1 \cup S_3 \cup S_4$, $H \cap S_1 \neq \emptyset$ and there are two fixed elements b^*, b^{**}, $b^* \neq b^{**}$ with $\{a^*, a^{**}\} = \{b^*, b^{**}\}$.*

Proof. Let H be the universe of a right-zero two-constant subsemigroup of $\mathcal{S}$. We show first that $H \cap S_2 = \emptyset$. Indeed, if $H \cap S_2 \neq \emptyset$, then there is an element $a_{2j} \in H \cap S_2$ for some $j \in \{1, \ldots, n_r\}$ and then $a_{2j} *_S b = \varphi(b)$ for any $b \in H$ with

$\varphi(b) \neq b$ and b can be chosen in such a way that $\varphi(b) \neq b^* \in H_2$, $\varphi(b) \neq b^{**} \in H_3$. But by the definition of a right-zero two-constant semigroup we must have

$$a_{2j} *_S b = a_{2j} *_H b = \begin{cases} b & \text{if } a_{2j} \in H_1 \\ b^* & \text{if } a_{2j} \in H_2 \\ b^{**} & \text{if } a_{2j} \in H_3, \end{cases}$$

a contradiction. This shows that $H \subseteq S_1 \cup S_3 \cup S_4$. Now we show that $H \cap S_1 \neq \emptyset$. If $H \cap S_1 = \emptyset$, then $H \subseteq S_3 \cup S_4$ and then

$$a_{ij} *_S a_{lk} = \begin{cases} b^* & \text{if } a_{ij} \in S_3 \\ b^{**} & \text{if } a_{ij} \in S_4 \end{cases}$$

for all $a_{ij}, a_{lk} \in H$. But if $a_{ij} \in H_1 \neq \emptyset$, $a_{ij} \in S_3$, then we have $a_{ij} *_S b^{**} = b^* \neq b^{**} = a_{ij} *_H b^{**}$, a contradiction. If $a_{ij} \in H_1 \neq \emptyset$, but $a_{ij} \in S_4$ we have $a_{ij} *_S b^* = b^{**} \neq b^* = a_{ij} *_H b^*$. This shows that $H \cap S_1 \neq \emptyset$. Since $b^* \in H_2$ we have $b^* *_H b^{**} = b^*$. Since $H_2 \subseteq S_1 \cup S_3 \cup S_4$, we consider the following three possibilities for b^*:

1. $b^* \in S_1$, then $b^* *_S b^{**} = b^{**} \neq b^*$, a contradiction.

2. $b^* \in S_3$, then $b^* *_S b^{**} = a^*$ and thus $a^* = b^*$ or

3. $b^* \in S_4$, then $b^* *_S b^{**} = a^{**} = b^*$.

Therefore $\{a^*, a^{**}\} = \{b^*, b^{**}\}$.

Conversely, assume that $H \subseteq S$ is a subset of the universe S of a four-part semigroup $\mathcal{S}$ with $H \subseteq S_1 \cup S_3 \cup S_4$, $H \cap S_1 \neq \emptyset$ and that there are two elements $b^*, b^{**} \in H$ satisfying $b^* \neq b^{**}$ and $\{a^*, a^{**}\} = \{b^*, b^{**}\}$. We show that $\mathcal{H}$ is a right-zero two-constant subsemigroup of $\mathcal{S}$. We show that H is closed under the multiplication in S. If $a \in H \cap S_1$, then for any $b \in H$ we have $a *_S b = b \in H$ and in all other cases we get elements in $\{a^*, a^{**}\}$, which are in H since $\{a^*, a^{**}\} = \{b^*, b^{**}\}$. Therefore H is a subsemigroup of $\mathcal{S}$ with $H \cap S_1 \neq \emptyset$. Further we have $H \cap S_3 \neq \emptyset$ and $H \cap S_4 \neq \emptyset$ since $\{b^*, b^{**}\} \subseteq H$. Now we set $H_1 := H \cap S_1, H_2 := H \cap S_3, H_3 := H \cap S_4$. Then $H = H_1 \cup H_2 \cup H_3$. We have to show that $b^* \in H_2$ and $b^{**} \in H_3$ or conversely and that $*_S|_H$ is the multiplication of a right-zero two-constant semigroup. If $a \in H_2$, then we have $a *_S b = a *_H b = a^* \in H \cap S_3 = H_2$ for all $b \in H$ and if $a \in H_3$, then we have $a *_S b = a *_H b = a^{**} \in H \cap S_4 = H_3$ for all $b \in H$. Now we can set $a^* = b^* \in H_2$ or $b^* = a^{**}$ or conversely. If $a \in H_1 = S_1 \cap H$, then $a *_H b = a *_S b = b$ for all $b \in H$. This shows that $\mathcal{H}$ is a right-zero two-constant subsemigroup of $\mathcal{S}$. ■

Proposition 7. *A subset $H \subseteq S$ of the universe of a four-part semigroup is the universe of a two-constant subsemigroup of $\mathcal{S}$ if and only if $H \subseteq S_3 \cup S_4$ and there are two elements $b^*, b^{**} \in H$, $b^* \neq b^{**}$ such that $\{a^*, a^{**}\} = \{b^*, b^{**}\}$.*

Proof. If $\mathcal{H}$ is a two-constant semigroup, then $H \cap S_1 = \emptyset$ and $H \cap S_2 = \emptyset$ since, if $a_{ij} \in S_1$, then $a_{ij} *_S b^* = b^*$ and $a_{ij} *_S b^{**} = b^{**}$, $a_{ij} \in H \cap S_1$ means $a_{ij} \in H_1$ or $a_{ij} \in H_2$. In the first case we have $a_{1j} *_H b^{**} = b^*$ and in the second case, $a_{1j} *_H b^* = b^{**}$. In both cases, we have a contradiction. If $a_{2j} \in H \cap S_2$, then again we have $a_{2j} \in H_1$ or $a_{2j} \in H_2$ and $a_{2j} *_S b^* = \varphi(b^*) = b^{**}$ and $a_{2j} *_S b^{**} = \varphi(b^{**}) = b^*$. If $a_{2j} \in H_1$, then $a_{2j} *_H b^* = b^*$ and if $a_{2j} \in H_2$, then $a_{2j} *_H b^{**} = b^{**}$. In both cases we have a contradiction. This shows $H \subseteq S_3 \cup S_4$. The second condition is clear. Assume now that H is a subset of $S_3 \cup S_4$. We define $H_1 := H \cap S_3$ and $H_2 := H \cap S_4$, $b^* := a^*, b^{**} := a^{**}$. If $a \in H_1$, then $a * b = b^* = a^* \in H$ and if $a \in H_2$, then $a *_S b = b^{**} \in H$ for all $b \in H$. This shows that $\mathcal{H}$ is a two-constant semigroup. ∎

Proposition 8. *Let $H \subseteq S$ be a subset of the universe of a subsemigroup of the four-part semigroup $\mathcal{S}$. Then H is the universe of a right-zero constant subsemigroup of $\mathcal{S}$ if and only if either $H \subseteq S_1 \cup S_3, H \cap S_1 \neq \emptyset$ and $a^* \in H$ or $H \subseteq S_1 \cup S_4, H \cap S_1 \neq \emptyset$ and $a^{**} \in H$.*

Proof. Assume that $\mathcal{H}$ is a right-zero constant subsemigroup of $\mathcal{S}$.

Claim.

(i) Either $H \cap S_3 = \emptyset$ or $H \cap S_4 = \emptyset$.

(ii) $H \not\subseteq S_1, H \not\subseteq S_2, H \not\subseteq S_3, H \not\subseteq S_4$.

Proof of Claim. (i) For the fixed element b^* and for all $a \in H$ we have $a * b^* = b^*$. If $a \in S_3$, then $a *_S b^* = a^*$ and then $a^* = b^*$ and for $a \in S_4$ we have $a *_S b^* = a^{**} = b^*$. Thus if $S_3 \cap H \neq \emptyset$, then $S_4 \cap H = \emptyset$ and if $S_4 \cap H \neq \emptyset$, then $S_3 \cap H = \emptyset$ and this proves (i).

(ii) We use that $|H| \geq 2$. Assume that $H \subseteq S_1$, then for all $a, b \in H$ we have $a *_H b = b$ and if $a \in H_2$ we get $a *_S b = b^*$, i.e., $b = b^*$ for all $b \in S$, a contradiction, which shows $H \not\subseteq S_1$. Assume that $H \subseteq S_2$, then $a *_H b = \varphi(b)$ for all $a \in H$ and $a *_S b = b$ if $a \in H_1$, i.e., $\varphi(b) = b$ for all $b \in H$, a contradiction. Assume that $H \subseteq S_3$. If $a \in H_1$, then $a *_H b = b = a^* = a *_S b$ for all $b \in H$, i.e., $|H| = 1$, a contradiction. In a similar way we show that $H \not\subseteq S_4$.

Now we prove that $\mathcal{H} \subseteq \mathcal{S}$ is a right-zero constant subsemigroup if the conditions are satisfied. We show that $H \subseteq S$ is closed under $*_S$. Assume that $a \in S_1 \cap H$, then $a *_S b = b \in H$. If $a \in H \cap S_3$, then $a *_S b = a^* \in H$. In the second case we conclude in the same way. We set $H_1 = H \cap S_1 \neq \emptyset$

and $H_2 = H \cap S_3 \neq \emptyset$ and $b^* = a^*$ since $a^* \in H \cap S_3$ in the first case and $H_1 := H \cap S_1 \neq \emptyset$ and $H_2 = H \cap S_4 \neq \emptyset$ and $b^* = a^{**}$ in the second one. Then all conditions for a right-zero subsemigroup are satisfied. Now we assume that $\mathcal{H}$ is a right-zero constant semigroup. By the claim we have $H \subseteq S_1 \cup S_2 \cup S_4$ or $H \subseteq S_1 \cup S_2 \cup S_3$. We know also that $H \not\subseteq S_1, H \not\subseteq S_2, H \not\subseteq S_3, H \not\subseteq S_4$. Note that H cannot contain an element b from S_3 together with an element $a \in S_2$ since otherwise $a *_H b = \varphi(b) \in S_4 \cap H$ which contradicts the claim. A similar argument shows that H cannot contain an element from S_4 together with an element from S_2. Then for H we have precisely the following cases:

1. $H \subseteq S_3 \cup S_1, H \cap S_3 \neq \emptyset, H \cap S_1 \neq \emptyset$ or
2. $H \subseteq S_4 \cup S_1, H \cap S_4 \neq \emptyset, H \cap S_1 \neq \emptyset$ or
3. $H \subseteq S_1 \cup S_2, H \cap S_1 \neq \emptyset, H \cap S_2 \neq \emptyset$.

If $a \in S_2, b^* \in S_1$, then $a *_S b^* = \varphi(b^*) = b^* = a *_H b^*$, a contradiction and for $a \in H, b^* \in S_2$ we get $a *_S b^* = b^*$, which is also a contradiction. Therefore the third case can be excluded. In the first case from an element in $H \cap S_1 \neq \emptyset$ and an element from $H \cap S_3 \neq \emptyset$ we can produce a^*, namely by $a *_H b = a^*$ and have $a^* \in H$. In the second case we get $a^{**} \in H$. By the claim both conditions exclude each other. ■

Proposition 9. *Let $H \subseteq S$ be a subset of the universe of a subsemigroup of a four-part semigroup $\mathcal{S}$. Then H is the universe of a right-zero semigroup if and only if $H \subseteq S_1$ or $H = \{a^*\}$ or $H = \{a^{**}\}$.*

Proof. Assume that $H \subseteq S_1$ or $\{a^*\}$ or $\{a^{**}\}$. Then H is closed under the multiplication of $\mathcal{S}$ and forms a right-zero semigroup. Conversely, let $\mathcal{H}$ be a right-zero subsemigroup of $\mathcal{S}$. Assume that $H \not\subseteq S_1$, then there exists $a \in H \cap (S_2 \cup S_3 \cup S_4)$. But if $a \in S_2$ then we have $a *_H a = a \neq \varphi(a) = a *_S a$, a contradiction. Therefore $H \subseteq S_3 \cup S_4$. We show that $H \cap S_3 = \{a^*\}$ and $H \cap S_4 = \{a^{**}\}$. Let $a \in H \cap S_3, a \neq a^*$. Then $a *_H a = a^*$ which contradicts the definition of a right-zero semigroup. Therefore $H \cap S_3 = \{a^*\}$. Similarly we obtain $H \cap S_4 = \{a^{**}\}$. Hence $H \subseteq \{a^*, a^{**}\}$ and we obtain $H = \{a^*\}$ or $H = \{a^{**}\}$ or $H = \{a^*, a^{**}\}$. The latter case is impossible since otherwise $a^* *_H a^{**} = a^* *_S a^{**} = a^*$. ■

Proposition 10. *Let $H \subseteq S$ be a subset of the universe of a subsemigroup of a four-part semigroup $\mathcal{S}$. Then H is the universe of a constant semigroup if and only if $H \subseteq S_3$ and $a^* \in H$ or $H \subseteq S_4$ and $a^{**} \in H$ or $H = \{a\}, a \in S_1$.*

Proof. If the condition is satisfied, then $a *_S b = a^* \in H$ if $a, b \in H \cap S_3$ or $a *_S b = a^{**}$ if $a, b \in S_4 \cap H$. Therefore the set H is closed under multiplication and forms a constant subsemigroup of $\mathcal{S}$. If $\mathcal{H} \subseteq \mathcal{S}$ is a constant subsemigroup of S and $|H| \geq 2$, then $H \cap S_1 = \emptyset, H \cap S_2 = \emptyset, H \cap S_4 = \emptyset$ or $H \cap S_1 = \emptyset, H \cap S_2 = \emptyset, H \cap S_3 = \emptyset$. Indeed, if $a \in H \cap S_1$ and $b \neq a, b \in H$, then $a *_H b = b$, but $a *_H a = a \neq b$ and $\mathcal{H}$ is not a constant semigroup, therefore $H \cap S_1 = \emptyset$. Let $H \cap S_2 \neq \emptyset$ and $a \in H \cap S_2$. Then $a *_S b = \varphi(b)$ and $a *_S \varphi(b) = b$. Because of $\varphi(b) \neq b$ (φ is a fixed point free mapping) is $\mathcal{H}$ not a constant semigroup. Therefore $H \cap S_2 = \emptyset$. If $a \in H \cap S_4$ and assume that $b \in S_3$. Then $a *_S b = a^{**}$ and $b * a = a^* \in A$. Because of $a^* \neq a^{**}$, $\mathcal{H}$ cannot be constant. In the second case we conclude in a similar way. Moreover, we cannot have elements from S_3 and from S_4 since otherwise $a *_H b = a^*$ if $a \in S_3$ and $a *_H b = a^{**}$ if $a \in S_4$ and this contradicts the assumption that $\mathcal{H}$ is a constant semigroup. These equation show also that $a^* \in H$ if $H \subseteq S_3$ or $a^{**} \in H$ if $H \subseteq S_4$. If $|H| = 1$ then the only element must be idempotent, i.e $a \in S_1$ or $a \in \{a^*, a^{**}\}$. But the second case is already included in the previous cases. ∎

Proposition 11. *Let $\mathcal{S}$ be a four-part semigroup. Then a non-empty subset $H \subseteq S$ is the universe of a right-zero φ-subsemigroup of $\mathcal{S}$ if and only if $H \subseteq S_1 \cup S_2, H \cap S_1 \neq \emptyset$ and $H \cap S_2 \neq \emptyset$ and H is closed under φ, i.e., if $a \in H$, then $\varphi(a) \in H$ for all $a \in H$.*

Proof. Let H be the universe of a right-zero φ-subsemigroup of $\mathcal{S}$. We prove that $H \cap S_3 = \emptyset$ and $H \cap S_4 = \emptyset$. If $H \cap S_3 \neq \emptyset$ and $a \in S_3$, then $a *_H b = b$ if $a \in H_1$ or $a *_H b = \varphi(b)$ if $a \in H_2$, but $a *_S b = a^*$ for any $b \in H$, i.e., $b = a^*$ for any $b \in H$, a contradiction or $\varphi(b) = a^*$ for any $b \in H$, which is also a contradiction. Similarly we get a contradiction if $H \cap S_4 \neq \emptyset$. Altogether, we have $H \subseteq S_1 \cup S_2$.

Suppose that $H \cap S_1 = \emptyset$. Then $H \subseteq S_2$ and since $H \neq \emptyset$, there is an element $a \in H \cap S_2$. Then $a *_S a = a *_H a = \varphi(a)$, where φ is the idempotent, fixed point free bijective mapping from $\mathcal{S}$. Since $a \in S_2$, the image $\varphi(a)$ belongs to S_1, a contradiction. If $H \cap S_2 = \emptyset$, then $H \subseteq S_1$ and with $a \in H \cap S_1$ and $b \in H_2$ we have $b *_S a = b *_H a = \varphi(a) \in H_1$, where φ is the fixed point free, bijective mapping from $\mathcal{H}$. The element $\varphi(a)$ belongs to S and since $a \in S_1$, we have $\varphi(a) \in S_2$, a contradiction. With $b \in H_2$ for any $a \in H$ we have $b *_H a = \varphi(a) \in H$, i.e., H is closed under φ.

Since $\mathcal{H} \subseteq \mathcal{S}$ is a subsemigroup let conversely, $H \subseteq S$ be a subset which satisfies $H \subseteq S_1 \cup S_2, H \cap S_1 \neq \emptyset, H \cap S_2 \neq \emptyset$. Then we define $H_1 := H \cap S_1$ and $H_2 := H \cap S_2$ and use as fixed point free, bijective mapping from H the restriction of the corresponding mapping of H since H is closed under φ. Now we have

$$a *_H b = \begin{cases} b & \text{if } a \in H_1 \\ \varphi(b) & \text{if } a \in H_2 \end{cases}$$

and $\mathcal{H}$ is a right-zero φ-semigroup. ∎

3. Idempotent and regular subsemigroups of four-part semigroups

Proposition 12. *Let $\mathcal{S}$ be a four-part semigroup and let $a \in S$ be arbitrary. Then a is an idempotent element of $\mathcal{S}$ if and only if $a \in S_1 \cup \{a^*, a^{**}\}$.*

Proof. If $a \in S_1 \cup \{a^*, a^{**}\}$, then it is clear that $a * a = a$. Conversely, let $a \in S$ be idempotent. Assume that $a \notin S_1 \cup \{a^*, a^{**}\}$. If $a \in S_2$ then $a * a = \varphi(a) \neq a$, a contradiction. If $a \in (S_3 \cup S_4) \setminus \{a^*, a^{**}\}$, then $a * a \in \{a^*, a^{**}\}$ and thus $a * a \neq a$, a contradiction. This completes the proof. ■

Proposition 13. *Let $\mathcal{S}$ be a four-part semigroup and let $H \subseteq S$. Then $\mathcal{H}$ is an idempotent subsemigroup of $\mathcal{S}$ if and only if $H \subseteq S_1 \cup \{a^*, a^{**}\}$.*

Proof. If $H \subseteq S_1 \cup \{a^*, a^{**}\}$, then by definition $a * b \in H$ for every $a, b \in H$ and thus $\mathcal{H}$ is a subsemigroup of $\mathcal{S}$. By Proposition 12 it follows that $\mathcal{H}$ is an idempotent subsemigroup. Conversely, if $\mathcal{H}$ is an idempotent subsemigroup of $\mathcal{S}$, then by Proposition 12, $H \subseteq S_1 \cup \{a^*, a^{**}\}$. ■

Proposition 14. *Let $\mathcal{S}$ be a four-part semigroup and let $a \in S$ be arbitrary. Then a is a regular element of $\mathcal{S}$ if and only if $a \in S_1 \cup S_2 \cup \{a^*, a^{**}\}$.*

Proof. Let $a \in S_1 \cup S_2 \cup \{a^*, a^{**}\}$. If $a \in S_1$, then $(a * a) * a = a * a = a$, if $a_{2j} \in S_2$, then $(a_{2j} * a_{2j}) * a_{2j} = \varphi(a_{2j}) * a_{2j} = a_{1j} * a_{2j} = a_{2j}$ and if $a = a^*$ or $a = a^{**}$, then $a * a * a = a$. Thus any $a \in S_1 \cup S_2 \cup \{a^*, a^{**}\}$ is regular. Conversely, for arbitrary $a_{3j} \in S_3, a_{3j} \neq a^*$, $a_{4j} \in S_4, a_{4j} \neq a^{**}$ and for any $b \in S$ we have $(a_{3j} * b) * a_{3j} = a^* * a_{3j} = a^* \neq a_{3j}$ and $(a_{4j} * b) * a_{4j} = a^{**} * a_{4j} = a^{**} \neq a_{4j}$. Hence, $a \in (S_3 \cup S_4) \setminus \{a^*, a^{**}\}$ cannot be regular. Therefore if $a \in S$ is a regular element, then $a \in S_1 \cup S_2 \cup \{a^*, a^{**}\}$. ■

Proposition 15. *Let $\mathcal{S}$ be a four-part semigroup and let $H \subseteq S$. Then $\mathcal{H}$ is a regular subsemigroup of $\mathcal{S}$ if and only if $H \subseteq S_1 \cup \{a^*, a^{**}\}$ or $H \subseteq S_1 \cup S_2 \cup \{a^*, a^{**}\}$ such that $\varphi(a) \in H$ for all $a \in H$.*

Proof. If $H \subseteq S_1 \cup \{a^*, a^{**}\}$, then by Proposition 13, $\mathcal{H}$ is an idempotent subsemigroup and hence a regular subsemigroup. Now, let $H \subseteq S_1 \cup S_2 \cup \{a^*, a^{**}\}$ such that $\varphi(a) \in H$ for all $a \in H$. If $a \in S_1$, then for all $b \in H$ we have $a * b = b \in H$, if $a \in S_2$ and $b \in H$, then $a * b = \varphi(b) \in H$, if $a = a^*$ or $a = a^{**}$, then $a * b = a^* \in H$ or $a * b = a^{**}$ for all $b \in H$. Thus H is closed under multiplication and hence forms a subsemigroup and by Proposition 14, $\mathcal{H}$ is regular. Conversely, let $\mathcal{H}$ be a regular subsemigroup of $\mathcal{S}$ such that $H \not\subseteq S_1 \cup \{a^*, a^{**}\}$. Then by Proposition 14 we have $H \subseteq S_1 \cup S_2 \cup \{a^*, a^{**}\}$ and $H \cap S_2 \neq \emptyset$. Since $\mathcal{H}$ is a semigroup then for all $b \in H \cap S_2$ and $a \in H$ we have $b * a = \varphi(a) \in H$. This completes the proof. ■

4. Homomorphisms of four-part semigroups

Lemma 16. *Let $\mathcal{S} = (S;*)$ be a four-part semigroup with constant elements a^* and a^{**} and let $\mathcal{S}' = (S;*')$ be an arbitrary semigroup. Let $\phi : S \to S'$ be a homomorphism. Then the following Propositions are true for all $j, j' \in \{1, \dots, n_r\}$ and $k, k' \in \{1, \dots, n_s\}$.*

(i) *If there are $a_{1j}, a_{2j'} \in S$ such that $(a_{1j}, a_{2j'}) \in Ker\phi$, then $(a, \varphi(b)) \in Ker\phi$ for every $(a, b) \in Ker\phi$.*

(ii) *If there are $a_{1j}, a_{3k} \in S$ such that $(a_{1j}, a_{3k}) \in Ker\phi$, then ϕ is constant.*

(iii) *If there are $a_{1j}, a_{4k} \in S$ such that $(a_{1j}, a_{4k}) \in Ker\phi$, then ϕ is constant.*

(iv) *If there are $a_{2j}, a_{3k} \in S$ such that $(a_{2j}, a_{3k}) \in Ker\phi$, then ϕ is constant.*

(v) *If there are $a_{2j}, a_{4k} \in S$ such that $(a_{2j}, a_{4k}) \in Ker\phi$, then ϕ is constant.*

(vi) *If there are $a_{3k}, a_{4k'} \in S$ such that $(a_{3k}, a_{4k'}) \in Ker\phi$, then $(a^*, a^{**}) \in Ker\phi$.*

Proof. Let $\phi : S \to S'$ be a homomorphism.

(i) If $(a_{1j}, a_{2j'}) \in Ker\phi$, then $(a, \varphi(b)) = (a_{1j} * a, a_{2j'} * b) \in Ker\phi$ for every $(a, b) \in Ker\phi$.

(ii) If $(a_{1j}, a_{3k}) \in Ker\phi$, then for every $b \in S$ we have $(b, a^*) = (a_{1j} * b, a_{3k} * b) \in Ker\phi$ and therefore ϕ is constant.

(iii) If $(a_{1j}, a_{4k}) \in Ker\phi$, then for every $b \in S$ we have $(b, a^{**}) = (a_{1j} * b, a_{4k} * b) \in Ker\phi$ and therefore ϕ is constant.

(iv) If $(a_{2j}, a_{3k}) \in Ker\phi$, then $(a_{1j}, a^*) = (a_{2j} * a_{2j}, a_{3k} * a_{3k}) \in Ker\phi$ and by (ii), ϕ is constant.

(v) If $(a_{2j}, a_{4k}) \in Ker\phi$, then $(a_{1j}, a^{**}) = (a_{2j} * a_{2j}, a_{4k} * a_{4k}) \in Ker\phi$ and by (iii), ϕ is constant.

(vi) If $(a_{3k}, a_{4k'}) \in Ker\phi$, then $(a^*, a^{**}) = (a_{3k} * a_{3k}, a_{4k'} * a_{4k'}) \in Ker\phi$. ∎

Using the kernel $Ker\phi$ of a homomorphism ϕ we now give some more conditions for a homomorphism ϕ.

Theorem 17. *Let $\mathcal{S} = (S;*)$ be a four-part semigroup with constant elements a^* and a^{**} and let $\mathcal{S}' = (S;*')$ be an arbitrary semigroup. If the mapping $\phi : S \to S'$ is a homomorphism then*

(i) *ϕ is constant and maps every element of S to an idempotent element of S' or*

(ii) *ϕ satisfies $(\varphi(a), \varphi(b)) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $(a, b) \in Ker\phi$ if and only if $a, b \in S_i$ for $i = 1, 2, 3, 4$ and for any $a, b \in S$ or*

(iii) *ϕ satisfies $(a^*, a^{**}) \in Ker\phi$, $(\varphi(a), \varphi(b)) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $(a, b) \in Ker\phi$ if and only if $a, b \in S_3 \cup S_4$ or $a, b \in S_1$ or $a, b \in S_2$ for any $a, b \in S$ or*

(iv) *ϕ satisfies $(a, \varphi(b)) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $(a, b) \in Ker\phi$ if and only if $a, b \in S_1 \cup S_2$ or $a, b \in S_3 \cup S_4$ for any $a, b \in S$.*

Proof. Let $\phi : S \to S'$ be a homomorphism. Let $(a, b) \in Ker\phi$. We consider the following cases:

1. If there are $a_{1j}, a_{3k} \in S$ such that $(a_{1j}, a_{3k}) \in Ker\phi$ or there are $a_{1j}, a_{4k} \in S$ such that $(a_{1j}, a_{4k}) \in Ker\phi$ or there are $a_{2j}, a_{3k} \in S$ such that $(a_{2j}, a_{3k}) \in Ker\phi$ or there are $a_{2j}, a_{4k} \in S$ such that $(a_{2j}, a_{4k}) \in Ker\phi$, then by Lemma 16 (ii), (iii), (iv) and (v), ϕ is constant. Moreover, if ϕ maps all $a \in S$ to $c \in S'$, then $c = \phi(a * b) = \phi(a) *' \phi(b) = c *' c$, i.e., c is idempotent and we have (i).

2. If $(a_{1j}, a_{3k}), (a_{1j}, a_{4k}), (a_{2j}, a_{3k}), (a_{2j}, a_{4k}) \notin Ker\phi$ for all $j \in \{1, \ldots, n_r\}$ and for all $k \in \{1, \ldots, n_s\}$, then we consider the following subcases:

   a. If $(a_{1j}, a_{2j'}), (a_{3k}, a_{4k'}) \notin Ker\phi$ for all $j, j' \in \{1, \ldots, n_r\}$ and for all $k, k' \in \{1, \ldots, n_s\}$, then $(a, b) \in Ker\phi$ if and only if a and b are in the same set S_i for all $a, b \in S$. Moreover, $(\varphi(a), \varphi(b)) = (a_{2j} * a, a_{2j} * b) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $j \in \{1, \ldots, n_r\}$. Hence we have (ii).

   b. If $(a_{3k}, a_{4k'}) \in Ker\phi$ for some $k, k' \in \{1, \ldots, n_s\}$ and $(a_{1j}, a_{2j'}) \notin Ker\phi$ for every $j, j' \in \{1, \ldots, n_r\}$, then $(a^*, a^{**}) = (a_{3k} * a_{3k}, a_{4k'} * a_{4k'}) \in Ker\phi$. Moreover, $(\varphi(a), \varphi(b)) = (a_{2j} * a, a_{2j} * b) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $(a, b) \in Ker\phi$ if and only if $a, b \in S_3 \cup S_4$ or $a, b \in S_1$ or $a, b \in S_2$. Thus we have (iii).

   c. If there is $(a_{1j}, a_{2j'}) \in Ker\phi$ for some $j, j' \in \ldots, n_r$, then by Lemma 16 (i), $(a, \varphi(b)) \in Ker\phi$ for any $(a, b) \in Ker\phi$. Moreover, $(a, b) \in Ker\phi$ if and only if $a, b \in S_1 \cup S_2$ or $a, b \in S_3 \cup S_4$ and thus we have (iv).

The opposite direction is not true. The following easy example shows that there are mappings ϕ which satisfy (ii), but are not homomorphisms. Let $\phi : S \to \mathbb{Z}_4$ with $\mathcal{Z}_4 = (\{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}; \cdot)$ be defined by $\phi(S_1) = \bar{0}, \phi(S_2) = \bar{1}, \phi(S_3) = \bar{2}, \phi(S_4) = \bar{3}$. Then ϕ satisfies (ii) but is not a homomorphism since $\phi(a^* a^*) = \phi(a^*) = \bar{2}$, but $\phi(a^*)\phi(a^*) = \bar{2} \cdot \bar{2} = \bar{0}$. ∎

As a consequence we get the following description of congruence relations of four-part semigroups.

Proposition 18. *Let $\mathcal{S}$ be a four-part semigroup with a^* and a^{**} as the constant elements. Then the following equivalence relations are congruence relations on $\mathcal{S}$.*

(i) $\theta = S \times S$ *or*

(ii) $\theta = \theta_1 \cup \theta_2 \cup \theta_3 \cup \theta_4$ *where θ_i is an equivalence relation on S_i for all $i = 1, 2, 3, 4$ such that $(\varphi(a), \varphi(b)) \in \theta$ whenever $(a, b) \in \theta$ or*

(iii) $\theta = \theta_1 \cup \theta_2 \cup \theta_3$ *where θ_i is an equivalence relation on $S_3 \cup S_4$, on S_1 and on S_2, respectively such that $(a^*, a^{**}) \in \theta$ and $(\varphi(a), \varphi(b)) \in \theta$ whenever $(a, b) \in \theta$ or*

(iv) $\theta = \theta_1 \cup \theta_2$ *where θ_1, θ_2 are equivalence relations on $S_1 \cup S_2$ and on $S_3 \cup S_4$, respectively such that $(a, \varphi(a)) \in \theta$ for all $a \in S$.*

Now, we consider the particular case that $\mathcal{S}$ and $\mathcal{S}'$ both are four-part semigroups.

Lemma 19. *Let $\mathcal{S} = (S; *)$ and $\mathcal{S}' = (S'; *')$ be two four-part semigroups with the constant elements a^*, a^{**} and b^*, b^{**} respectively and let $\phi : S \to S'$ be an arbitrary homomorphism. Then the following Propositions hold:*

(i) *If $\phi(S_1) \nsubseteq S_1'$, then ϕ is constant and $\phi(S) \subseteq \{b^*, b^{**}\}$.*

(ii) *If $\phi(S_2) \nsubseteq S_2'$, then $(a, \varphi(a)) \in Ker\phi$ for all $a \in S$ or ϕ is constant and $\phi(S) \subseteq \{b^*, b^{**}\}$.*

(iii) *If $\phi(S_3) \nsubseteq S_3'$, then $\phi(a^*) = b^{**}$ or ϕ is constant and $\phi(S) \subseteq S_1' \cup \{b^*, b^{**}\}$.*

(iv) *If $\phi(S_4) \nsubseteq S_4'$, then $\phi(a^{**}) = b^*$ or ϕ is constant and $\phi(S) \subseteq S_1' \cup \{b^*, b^{**}\}$.*

Proof. Let $\phi : S \to S'$ be a homomorphism.

(i) Let $a \in S_1$ such that $\phi(a) \notin S_1'$. Then for all $b \in S$ we have $b = a * b$ and $\phi(b) = \phi(a * b) = \phi(a) *' \phi(b)$. If $\phi(a) \in S_2'$, then we have $\phi(b) = \phi(a) *' \phi(b) = \varphi'(\phi(b))$, a contradiction. If $\phi(a) \in S_3'$ or $\phi(a) \in S_4'$, then $\phi(b) = \phi(a) *' \phi(b) = b^*$ or $\phi(b) = \phi(a) *' \phi(b) = b^{**}$, i.e., ϕ is constant and $\phi(S) \subseteq \{b^*, b^{**}\}$.

(ii) Let $a_{2j} \in S_2$ such that $\phi(a_{2j}) \notin S_2'$. Then for all $a \in S$, we have $\varphi(a) = a_{2j} * a$ and $\phi(\varphi(a)) = \phi(a_{2j}) *' \phi(a)$. If $\phi(a_{2j}) \in S_1'$, then $\phi(\varphi(a)) = \phi(a_{2j}) *' \phi(a) = \phi(a)$, i.e., $(a, \varphi(a)) \in Ker\phi$. If $\phi(a_{2j}) \in S_3'$ or $\phi(a_{2j}) \in S_4'$, then $\phi(\varphi(a)) = \phi(a_{2j}) *' \phi(a) = b^*$ or $\phi(\varphi(a)) = \phi(a_{2j}) *' \phi(a) = b^{**}$, i.e., ϕ is constant and $\phi(S) \subseteq \{b^*, b^{**}\}$.

(iii) Let $a_{3j} \in S_3$ such that $\phi(a_{3j}) \notin S_3'$. Then for all $a \in S$ we have $a_{3j} * a = a^*$ and therefore $\phi(a_{3j}) *' \phi(a) = \phi(a_{3j} * a) = \phi(a^*)$. If $\phi(a_{3j}) \in S_1'$, then $\phi(a) = \phi(a^*)$, i.e., ϕ is a constant homomorphism such that $\phi(S) \subseteq S_1' \cup \{b^*, b^{**}\}$. If $\phi(a_{3j}) \in S_2'$, then $\phi(a^*) = \phi(a_{3j}) *' \phi(a) = \varphi'(\phi(a))$ and hence $\phi(a^*) = \varphi'(\phi(a))$.

But this is not possible for $a = a^*$ and therefore we have a contradiction. If $\phi(a_{3j}) \in S'_4$, then $b^{**} = \phi(a_{3j}) *' \phi(a) = \phi(a^*)$.

(iv) If there is $a_{4j} \in S_4$ such that $\phi(a_{4j}) \notin S'_4$, then in the same way as in (iii), we have that ϕ is a constant homomorphism such that $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$ or $\phi(a^{**}) = b^*$. ■

Lemma 20. *Let $\mathcal{S} = (S; *)$ and $\mathcal{S}' = (S'; *')$ be two four-part semigroups with the constant elements a^*, a^{**} and b^*, b^{**} respectively and let $\phi : S \to S'$ be an arbitrary homomorphism. If $(a, \varphi(a)) \in Ker\phi$ for all $a \in S$, then*

(i) *ϕ is constant such that $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$ or*

(ii) *$\phi(S_1) \subseteq S'_1$ and $\phi(a^*) = b^*$ or*

(iii) *$\phi(S_1) \subseteq S'_1$ and $\phi(a^*) = b^{**}$.*

Proof. Let $\phi : S \to S'$ be a homomorphism satisfying $\phi(a) = \phi(\varphi(a))$ for all $a \in S$. Then we have $\phi(S_1) = \phi(S_2)$ and $\phi(S_3) = \phi(S_4)$. Now we will consider $H = \phi(S_1) = \phi(S_2)$ and $K = \phi(S_3) = \phi(S_4)$. If $H \nsubseteq S'_1$, then by Lemma 19 (i), ϕ is constant and $\phi(S) \subseteq \{b^*, b^{**}\}$ and we obtain (i). If $H \subseteq S'_1$, then $\phi(S_2) = H \nsubseteq S'_2$ and thus by Lemma 19 (ii), ϕ is constant such that $\phi(S) \subseteq \{b^*, b^{**}\}$, i.e., (i) or $\phi(a) = \phi(\varphi(a))$ for all $a \in S$. In the second case, if $K \subseteq S'_3$, i.e., $\phi(S_4) \nsubseteq S'_4$ then by Lemma 19 (iv), ϕ is constant such that $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$ which is not possible or $\phi(a^{**}) = b^*$ implying $\phi(a^*) = \phi(\varphi(a^{**})) = \phi(a^{**}) = b^*$ and hence we obtain (ii). If $K \nsubseteq S'_3$, then by Lemma 19 (iii), ϕ is constant such that $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$ or $\phi(a^*) = b^{**}$. Thus we have (i) or (iii). ■

Proposition 21. *Let $\mathcal{S} = (S; *)$ and $\mathcal{S}' = (S'; *')$ be two four-part semigroups with the constant elements a^*, a^{**} and b^*, b^{**} respectively and let $\phi : S \to S'$ be an arbitrary mapping such that $\phi(S_i) \subseteq S'_i$ for all $i = 1, 2, 3, 4$. Then ϕ is a homomorphism if and only if $\phi(\varphi(a)) = \varphi'(\phi(a))$ for all $a \in S$ and $\phi(a^*) = b^*$.*

Proof. Let for a mapping $\phi : S \to S'$ the conditions be satisfied. Then we have $\phi(a^{**}) = \phi(\varphi(a^*)) = \varphi'(\phi(a^*)) = \varphi'(b^*) = b^{**}$ and thus

$$\phi(a * b) = \begin{cases} \phi(b) = \phi(a) *' \phi(b) & \text{if } a \in S_1 \\ \phi(\varphi(b)) = \varphi'(\phi(b)) = \phi(a) *' \phi(b) & \text{if } a \in S_2 \\ \phi(a^*) = b^* = \phi(a) *' \phi(b) & \text{if } a \in S_3 \\ \phi(a^{**}) = b^{**} = \phi(a) *' \phi(b) & \text{if } a \in S_4. \end{cases}$$

Hence ϕ is a homomorphism. Conversely, let $\phi : \mathcal{S} \to \mathcal{S}'$ be a homomorphism such that $\phi(S_i) \subseteq S'_i$. Then we have $\phi(a^*) = \phi(a_{3j} * a_{3j}) = \phi(a_{3j}) *' \phi(a_{3j}) = b^*$ and for every $a \in S$ we have $\phi(\varphi(a)) = \phi(a_{2j} * a) = \phi(a_{2j}) *' \phi(a) = \varphi'(\phi(a))$. ■

More generally, we have

Theorem 22. *Let $\mathcal{S} = (S; *)$ and $\mathcal{S}' = (S'; *')$ be two four-part semigroups with the constant elements a^*, a^{**} and b^*, b^{**}, respectively and let $\phi : S \to S'$ be an arbitrary mapping. Then ϕ is a homomorphism if and only if*

(i) *ϕ is constant such that $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$ or*

(ii) *$\phi(S_i) \subseteq S'_i$ for $i = 1, 2, 3, 4$ such that $\phi(\varphi(a)) = \varphi'(\phi(a))$ for all $a \in S$ and $\phi(a^*) = b^*$ or*

(iii) *$\phi(S_1) \subseteq S'_1$, $\phi(S_2) \subseteq S'_2$, $\phi(S_3) \subseteq S'_4$, $\phi(S_4) \subseteq S'_3$, $\phi(\varphi(a)) = \varphi'(\phi(a))$ for all $a \in S$ and $\phi(a^*) = b^{**}$ or*

(iv) *$\phi(S_1) \subseteq S'_1$, $\phi(S_3) \subseteq S'_3$, $\phi(a) = \phi(\varphi(a))$ for all $a \in S$ and $\phi(a^*) = b^*$* (or $\phi(S_1) \subseteq S'_1$, $\phi(S_3) \subseteq S'_4$, $\phi(a) = \phi(\varphi(a))$ for all $a \in S$ and $\phi(a^*) = b^{**}6$).

Proof. Let $\mathcal{S}, \mathcal{S}'$ be two four-part semigroups with a^*, a^{**} and b^*, b^{**} being constant elements of $\mathcal{S}$ and $\mathcal{S}'$, respectively. Let $\phi : S \to S'$ be a homomorphism. We will consider the different cases from Theorem 17:

1. If ϕ is constant and maps every element of S to an idempotent element of S', then by Proposition 12, $\phi(S) \subseteq S'_1 \cup \{b^*, b^{**}\}$. Thus we have (i)

2. Let ϕ satisfy $(\varphi(a), \varphi(b)) \in Ker\phi$ whenever $(a, b) \in Ker\phi$ and $(a, b) \in Ker\phi$ only if $a, b \in S_i$ for $i = 1, 2, 3, 4$ and for every $a, b \in S$. It is clear that ϕ is not constant and $(a, \varphi(a)) \notin Ker\phi$ for all $a \in S$. Then by Lemma 19 (i) and Lemma 19 (ii), $\phi(S_1) \subseteq S'_1$ and $\phi(S_2) \subseteq S'_2$. Now we consider the following cases:

   a. If $\phi(S_1) \subseteq S'_1$, $\phi(S_2) \subseteq S'_2$ and $\phi(S_3) \nsubseteq S'_3$, then by Lemma 19 (iii), $\phi(a^*) = b^{**}$. In this case, $a_{3j} * a_{3j} = a^*$ implies $\phi(a_{3j}) *' \phi(a_{3j}) = \phi(a^*) = b^{**}$, i.e., $\phi(a_{3j}) \in S'_4$ and hence $\phi(S_3) \subseteq S'_4$. For any $a_{4j} \in S_4$ and for $a_{2j} \in S_2$ we obtain $\phi(a_{4j}) = \phi(a_{2j} * a_{3j}) = \phi(a_{2j}) *' \phi(a_{3j}) = \varphi'(\phi(a_{3j})) \in \varphi'(S'_4) = S'_3$, i.e., $\phi(S_4) \subseteq S'_3$. Moreover, for every $a \in S$ and for $a_{2j} \in S_2$ we get $\varphi(a) = a_{2j} * a$ and hence $\phi(\varphi(a)) = \phi(a_{2j}) *' \phi(a) = \varphi'(\phi(a))$. Therefore we have (iii).

   b. If $\phi(S_1) \subseteq S'_1$, $\phi(S_2) \subseteq S'_2$, $\phi(S_3) \subseteq S'_3$ and $\phi(S_4) \nsubseteq S'_4$, then by Lemma 19 (iv) we get $\phi(a^{**}) = b^*$. In this case, we have $\phi(a^{**}) = \phi(a_{4j} * a_{4j}) = \phi(a_{4j}) *' \phi(a_{4j}) = b^*$ for every $a_{4j} \in S_4$. This is possible iff $\phi(a_{4j}) \in S'_3$ and hence $\phi(S_4) \subseteq S'_3$. Therefore for every $a_{2j} \in S_2$ and $a_{3j} \in S_3$ we obtain $\phi(a_{3j}) = \phi(a_{2j} * a_{4j}) = \phi(a_{2j}) *' \phi(a_{4j}) = \varphi'(\phi(a_{4j})) \in \varphi'(S'_3) = S'_4$, i.e., $\phi(S_3) \subseteq S'_4$, a contradiction.

c. If $\phi(S_1) \subseteq S_1'$, $\phi(S_2) \subseteq S_2'$, $\phi(S_3) \subseteq S_3'$ and $\phi(S_4) \subseteq S_4'$, then by Proposition 21, we have (ii).

3. Let ϕ satisfy $(a^*, a^{**}) \in Ker\phi$, $(\varphi(a), \varphi(b)) \in Ker\phi$ whenever $(a,b) \in Ker\phi$ and $(a,b) \in Ker\phi$ only if $a,b \in S_3 \cup S_4$ or $a,b \in S_1$ or $a,b \in S_2$ for every $a,b \in S$. It is clear that ϕ is not constant and $(a_{1j}, \varphi(a_{1j})) \notin Ker\phi$ for $a_{1j} \in S_1$. Then by Lemma 19 (i) and Lemma 19 (ii), $\phi(S_1) \subseteq S_1'$ and $\phi(S_2) \subseteq S_2'$. Now we will consider all possible cases:

   a. If $\phi(S_1) \subseteq S_1'$, $\phi(S_2) \subseteq S_2'$ and $\phi(S_3) \not\subseteq S_3'$, then by Lemma 19 (iii), $\phi(a^*) = b^{**}$ and we have $b^{**} = \phi(a^*) = \phi(a^{**})$. Then for every $a_{3j} \in S_3$ and for every $a_{4j} \in S_4$ we have $\phi(a_{3j}) *' \phi(a_{3j}) = \phi(a_{3j} * a_{3j}) = \phi(a^*) = b^{**}$ and $\phi(a_{4j}) *' \phi(a_{4j}) = \phi(a_{4j} * a_{4j}) = \phi(a^{**}) = b^{**}$ i.e., $\phi(a_{3j}), \phi(a_{4j}) \in S_4'$. Hence we obtain $\phi(a_{2j} * a_{3j}) = \phi(a_{4j}) \in S_4'$ and $\phi(a_{2j} * a_{3j}) = \phi(a_{2j}) *' \phi(a_{3j}) \in S_3'$, a contradiction.

   b. If $\phi(S_1) \subseteq S_1'$, $\phi(S_2) \subseteq S_2'$, $\phi(S_3) \subseteq S_3'$ and $\phi(S_4) \not\subseteq S_4'$, then by Lemma 19 (iv), $\phi(a^{**}) = b^*$. Thus we have $b^* = \phi(a^*) = \phi(a^{**})$. Hence for every $a_{3j} \in S_3$ and for every $a_{4j} \in S_4$ we have $\phi(a_{3j}) *' \phi(a_{3j}) = \phi(a_{3j} * a_{3j}) = \phi(a^*) = b^*$ and $\phi(a_{4j}) *' \phi(a_{4j}) = \phi(a_{4j} * a_{4j}) = \phi(a^{**}) = b^*$ i.e., $\phi(a_3 j), \phi(a_{4j}) \in S_3'$. Therefore we obtain $\phi(a_{2j} * a_{3j}) = \phi(a_{4j}) \in S_3'$ and $\phi(a_{2j} * a_{3j}) = \phi(a_{2j}) *' \phi(a_{3j}) \in S_4'$, a contradiction.

   c. If $\phi(S_1) \subseteq S_1'$, $\phi(S_2) \subseteq S_2'$, $\phi(S_3) \subseteq S_3'$ and $\phi(S_4) \subseteq S_4'$, then we have a contradiction to $(a^*, a^{**}) \in Ker\phi$.

4. Let ϕ satisfy $(a, \varphi(b)) \in Ker\phi$ whenever $(a,b) \in Ker\phi$ and $(a,b) \in Ker\phi$ only if $a,b \in S_1 \cup S_2$ or $a,b \in S_3 \cup S_4$ for every $a,b \in S$. It is obvious that $(a, \varphi(a)) \in Ker\phi$ for all $a \in S$. Thus by Lemma 20, we have two possible cases $\phi(S_1) \subseteq S_1'$ and $\phi(a^*) = b^*$ or $\phi(S_1) \subseteq S_1'$ and $\phi(a^*) = b^{**}$. For every $a_{3j} \in S_3$, in the first case we obtain $\phi(a_{3j}) *' \phi(a_{3j}) = \phi(a_{3j} * a_{3j}) = \phi(a^*) = b^*$, i.e., $\phi(a_{3j}) \in S_3'$ and hence $\phi(S_3) \subseteq S_3'$ and in the second case we have $\phi(a_{3j}) *' \phi(a_{3j}) = \phi(a_{3j} * a_{3j}) = \phi(a^*) = b^{**}$, i.e., $\phi(a_{3j}) \in S_4'$ and hence $\phi(S_3) \subseteq S_4'$. Therefore we have (iv).

Conversely, let $\phi : S \to S'$ be a mapping. If ϕ satisfies (i) and $\phi(a) = c$ for all $a \in S$ with $c \in S_1' \cup \{b^*, b^{**}\}$, then we get $\phi(a*b) = c = c*c = \phi(a)*\phi(b)$ and hence ϕ is a homomorphism. If ϕ satisfies (ii), then ϕ is a homomorphism by Proposition 21. If ϕ satisfies (iii), then $\phi(a^{**}) = \phi(\varphi(a^*)) = \varphi'(\phi(a^*)) = \varphi'(b^{**}) = b^*$ and we obtain

$$\phi(a * b) = \begin{cases} \phi(b) = \phi(a) *' \phi(b) & \text{if } a \in S_1 \\ \phi(\varphi(b)) = \varphi'(\phi(b)) = \phi(a) *' \phi(b) & \text{if } a \in S_2 \\ \phi(a^*) = b^{**} = \phi(a) *' \phi(b) & \text{if } a \in S_3 \\ \phi(a^{**}) = b^* = \phi(a) *' \phi(b) & \text{if } a \in S_4, \end{cases}$$

i.e., ϕ is a homomorphism. If ϕ satisfies (iv), i.e., $\phi(a) = \phi(\varphi(a))$ for all $a \in S$, $\phi(S_1) \subseteq S'_1$, $\phi(S_3) \subseteq S'_3$ and $\phi(a^*) = b^*$, then we have $\phi(S_2) = \phi(\varphi(S_1)) = \phi(S_1) \subseteq S'_1$, $\phi(S_4) = \phi(\varphi(S_3)) = \phi(S_3) \subseteq S'_3$ and $\phi(a^{**}) = \phi(\varphi(a^*)) = \phi(a^*) = b^*$. Therefore we have

$$\phi(a * b) = \begin{cases} \phi(b) = \phi(a) *' \phi(b) & \text{if } a \in S_1 \\ \phi(\varphi(b)) = \phi(b) = \phi(a) *' \phi(b) & \text{if } a \in S_2 \\ \phi(a^*) = b^* = \phi(a) *' \phi(b) & \text{if } a \in S_3 \\ \phi(a^{**}) = b^* = \phi(a) *' \phi(b) & \text{if } a \in S_4. \end{cases}$$

Hence ϕ is a homomorphism. Similarly, ϕ is a homomorphism if $\phi(a) = \phi(\varphi(a))$ for all $a \in S$, $\phi(S_1) \subseteq S'_1$, $\phi(S_3) \subseteq S'_4$ and $\phi(a^*) = b^{**}$. This completes the proof. ■

Proposition 23. *Let $\mathcal{S} = (S; *)$ and $\mathcal{S}' = (S'; *')$ be two four-part semigroups with the constant elements a^*, a^{**} and b^*, b^{**} respectively and let $\phi : S \to S'$ be a homomorphism. Then the following Propositions are true.*

(i) *If ϕ is a homomorphism of the first type of Theorem 22, then $Im\phi$ forms a constant subsemigroup of $\mathcal{S}'$.*

(ii) *If ϕ is a homomorphism of the second type or of the third type of Theorem 22, then $Im\phi$ forms a four-part subsemigroup of $\mathcal{S}'$.*

(iii) *If ϕ is a homomorphism of the fourth type of Theorem 22, then $Im\phi$ forms a right-zero constant subsemigroup of $\mathcal{S}'$.*

Proof. (i) is obvious.

(ii) Let $\phi : S \to S'$ be a homomorphism of the third type of Theorem 22, i.e., $\phi(S_i) \subseteq S'_i$ for $i = 1, 2$, $\phi(S_3) \subseteq S'_4$, $\phi(S_4) \subseteq S'_3$, $\phi(\varphi(a)) = \varphi'(\phi(a))$ for all $a \in S$, $\phi(a^*) = b^{**}$ and $\phi(a^{**}) = b^*$. Then it is clear that $Im\phi \cap S'_2 \neq \emptyset$ and $b^*, b^{**} \in Im\phi$. Moreover, if $b \in Im\phi$, then there is $a \in S$ such that $b = \phi(a)$. Thus, by assumption, we obtain $b = \phi(a) = \phi(a_{2j} * \varphi(a)) = \phi(a_{2j}) *' \phi(\varphi(a)) = \varphi'(\phi(\varphi(a)))$ for $a_{2j} \in S_2$ and hence $\varphi'(b) = \phi(\varphi(a)) \in Im\phi$. Therefore $Im\phi$ satisfies the two conditions in Proposition 5 and hence forms a four-part subsemigroup of $\mathcal{S}'$. By the same argumentation, if ϕ is a homomorphism of the second type of Theorem 22, then $Im\phi$ forms a four-part subsemigroup of $\mathcal{S}'$.

(iii) Let $\phi : S \to S'$ be a homomorphism of the fourth type of Theorem 22, i.e., $\phi(a) = \phi(\varphi(a))$ for all $a \in S$ such that $\phi(S_1) \subseteq S'_1$, $\phi(S_3) \subseteq S'_3$ and $\phi(a^*) = b^*$ (or $\phi(a) = \phi(\varphi(a))$ for all $a \in S$ such that $\phi(S_1) \subseteq S'_1$, $\phi(S_3) \subseteq S'_4$ and $\phi(a^*) = b^*$). Then $Im\phi \subseteq S'_1 \cup S'_3$, $Im\phi \cap S'_1 \neq \emptyset$ and $b^* \in Im\phi$ (or $Im\phi \subseteq S'_1 \cup S'_4$, $Im\phi \cap S'_1 \neq \emptyset$ and $b^{**} \in Im\phi$). Thus by Proposition 8, $Im\phi$ forms a right-zero constant subsemigroup of $\mathcal{S}'$. ■

5. Green's relations on four-part semigroups

Let a and b be two elements in the semigroup $\mathcal{S} = (S;*)$. Recall that Green's relations are defined in the following way: $a\mathcal{L}b$ iff $a = b$ or there exist $c, d \in S$ such that $c * a = b$ and $d * b = a$, $a\mathcal{R}b$ iff $a = b$ or there exist $c, d \in S$ such that $a * c = b$ and $b * d = a$, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. It is well-known that for a finite semigroup $\mathcal{D}$ and $\mathcal{J}$ are the same.

Proposition 24. *Let $\mathcal{S} = (S;*)$ be a four-part semigroup with a^* and a^{**} as constant elements. Then $\mathcal{L}_a = \{a, \varphi(a)\}$ for all $a \in S$.*

Proof. Let $a, b \in S$ such that $a \neq b$ satisfy $a\mathcal{L}b$. Thus there are $c, d \in S$ such that $c * a = b$ and $d * b = a$. Assume that $b \neq \varphi(a)$. If $a = a_{1j} \in S_1$, then $a_{1j} \neq b \neq \varphi(a_{1j}) = a_{2j} \in S_2$. Thus we have

$$c * a = c * a_{1j} = \begin{cases} a_{1j} \neq b & \text{if } c \in S_1 \\ \varphi(a_{1j}) = a_{2j} \neq b & \text{if } c \in S_2 \\ a^* & \text{if } c \in S_3 \\ a^{**} & \text{if } c \in S_4. \end{cases}$$

Therefore $c * a = b$ is only possible for $b = a^*$ or $b = a^{**}$. But if $b = a^*$, then we have

$$d * b = d * a^* = \begin{cases} a^* \neq a_{1j} = a & \text{if } d \in S_1 \\ \varphi(a^*) = a^{**} \neq a_{1j} = a & \text{if } d \in S_2 \\ a^* \neq a_{1j} = a & \text{if } d \in S_3 \\ a^{**} \neq a_{1j} = a & \text{if } d \in S_4, \end{cases}$$

a contradiction. Similarly, we have a contradiction when $b = a^{**}$. If $a = a_{2j} \in S_2$, then in the same way we also obtain a contradiction.

Now, if $a = a_{3j} \in S_3$, then $a_{3j} \neq b \neq \varphi(a_{3j}) = a_{4j}$. Thus we have

$$c * a = c * a_{3j} = \begin{cases} a_{3j} \neq b & \text{if } c \in S_1 \\ \varphi(a_{3j}) = a_{4j} \neq b & \text{if } c \in S_2 \\ a^* & \text{if } c \in S_3 \\ a^{**} & \text{if } c \in S_4. \end{cases}$$

Thus $c * a = b$ is only possible for $b = a^*$ or $b = a^{**}$. But if $b = a^*$, then we have

$$d * b = d * a^* = \begin{cases} a^* & \text{if } d \in S_1 \\ \varphi(a^*) = a^{**} \neq a_{3j} = a & \text{if } d \in S_2 \\ a^* & \text{if } d \in S_3 \\ a^{**} \neq a_{3j} = a & \text{if } d \in S_4, \end{cases}$$

and therefore $d*b = a$ is possible only when $a = a_{3j} = a^*$ and we have $a = a^* = b$, a contradiction. If $b = a^{**}$, then we obtain

$$d * b = d * a^{**} = \begin{cases} a^{**} \neq a_{3j} = a & \text{if } d \in S_1 \\ \varphi(a^{**}) = a^* & \text{if } d \in S_2 \\ a^* & \text{if } d \in S_3 \\ a^{**} \neq a_{3j} = a & \text{if } d \in S_4, \end{cases}$$

and therefore $d * b = a$ is possible only when $a = a_{3j} = a^*$ and thus $b = a^{**} = \varphi(a^*) = \varphi(a)$, a contradiction. Similarly, we also have a contradiction for the case $a = a_{4j} \in S_4$. Therefore $b = \varphi(a)$ and hence $\mathcal{L} = \{a, \varphi(a)\}$. ■

Proposition 25. *Let $\mathcal{S} = (S; *)$ be a four-part semigroup with a^* and a^{**} as constant elements and let $a \in S$. Then $\mathcal{R}_a = \{a\}$ or $\mathcal{R}_a = S_1 \cup S_2$.*

Proof. First we show that $a\mathcal{R}b$ for every $a, b \in S_1 \cup S_2$. Let $a \neq b$. If $a, b \in S_1$, then clearly $a\mathcal{R}b$ with $c = d, b = a$. If $a, b \in S_2$, then with $c = \varphi(b)$ and $d = \varphi(a)$ we have $a * c = a * \varphi(b) = \varphi(\varphi(b)) = b$ and $b * d = b * \varphi(a) = \varphi(\varphi(a)) = a$ and hence $a\mathcal{R}b$. If $a \in S_1$ and $b \in S_2$, then $a * c = b$ and $b * d = a$ for $c = b$ and $d = \varphi(a)$ and thus $a\mathcal{R}b$. Now, we show that $\mathcal{R}_a = \{a\}$ if $a \in S_3 \cup S_4$. Let $a \in S_3$ and assume that $\mathcal{R}_a \neq \{a\}$, i.e., there is $b \in \mathcal{R}_a$ such that $b \neq a$. Hence for every $c, d \in S$ satisfying $a * c = b$ and $b * d = a$, we obtain $a^* = a * c = b$ and therefore $a = b * d = a^* * d = a^* = b$, a contradiction. Thus there is no $a \neq b \in S$ such that $b \in \mathcal{R}_a$. Similarly, there is no $b \neq a$ such that $a\mathcal{R}b$ for $a \in S_4$. Hence $\mathcal{R}_a = \{a\}$ for $a \in S_3 \cup S_4$. This completes the proof. ■

Proposition 26. *Let $\mathcal{S} = (S : *)$ be a four-part semigroup and let $a \in S$. Then $\mathcal{H}_a = \{a, \varphi(a)\}$ if $a \in S_1 \cup S_2$ and $\mathcal{H}_a = \{a\}$ if $a \in S_3 \cup S_4$.*

Proof. Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, by Proposition 24 and Proposition 25, we have that $\mathcal{H}_a = \{a, \varphi(a)\}$ if $a \in S_1 \cup S_2$ and $\mathcal{H}_a = \{a\}$ for every $a \in S_3 \cup S_4$. ■

Proposition 27. *Let $\mathcal{S} = (S; *)$ be a four-part semigroup. Then $\mathcal{D}_a = \mathcal{J}_a = \{a, \gamma a\}$ or $\mathcal{D}_a = \mathcal{J}_a = S_1 \cup S_2$.*

Proof. We show that $a\mathcal{D}b$ ($a\mathcal{J}b$) for all $a, b \in S_1 \cup S_2$. If $a, b \in S_1 \cup S_2$, then by taking $c = a$ we have $a\mathcal{L}c$ and $c\mathcal{R}b$ by Proposition 24 and Proposition 25, i.e., $a\mathcal{D}b$. If $a \notin S_1 \cup S_2$, by taking $c = \gamma a$ we have $a\mathcal{L}c$ and $c\mathcal{R}c$ by Proposition 24 and Proposition 25, i.e., $a\mathcal{D}\gamma a$. Now, let $a, b \notin S_1 \cup S_2$ and $a\mathcal{D}b$. Then there exists $c \in S$ such that $a\mathcal{L}c$ and $c\mathcal{R}b$. By Proposition 24, we have $c = a$ or $c = \gamma a$ and by Proposition 25, we have $c = b$ since $b \notin S_1 \cup S_2$. Therefore we have two possibilities $a = c = b$ or $b = c = \gamma a$. Thus $\mathcal{D}_a = \{a, \gamma a\}$. By the finiteness of S, we have $\mathcal{D} = \mathcal{J}$. This completes the proof. ■

6. Representation of four-part semigroups

Theorem 28. *Let $\mathcal{S} = (S; *)$ be an arbitrary four-part semigroup with the constant elements a^* and a^{**}. Then there is a natural number $n \geq 1$ such that $\mathcal{S}$ is isomorphic to a four-part subsemigroup of $(O^n(\{0,1\}); +)$.*

Proof. Let $\mathcal{S}$ be a four-part semigroup with $|S_1| = |S_2| = n_r$ and $|S_3| = |S_4| = n_s$. We choose a natural number n such that $max(n_r, n_s) \leq 2^{2^n-2}$ and consider $O^n(\{0,1\})$. Now, define a one-to-one mappings $\phi_1 : S_1 \to C_4^n \subseteq O^n(\{0,1\})$ and $\phi_3 : S_3 \to K_0^n \subseteq O^n(\{0,1\})$ such that $\phi_3(a^*) = c_0^n$ and define mappings $\phi_2 : S_2 \to \neg C_4^n \subseteq O^n(\{0,1\})$ and $\phi_4 : S_4 \to K_1^n \subseteq O^n(\{0,1\})$ by $\phi_2(a_{2j}) = \neg\phi_1(a_{1j})$ and $\phi_4(a_{4j}) = \neg\phi_3(a_{3j})$. It is clear that $\phi : S \to O^n(\{0,1\})$ defined by $\phi(a_{ij}) = \phi_i(a_{ij})$ is a one-to-one mapping satisfying $\phi(\varphi(a)) = \neg\phi(a)$ for all $a \in S$. Therefore, $\neg\phi(a) \in \phi(S)$ for every $a \in S$. Moreover, $S_1' := \phi(S_1) \subseteq C_4^n$, $S_2' := \phi(S_2) = \neg\phi(S_1) \subseteq \neg C_4^n$, $S_3' := \phi(S_3) \subseteq K_0^n$ and $S_4' := \phi(S_4) = \neg\phi(S_3) \subseteq K_1^n$ and for $a, b \in \phi(S) = S_1' \cup S_2' \cup S_3' \cup S_4'$ we have

$$a + b = \begin{cases} b \in \phi(S) & \text{if } a \in S_1' \\ \neg b \in \phi(S) & \text{if } a \in S_2' \\ c_0^n \in \phi(S) & \text{if } a \in S_3' \\ c_1^n \in \phi(S) & \text{if } a \in S_4', \end{cases}$$

i.e., $\phi(S) = S_1' \cup S_2' \cup S_3' \cup S_4'$ forms a four-part subsemigroup of $(O^n(\{0,1\}); +)$. Furthermore, considering the unary operation $\neg$ as φ' in $O^n(\{0,1\})$, then by Theorem 22 (ii), $\phi : S \to O^n(\{0,1\})$ is a homomorphism. Therefore $\mathcal{S} \cong \phi(\mathcal{S}) \subseteq \mathcal{O}^n(\{0,1\})$. ∎

References

[1] R. Butkote and K. Denecke, *Semigroup Properties of Boolean Operations*, Asian-Eur. J. Math. **1** (2008) 157–176.

[2] R. Butkote, Universal-algebraic and Semigroup-theoretical Properties of Boolean Operations (Dissertation Universität Potsdam, 2009).

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