*Discussiones Mathematicae*
*General Algebra and Applications 31 (2011) 185–199*

# POLYNOMIALS OF MULTIPARTITIONAL TYPE AND INVERSE RELATIONS

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**Abstract**

Chou, Hsu and Shiue gave some applications of Faà di Bruno's formula to characterize inverse relations. Our aim is to develop some inverse relations connected to the multipartitional type polynomials involving to binomial type sequences.

**Keywords:** Bell polynomials, inverses relations, polynomials of multipartitional type, binomial type sequences.

**2010 Mathematics Subject Classification:** Primary 05A10; Secondary 11B83, 11B73, 40E15.

## 1. Introduction

The partial (exponential) multipartitional polynomial

$$B_{n_1,\ldots,n_r;k}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right)$$

in the variables $x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}$ is defined by the sum

(1) $$B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1},x_{0,\ldots,0,2},\ldots,x_{n_1,\ldots,n_r}\right)$$

$$:=\sum\frac{n_1!\cdots n_r!}{k_{0,\ldots,0,1}!k_{0,\ldots,0,2}!\cdots k_{n_1,\ldots,n_r}!}\left(\frac{x_{0,\ldots,0,1}}{0!\cdots 0!1!}\right)^{k_{0,\ldots,0,1}}\cdots\left(\frac{x_{n_1,\ldots,n_r}}{n_1!\cdots n_r!}\right)^{k_{n_1,\ldots,n_r}},$$

where the summation is extended over all partitions of the multipartite number $(n_1, \ldots, n_r)$ into $k$ parts, that is, over all nonnegative integers $k_{0,\ldots,0,1}, \ldots, k_{n_1,\ldots,n_r}$ solution of the equations

$$\sum_{i_1,\ldots,i_r=0}^{n_1,\ldots,n_r} i_j k_{i_1,\ldots,i_r} = n_j, \quad j = 1, \ldots, r, \quad \sum_{i_1,\ldots,i_r=0}^{n_1,\ldots,n_r} k_{i_1,\ldots,i_r} = k \tag{2}$$

with convention $k_{0,\ldots,0} = 0$.

The exponential generating function for $B_{n_1,\ldots,n_r,k}$ is given by

$$\begin{aligned} &\sum_{n_1+\cdots+n_r \geq k} B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1}, \ldots, x_{n_1,\ldots,n_r}\right) \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!} \\ &= \frac{1}{k!} \left( \sum_{i_1+\cdots+i_r \geq 1} x_{i_1,\ldots,i_r} \frac{t_1^{i_1}}{i_1!} \cdots \frac{t_r^{i_r}}{i_r!} \right)^k . \end{aligned} \tag{3}$$

From the above definition, the partial multipartitional polynomials generalize the partial Bell polynomials introduced by Bell [3], see also [5, 9] and [10]. Chou *et al.* [4] gave some inverse relations related to Faà di Bruno's formula. They have proved that any function $F$ having power formal series with compositional inverse $F^{\langle -1 \rangle}$, satisfy

$$\begin{aligned} y_n &= \sum_{k=1}^{n} D_{x=a}^{k} F(x)\, B_{n,k}(x_1, x_2, \ldots), \\ x_n &= \sum_{k=1}^{n} D_{x=F(a)}^{k} F^{\langle -1 \rangle}(x)\, B_{n,k}(y_1, y_2, \ldots). \end{aligned}$$

In this paper, we show the role of the binomial type sequences to develop some inverse relations related to the multipartitional polynomials. In other words, we connect some inverse relations with the multipartitional polynomials and with some results given in [6] and [8] related to partial Bell polynomials.

## 2. Multipartitional polynomials and inverse relations

**Theorem 1.** *Let $a$ be a real number and $F$ be a function such that $F(a+x)$ admits formal power series in $x$ and let $F^{\langle -1 \rangle}$ denote the compositional inverse* *of $F$ so that $\left(F \circ F^{\langle -1\rangle}\right)(x) = \left(F^{\langle -1\rangle} \circ F\right)(x) = x$. Let*

$$\varphi_x (t_1, \ldots, t_r) := \sum_{n_1+\cdots+n_r \geq 1} x_{n_1,\ldots,n_r} \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!},$$

$$\varphi_y (t_1, \ldots, t_r) := \sum_{n_1+\cdots+n_r \geq 1} y_{n_1,\ldots,n_r} \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!}.$$

*Then we have the pair of inverse relations*

$$y_{n_1,\ldots,n_r} = \sum_{k=1}^{n_1+\cdots+n_r} D_{x=a}^k F(x) B_{n_1,\ldots,n_r,k} \left(x_{0,\ldots,0,1}, \ldots, x_{n_1,\ldots,n_r}\right), \tag{4}$$

$$x_{n_1,\ldots,n_r} = \sum_{k=1}^{n_1+\cdots+n_r} D_{x=F(a)}^k F^{\langle -1\rangle}(x) B_{n_1,\ldots,n_r,k} \left(y_{0,\ldots,0,1}, \ldots, y_{n_1,\ldots,n_r}\right),$$

*or equivalently*

$$F\left(a + \varphi_x (t_1, \ldots, t_r)\right) = F(a) + \varphi_y (t_1, \ldots, t_r). \tag{5}$$

***Proof.*** The given conditions of the theorem ensure that there holds a pair of formal series expansions

$$\begin{aligned}
\varphi_y (t_1, \ldots, t_r) &= \sum_{n_1+\cdots+n_r \geq 1} \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!} \sum_{k=1}^{n_1+\cdots+n_r} D_{x=a}^k F(x) \times \\
&\qquad\qquad B_{n_1,\ldots,n_r,k} \left(x_{0,\ldots,0,1}, \ldots, x_{n_1,\ldots,n_r}\right) \\
&= \sum_{k=1}^{\infty} D_{x=a}^k F(x) \sum_{n_1+\cdots+n_r \geq k} \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!} \times \\
&\qquad\qquad B_{n_1,\ldots,n_r,k} \left(x_{0,\ldots,0,1}, \ldots, x_{n_1,\ldots,n_r}\right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=1}^{\infty} D_{x=a}^k F(x) \frac{1}{k!} \left( \sum_{i_1+\cdots+i_r \geq 1} x_{i_1,\ldots,i_r} \frac{t_1^{i_1}}{i_1!} \cdots \frac{t_r^{i_r}}{i_r!} \right)^k \\
&= \sum_{k=1}^{\infty} D_{x=a}^k F(x) \frac{\left(\varphi_x (t_1, \ldots, t_r)\right)^k}{k!} \\
&= F\left(a + \varphi_x (t_1, \ldots, t_r)\right) - F(a),
\end{aligned}$$

which states that

$$\begin{aligned}\varphi_x(t_1,\dots,t_r) &= F^{\langle -1\rangle}\left(F(a)+\varphi_y(t_1,\dots,t_r)\right)-a\\ &= \sum_{k=1}^{\infty} D^k_{x=F(a)}F^{\langle -1\rangle}(x)\,\frac{\left(\varphi_y(t_1,\dots,t_r)\right)^k}{k!},\end{aligned}$$

so, we have

$$\begin{aligned}&\sum_{n_1+\cdots+n_r\geq 1} x_{n_1,\dots,n_r}\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}\\ &= \sum_{n_1+\cdots+n_r\geq 1}\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}\sum_{k=1}^{n_1+\cdots+n_r} D^k_{x=F(a)}F^{\langle -1\rangle}(x)\\ &\quad\times B_{n_1,\dots,n_r,k}\left(y_{0,\dots,0,1},\dots,y_{n_1,\dots,n_r}\right).\end{aligned}$$

Then $x_{n_1,\dots,n_r}=\sum\limits_{k=1}^{n_1+\cdots+n_r} D^k_{x=F(a)}F^{\langle -1\rangle}(x)B_{n_1,\dots,n_r,k}(y_{0,\dots,0,1},\dots,y_{n_1,\dots,n_r})$.

■

**Remark 1.** Replacing

$$x_{n_1,\dots,n_r} \text{ by } n_1!\cdots n_r!x_{n_1,\dots,n_r} \text{ and } y_{n_1,\dots,n_r} \text{ by } n_1!\cdots n_r!y_{n_1,\dots,n_r},$$

relation (4) may be expressed in terms of the multivariate Faà di Bruno's formula, as follows

$$y_{n_1,\dots,n_r}=\sum\frac{D^k_{x=a}F(x)}{k_{0,\dots,0,1}!\cdots k_{n_1,\dots,n_r}!}\left(x_{0,\dots,0,1}\right)^{k_{0,\dots,0,1}}\cdots\left(x_{n_1,\dots,n_r}\right)^{k_{n_1,\dots,n_r}},$$

$$x_{n_1,\dots,n_r}=\sum\frac{D^k_{x=F(a)}F^{\langle -1\rangle}(x)}{k_{0,\dots,0,1}!\cdots k_{n_1,\dots,n_r}!}\left(y_{0,\dots,0,1}\right)^{k_{0,\dots,0,1}}\cdots\left(y_{n_1,\dots,n_r}\right)^{k_{n_1,\dots,n_r}},$$

where the summation is extended over all nonnegative integers $k_{0,\dots,0,1}$, $\dots$, $k_{n_1,\dots,n_r}$ solution of equations (2) and over $k=1,\dots,n_1+\cdots+n_r$.

Also, the pair of inverse relations given by relation (4) implies that for any sequence of real numbers $(x_n)$ we have the following identity:

$$\sum_{k=1}^{n} D^k_{x=F(a)}F^{\langle -1\rangle}(x)B_{n,k}\left(y_1,\dots,y_n\right)=x_n.$$

where $y_n=\sum_{j=1}^{n} D^j_{x=a}F(x)B_{n,j}(x_1,\dots,x_n)$.

**Corollary 2.** *We have the pair of inverse relations*

$$\text{(6)}\qquad y_{n_1,\ldots,n_r} = \sum_{j=1}^{n_1+\cdots+n_r} B_{n_1,\ldots,n_r,j}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right),$$

$$x_{n_1,\ldots,n_r} = \sum_{j=1}^{n_1+\cdots+n_r} (-1)^{j-1}(j-1)!B_{n_1,\ldots,n_r,j}\left(y_{0,\ldots,0,1},\ldots,y_{n_1,\ldots,n_r}\right).$$

***Proof.*** It suffices to take in (4) $F(x)=\exp(x)$ and $a=0$. ∎

**Corollary 3.** *We have the pair of inverse relations*

$$\text{(7)}\qquad y_{n_1,\ldots,n_r} = \sum_{k=1}^{n_1+\cdots+n_r} (\alpha)_k B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right),$$

$$x_{n_1,\ldots,n_r} = \sum_{k=1}^{n_1+\cdots+n_r} \left(\frac{1}{\alpha}\right)_k B_{n_1,\ldots,n_r,k}\left(y_{0,\ldots,0,1},\ldots,y_{n_1,\ldots,n_r}\right),$$

*where* $(\alpha)_k=\alpha(\alpha-1)\cdots(\alpha-k+1)$, $k\geq 1$, *and* $(\alpha)_0=1,\quad \alpha\in\mathbb{C}$.

***Proof.*** It suffices to take in (4) $F(x)=x^\alpha$ and $a=1$. ∎

**Remark 2.** For $\alpha=-1$ in (7), we obtain

$$\text{(8)}\qquad y_{n_1,\ldots,n_r} = \sum_{k=1}^{n_1+\cdots+n_r} (-1)^k k!B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right),$$

$$x_{n_1,\ldots,n_r} = \sum_{k=1}^{n_1+\cdots+n_r} (-1)^k k!B_{n_1,\ldots,n_r,k}\left(y_{0,\ldots,0,1},\ldots,y_{n_1,\ldots,n_r}\right).$$

The inverse relations (8) are equivalent to

$$\left(1+\sum x_{n_1,\ldots,n_r}\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}\right)\left(1+\sum y_{n_1,\ldots,n_r}\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}\right)=1,$$

where the summations are taken over all nonnegative integers such that $n_1+\cdots+n_r\geq 1$.

Corollary 3 can be generalized by the following:

**Corollary 4.** *We have the pair of inverse relations*

$$y_{n_1,\ldots,n_r} = \frac{1}{\beta}\sum_{k=1}^{n_1+\cdots+n_r} \left(\frac{\beta}{\alpha}\right)_k \alpha^k B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right),$$

$$x_{n_1,\ldots,n_r} = \frac{1}{\alpha}\sum_{k=1}^{n_1+\cdots+n_r} \left(\frac{\alpha}{\beta}\right)_k \beta^k B_{n_1,\ldots,n_r,k}\left(y_{0,\ldots,0,1},\ldots,y_{n_1,\ldots,n_r}\right).$$

***Proof.*** It suffices de take in (4) $F(x) = \frac{(1+\alpha t)^{\beta/\alpha}-1}{\beta}$, $\alpha, \beta \in \mathbb{C}$, $\alpha\beta \neq 0$, and $a = 0$. ∎

**Theorem 5.** *Let $\{f_n(x)\}$ be a polynomial sequence of binomial type. Then we have the pair of inverse relations*

$$(9) \quad y_{n_1,\ldots,n_r} = \sum kx \frac{f_{k-1}(a(k-1)+x)}{a(k-1)+x} B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right),$$

$$x_{n_1,\ldots,n_r} = -\sum kx \frac{f_{k-1}(a(k-1)-kx)}{a(k-1)-kx} B_{n_1,\ldots,n_r,k}\left(y_{0,\ldots,0,1},\ldots,y_{n_1,\ldots,n_r}\right),$$

*where the summations are taken over $k$ from $1$ to $n_1+\cdots+n_r$.*

***Proof.*** From Corollary 2 given in Mihoubi [7], the compositional inverse function of

$$F(t) = \sum_{n\geq 1} \frac{nx}{a(n-1)+x} f_{n-1}(a(n-1)+x) \frac{t^n}{n!}$$

is given by

$$F^{\langle -1\rangle}(t) = -\sum_{n\geq 1} \frac{nx}{a(n-1)-nx} f_{n-1}(a(n-1)-nx) \frac{t^n}{n!}.$$

Then, it suffices to use the last function $F$ in the pair of inverse relations given by (4). ∎

**Remark 3.** For $x = 2a$ in (9), we obtain the remarkable inverse relations

$$y_{n_1,\ldots,n_r} = \sum_{k=1}^{n_1+\cdots+n_r} \frac{2k}{k+1} f_{k-1}(a(k+1)) B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right),$$

$$x_{n_1,\ldots,n_r} = \sum_{k=1}^{n_1+\cdots+n_r} \frac{2k}{k+1} f_{k-1}(-a(k+1)) B_{n_1,\ldots,n_r,k}\left(y_{0,\ldots,0,1},\ldots,y_{n_1,\ldots,n_r}\right).$$

We have binomial-type sequence of polynomials for

$$f_n(x) = x^n,$$

$$f_n(x) = (\alpha x)_{(n)} := \alpha x(\alpha x-1)\cdots(\alpha x-n+1) \text{ for } n \geq 1 \text{ and } (x)_{(0)} := 1,$$

$$f_n(x) = (\alpha x)^{(n)} := \alpha x(\alpha x+1)\cdots(\alpha x+n-1) \text{ for } n \geq 1 \text{ and } (x)^{(0)} := 1,$$

$$f_n(x) = n!\binom{x}{n}_q := \sum_{j=o}^{n} B_{n,j}(\binom{1}{1}_q,\ldots,i!\binom{1}{i}_q,\ldots)(x)_{(j)},$$

$$f_n(x) = B_n(x) := \sum_{j=o}^{n} \left\{ {n \atop k} \right\} x^k,$$

where $B_n(.)$, $\left\{ {n \atop k} \right\}$ and $\binom{k}{n}_q$ are, respectively, the single variable Bell polynomials, the Stirling numbers of second kind and the ordinary multinomials.

The ordinary multinomials, see Belbachir *et al.* [1], are defined by the following

$$(1+x+x^2+\cdots+x^q)^k=\sum_{n\geq 0}\binom{k}{n}_q x^n,$$

with $\binom{L}{a}_1=\binom{L}{a}$ ($\binom{L}{a}$ being the usual binomial coefficient) and $\binom{L}{a}_q=0$ for $a>qL$. Using the classical binomial coefficient, one has

$$\binom{L}{a}_q=\sum_{j_1+j_2+\cdots+j_q=a}\binom{L}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{q-1}}{j_q}.$$

This last identity justify the extension of ordinary multinomials to real or complex values by replacing $L$ by a real or complex value, one can see [2].
We deduce the following results:

**Corollary 6.** *Let $a$, $\alpha(\neq 0)$ and $x$ be real numbers, for $f_n(x)=x^n$, we get*

$$\begin{aligned} y_{n_1,\dots,n_r} &= \sum_{k=1}^{n_1+\cdots+n_r} xk\left(a\left(k-1\right)+x\right)^{k-2}\times \\ &\qquad B_{n_1,\dots,n_r,k}\left(x_{0,\dots,0,1},\dots,x_{n_1,\dots,n_r}\right), \\ x_{n_1,\dots,n_r} &= -\sum_{k=1}^{n_1+\cdots+n_r} xk\left(a\left(k-1\right)-kx\right)^{k-2}\times \\ &\qquad B_{n_1,\dots,n_r,k}\left(y_{0,\dots,0,1},\dots,y_{n_1,\dots,n_r}\right), \end{aligned} \tag{10}$$

*for $f_n(x)=(x)_{(n)}$, we get*

$$\begin{aligned} y_{n_1,\dots,n_r} &= \sum_{k=1}^{n_1+\cdots+n_r} \frac{kx}{a\left(k-1\right)+x}\left(a\left(k-1\right)+x\right)_{(k-1)}\times \\ &\qquad B_{n_1,\dots,n_r,k}\left(x_{0,\dots,0,1},\dots,x_{n_1,\dots,n_r}\right), \\ x_{n_1,\dots,n_r} &= -\sum_{k=1}^{n_1+\cdots+n_r} \frac{kx}{a\left(k-1\right)-kx}\left(a\left(k-1\right)-kx\right)_{(k-1)}\times \\ &\qquad B_{n_1,\dots,n_r,k}\left(y_{0,\dots,0,1},\dots,y_{n_1,\dots,n_r}\right), \end{aligned} \tag{11}$$

*for* $f_n(x) = (x)^{(n)}$, *we get*

$$
\begin{aligned}
y_{n_1,\ldots,n_r} &= \sum_{k=1}^{n_1+\cdots+n_r} \frac{kx}{a(k-1)+x}\left(a(k-1)+x\right)^{(k-1)} \times \\
&\qquad B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right), \\
x_{n_1,\ldots,n_r} &= -\sum_{k=1}^{n_1+\cdots+n_r} \frac{kx}{a(k-1)-kx}\left(a(k-1)-kx\right)^{(k-1)} \times \\
&\qquad B_{n_1,\ldots,n_r,k}\left(y_{0,\ldots,0,1},\ldots,y_{n_1,\ldots,n_r}\right),
\end{aligned}
\tag{12}
$$

*for* $f_n(x) = n!\binom{x}{n}_q$, *we get*

$$
\begin{aligned}
y_{n_1,\ldots,n_r} &= \sum_{k=1}^{n_1+\cdots+n_r} \frac{k!x}{a(k-1)+x}\binom{a(k-1)+x}{k-1}_q \times \\
&\qquad B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right), \\
x_{n_1,\ldots,n_r} &= -\sum_{k=1}^{n_1+\cdots+n_r} \frac{k!x}{a(k-1)-kx}\binom{a(k-1)-kx}{k-1}_q \times \\
&\qquad B_{n_1,\ldots,n_r,k}\left(y_{0,\ldots,0,1},\ldots,y_{n_1,\ldots,n_r}\right),
\end{aligned}
\tag{13}
$$

*and for* $f_n(x) = B_n(x)$, *we get*

$$
\begin{aligned}
y_{n_1,\ldots,n_r} &= \sum_{k=1}^{n_1+\cdots+n_r} \frac{kx}{a(k-1)+x} B_{k-1}\left(a(k-1)+x\right) \times \\
&\qquad B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right), \\
x_{n_1,\ldots,n_r} &= -\sum_{k=1}^{n_1+\cdots+n_r} \frac{kx}{a(k-1)-kx} B_{k-1}\left(a(k-1)-kx\right) \times \\
&\qquad B_{n_1,\ldots,n_r,k}\left(y_{0,\ldots,0,1},\ldots,y_{n_1,\ldots,n_r}\right).
\end{aligned}
\tag{14}
$$

**Example 1.** For $a = 0,\ x = 1$ in (11) and using the fact that

$$B_{n_1,\ldots,n_r,1}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right) = x_{n_1,\ldots,n_r} \quad \text{and}$$

$$B_{n_1,\ldots,n_r,2}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right)=\frac{1}{2}\sum_{1\leq i_1+\cdots+i_r\leq n_1+\cdots+n_r-1}\binom{n_1}{i_1}\cdots\binom{n_r}{i_r}\times x_{i_1,\ldots,i_r}x_{n_1-i_1,\ldots,n_r-i_r},$$

we obtain

$$y_{n_1,\ldots,n_r}=x_{n_1,\ldots,n_r}+\sum_{1\leq i_1+\cdots+i_r\leq n_1+\cdots+n_r-1}\binom{n_1}{i_1}\cdots\binom{n_r}{i_r}x_{i_1,\ldots,i_r}x_{n_1-i_1,\ldots,n_r-i_r},$$

$$x_{n_1,\ldots,n_r}=\sum_{k=1}^{n_1+\cdots+n_r}(-1)^{k-1}\frac{(2k-2)!}{(k-1)!}B_{n_1,\ldots,n_r,k}\left(y_{0,\ldots,0,1},\ldots,y_{n_1,\ldots,n_r}\right).$$

Now, consider $\delta:\mathbb{N}\rightarrow\{0,1\}$ be the application satisfying

$$\delta(0)=0\ \text{ and }\ \delta(n)=1\ \text{ for }\ n\geq 1.$$

**Theorem 7.** *Let $u,r,s$ be nonnegative integers $urs\neq 0$, and $\{u_n\}$ be a sequence of real numbers with $u_1=1$. Then*

$$\begin{aligned}y_{n_1,\ldots,n_r}&=s\sum_{k=1}^{n_1+\cdots+n_r}\frac{k}{U_k}\frac{B_{U_k+k-1,U_k}\left(1,u_2,u_3,\ldots\right)}{\binom{U_k+k-1}{U_k}}\times\\&\qquad B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1},\ldots,x_{n_1,\ldots,n_r}\right),\\x_{n_1,\ldots,n_r}&=y_{n_1,\ldots,n_r}-\delta\times s\sum_{k=2}^{n_1+\cdots+n_r}\frac{k}{V_k}\frac{B_{V_k+k-1,V_k}\left(1,u_2,u_3,\ldots\right)}{\binom{V_k+k-1}{V_k}}\times\\&\qquad B_{n_1,\ldots,n_r,k}\left(y_{0,\ldots,0,1},\ldots,y_{n_1,\ldots,n_r}\right),\end{aligned}\tag{15}$$

*where $\delta=\delta(n_1+\cdots+n_r-1)$, $U_k=(r+2s)(k-1)+s$ and $V_k=(r+s)(k-1)-s$.*

***Proof.*** Let $z_n(r):=\frac{1}{nr}\binom{(r+1)n}{nr}^{-1}B_{(r+1)n,nr}\left(1,u_2,u_3,\ldots\right)$, $n\geq 1$, $r$ integer $\geq 1$, and let the binomial type polynomials $\{f_n(x)\}$ defined by

$$f_n(x):=\sum_{k=1}^{n}B_{n,k}\left(z_1(r),z_2(r),\ldots\right)x^k,\ n\geq 1,\ \text{with}\ f_0(x)=1,$$

see [10]. Then from [6, 8] or [7], we have the identity

$$\sum_{k=1}^{n} B_{n,k}\left(z_1\left(r\right), z_2\left(r\right), \ldots\right) s^k$$
$$= \frac{s}{nr+s}\binom{\left(r+1\right)n+s}{nr+s}^{-1} B_{(r+1)n+s,\ nr+s}\left(1, u_2, u_3, \ldots\right),$$

where $r$ is an integer $\geq 1$ and $s$ is an integer, such that $nr+s \geq 1$. Then we get

$$f_n\left(s\right) = \frac{s}{nr+s}\binom{\left(r+1\right)n+s}{nr+s}^{-1} B_{(r+1)n+s,nr+s}\left(1, u_2, u_3, \ldots\right). \tag{16}$$

To obtain (15), it suffices to replace in (9) $a$ by zero, $f_n\left(x\right)$ by the expression of $f_n\left(s\right)$ given by (16) and $r$ by $r+2s$. ∎

**Example 2.** From the well-known identity $B_{n,k}\left(1!, \ldots, i!, \ldots\right) = \binom{n}{k}\frac{(n-1)!}{(k-1)!}$, we get

$$y_{n_1,\ldots,n_r} = s\sum_{k=1}^{n_1+\cdots+n_r} k\frac{\left(\left(r+2s+1\right)\left(k-1\right)+s-1\right)!}{\left(\left(r+2s\right)\left(k-1\right)+s\right)!}\times$$
$$B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1}, \ldots, x_{n_1,\ldots,n_r}\right)$$
$$x_{n_1,\ldots,n_r} = y_{n_1,\ldots,n_r} - \delta\times s\sum_{k=2}^{n_1+\cdots+n_r} k\frac{\left(\left(r+s+1\right)\left(k-1\right)-s-1\right)!}{\left(\left(r+s\right)\left(k-1\right)-s\right)!}\times$$
$$B_{n_1,\ldots,n_r,k}\left(y_{0,\ldots,0,1}, \ldots, y_{n_1,\ldots,n_r}\right),$$

where $\delta = \delta\left(n_1+\cdots+n_r-1\right)$.

We can obtain similar relations for the Stirling numbers (respectively the absolute Stirling numbers) of the first kind and the Stirling numbers of the second kind by setting $u_n = \left(-1\right)^{n-1}\left(n-1\right)!$ (respectively $u_n = \left(n-1\right)!$) and $u_n = 1,\ n\geq 1$.

**Corollary 8.** *Let $u, r, s$ be nonnegative integers, $a, \alpha$ be real numbers, such that $\alpha urs \neq 0$, and $\left\{f_n\left(x\right)\right\}$ be a polynomial sequence of binomial type. Then*

$$y_{n_1,\ldots,n_r} = s\sum_{k=1}^{n_1+\cdots+n_r} \frac{kD_{z=0}^{T_k}\left(e^{\alpha z}f_{k-1}\left(T_k x+z;a\right)\right)}{\alpha^{T_k-u(k-1)}T_k}\times$$
$$B_{n_1,\ldots,n_r,k}\left(x_{0,\ldots,0,1}, \ldots, x_{n_1,\ldots,n_r}\right),$$
$$x_{n_1,\ldots,n_r} = y_{n_1,\ldots,n_r} - \delta s\sum_{k=2}^{n_1+\cdots+n_r} \frac{kD_{z=0}^{R_k}\left(e^{\alpha z}f_{k-1}\left(R_k x+z;a\right)\right)}{\alpha^{R_k-u(k-1)}R_k}\times$$
$$B_{n_1,\ldots,n_r,k}\left(y_{0,\ldots,0,1}, \ldots, y_{n_1,\ldots,n_r}\right),$$

*where* $\delta = \delta(n_1 + \cdots + n_r - 1)$, $T_k = (u + r + 2s)(k-1) + s$ *and* $R_k = (u + r + s)(k-1) - s$.

***Proof.*** In Theorem 7 set

$$u_n = \frac{n}{(u(n-1)+1)\alpha} D_{z=0}^{u(n-1)+1} \left(e^{\alpha z} f_{n-1}\left((u(n-1)+1)x + z; a\right)\right)$$

and use the first identity of [8, Theorem 2]. ■

**Corollary 9.** *Let* $u, r, s$ *be nonnegative integers, such that* $urs \neq 0$, $a$ *be a real number and* $\{f_n(x)\}$ *be a binomial type polynomials. Then*

$$y_{n_1,\ldots,n_r} = s \sum_{k=1}^{n_1+\cdots+n_r} \frac{k! D_{z=0}^{T_k} f_{T_k+k-1}(T_k x + z; a)}{\alpha^{T_k - u(k-1)} (T_k + k - 1)! T_k} \times B_{n_1,\ldots,n_r,k}(x_{0,\ldots,0,1}, \ldots, x_{n_1,\ldots,n_r}),$$

$$x_{n_1,\ldots,n_r} = y_{n_1,\ldots,n_r} - \delta \times s \sum_{k=2}^{n_1+\cdots+n_r} \frac{k! D_{z=0}^{R_k} f_{R_k+k-1}(R_k x + z; a)}{\alpha^{R_k - u(k-1)} (R_k + k - 1)! R_k} \times B_{n_1,\ldots,n_r,k}(y_{0,\ldots,0,1}, \ldots, y_{n_1,\ldots,n_r}),$$

*where* $\delta = \delta(n_1 + \cdots + n_r - 1)$, $T_k = (u + r + 2s)(k-1) + s$ *and* $R_k = (u + r + s)(k-1) - s$.

***Proof.*** In Theorem 7 set

$$u_n = \frac{n! D_{z=0}^{u(n-1)+1} f_{((u+1)(n-1)+1)}\left((u(n-1)+1)x + z; a\right)}{((u+1)(n-1)+1)!\,(u(n-1)+1)\alpha}$$

and use the second identity of [8, Theorem 2]. ■

## 3. POLYNOMIALS OF MULTIPARTITIONAL TYPE AND INVERSE RELATIONS

For any sequence $u = (u_{n_1,\ldots,n_r})$ of real numbers, let

$$\varphi_u(\mathbf{t}) := \varphi_u(t_1, \ldots, t_r) = \sum_{n_1+\cdots+n_r \geq 1} u_{n_1,\ldots,n_r} \frac{t_1^{n_1}}{n_1!} \cdots \frac{t_r^{n_r}}{n_r!}$$

and for any sequences $u^{(1)} = \left(u^{(1)}_{n_1,\ldots,n_r}\right), \ldots, u^{(s)} = \left(u^{(s)}_{n_1,\ldots,n_r}\right)$ of real numbers, let be defined the polynomials of multipartitional type

$$B_{n_1,\ldots,n_r \;/\; j_1,\ldots,j_s}\left(u^{(1)}, \ldots, u^{(s)}\right)$$

by

$$\sum_{n_1+\cdots+n_r\geq j_1+\cdots+j_s\geq 1} B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(u^{(1)},\ldots,u^{(s)}\right)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!} := \frac{\left(\varphi_{u^{(1)}}\left(t_1,\ldots,t_r\right)\right)^{j_1}}{j_1!}\cdots\frac{\left(\varphi_{u^{(s)}}\left(t_1,\ldots,t_r\right)\right)^{j_s}}{j_s!}, \tag{17}$$

with

$$B_{n_1,\ldots,n_r\ /\ 0,\ldots,0}\left(u^{(1)},\ldots,u^{(s)}\right)=\begin{cases}1 & \text{if } n_1+\cdots+n_r=0,\\ 0 & \text{if } n_1+\cdots+n_r\geq 1.\end{cases}$$

**Theorem 10.** *Let* $\mathbf{a}=(a_1,\ldots,a_s)$ *be a vector of* $\mathbb{R}^s$ *and* $F=(F_1,\ldots,F_s):\mathbb{I}\rightarrow\mathbb{J}$ *be a bijective function with* $\mathbb{I},\ \mathbb{J}\subseteq\mathbb{R}^s$ *such that* $F(\mathbf{a}+\mathbf{x})$ *admits formal power series in each composite* $x_k$ *of* $\mathbf{x}$ *and let* $G=(G_1,\ldots,G_s)$ *denote the compositional inverse of* $F$ *so that* $(F\circ G)(\mathbf{x})=(G\circ F)(\mathbf{x})=\mathbf{x}$*. Then, we have the vectorial pair of inverse relations*

$$\begin{aligned}
u^{(1)}_{n_1,\ldots,n_r} &= \sum_J \frac{\partial F_1^{j_1+\cdots+j_s}}{\partial x_1^{j_1}\cdots\partial x_s^{j_s}}(\mathbf{a})\,B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(v^{(1)},\ldots,v^{(s)}\right),\\
&\ \ \vdots\\
u^{(s)}_{n_1,\ldots,n_r} &= \sum_J \frac{\partial F_s^{j_1+\cdots+j_s}}{\partial x_1^{j_1}\cdots\partial x_s^{j_s}}(\mathbf{a})\,B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(v^{(1)},\ldots,v^{(s)}\right),
\end{aligned}$$

*if and only if*

$$\begin{aligned}
v^{(1)}_{n_1,\ldots,n_r} &= \sum_J \frac{\partial G_1^{j_1+\cdots+j_s}}{\partial x_1^{j_1}\cdots\partial x_s^{j_s}}(F(\mathbf{a}))\,B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(u^{(1)},\ldots,u^{(s)}\right),\\
&\ \ \vdots\\
v^{(s)}_{n_1,\ldots,n_r} &= \sum_J \frac{\partial G_s^{j_1+\cdots+j_s}}{\partial x_1^{j_1}\cdots\partial x_s^{j_s}}(F(\mathbf{a}))\,B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(u^{(1)},\ldots,u^{(s)}\right),
\end{aligned}$$

*where* $J=\{(j_1,\ldots,j_s)\in\mathbb{N}^s:1\leq j_1+\cdots+j_s\leq n_1+\cdots+n_r\}$.

***Proof.*** We have

$$\begin{aligned}
\varphi_{u^{(k)}}(\mathbf{t}) &= \sum_{n_1+\cdots+n_r\geq 1} u_{n_1,\ldots,n_r}\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!} \\
&= \sum_{n_1+\cdots+n_r\geq 1}\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}\sum_{j_1+\cdots+j_k\leq n_1+\cdots+n_r}\frac{\partial F_k^{j_1+\cdots+j_s}}{\partial x_1^{j_1}\cdots\partial x_s^{j_s}}(\mathbf{a}) \\
&\qquad \times B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(v^{(1)},\ldots,v^{(s)}\right) \\
&= \sum_{j_1+\cdots+j_k\geq 1}\frac{\partial F_k^{j_1+\cdots+j_s}}{\partial x_1^{j_1}\cdots\partial x_s^{j_s}}(\mathbf{a})\sum_{n_1+\cdots+n_r\geq j_1+\cdots+j_k}\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!} \\
&\qquad \times B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(v^{(1)},\ldots,v^{(s)}\right) \\
&= \sum_{j_1+\cdots+j_k\geq 1}\frac{\partial F_k^{j_1+\cdots+j_s}}{\partial x_1^{j_1}\cdots\partial x_s^{j_s}}(\mathbf{a})\frac{\left(\varphi_{v^{(1)}}(\mathbf{t})\right)^{j_1}}{j_1!}\cdots\frac{\left(\varphi_{v^{(s)}}(\mathbf{t})\right)^{j_s}}{j_s!} \\
&= F_k\left(\mathbf{a}+\left(\varphi_{v^{(1)}}(\mathbf{t}),\ldots,\varphi_{v^{(s)}}(\mathbf{t})\right)\right)-F_k(\mathbf{a})
\end{aligned}$$

which implies

$$\begin{aligned}
&\left(F_1(\mathbf{a})+\varphi_{u^{(1)}}(\mathbf{t}),\ldots,F_s(\mathbf{a})+\varphi_{u^{(s)}}(\mathbf{t})\right) \\
&=\left(F_1\left(\mathbf{a}+\left(\varphi_{v^{(1)}}(\mathbf{t}),\ldots,\varphi_{v^{(s)}}(\mathbf{t})\right)\right),\ldots,F_s\left(\mathbf{a}+\left(\varphi_{v^{(1)}}(\mathbf{t}),\ldots,\varphi_{v^{(s)}}(\mathbf{t})\right)\right)\right)
\end{aligned}$$

or equivalently

$$F(\mathbf{a})+\left(\varphi_{u^{(1)}}(\mathbf{t}),\ldots,\varphi_{u^{(s)}}(\mathbf{t})\right)=F\left(a_1+\varphi_{v^{(1)}}(\mathbf{t}),\ldots,a_s+\varphi_{v^{(s)}}(\mathbf{t})\right).$$

Then, applying $G$ to the two sides of this equality, we obtain

$$\begin{aligned}
\varphi_{v^{(k)}}(\mathbf{t}) &= \sum_{j_1+\cdots+j_s\geq 1} \frac{\partial G_k^{j_1+\cdots+j_s}}{\partial x_1^{j_1}\cdots\partial x_s^{j_s}}(F(\mathbf{a}))\frac{(\varphi_{u^{(1)}}(\mathbf{t}))^{j_1}}{j_1!}\cdots\frac{(\varphi_{u^{(s)}}(\mathbf{t}))^{j_s}}{j_s!} \\
&= \sum_{j_1+\cdots+j_s\geq 1} \frac{\partial G_k^{j_1+\cdots+j_s}}{\partial x_1^{j_1}\cdots\partial x_s^{j_s}}(F(\mathbf{a})) \sum_{n_1+\cdots+n_r\geq j_1+\cdots+j_s} \frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!}\times \\
&\qquad B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(u^{(1)},\ldots,u^{(s)}\right) \\
&= \sum_{n_1+\cdots+n_r\geq 1} \frac{t_1^{n_1}}{n_1!}\cdots\frac{t_r^{n_r}}{n_r!} \sum_{1\leq j_1+\cdots+j_s\leq n_1+\cdots+n_r} \frac{\partial G_k^{j_1+\cdots+j_s}}{\partial x_1^{j_1}\cdots\partial x_s^{j_s}}(F(\mathbf{a}))\times \\
&\qquad B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(u^{(1)},\ldots,u^{(s)}\right).
\end{aligned}$$

■

**Corollary 11.** *Let $A=(\alpha_{ij})_{1\leq i,j\leq s}$ be a nonsingular matrix of real numbers and $B=A^{-1}=(\beta_{ij})$ be the inverse matrix of $A$. Then we have the vectorial pair of inverse relations*

$$\begin{aligned}
u_{n_1,\ldots,n_r}^{(1)} &= \sum_J (\alpha_{11})_{j_1}\cdots(\alpha_{1s})_{j_s} B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(v^{(1)},\ldots,v^{(s)}\right), \\
&\ \ \vdots \\
u_{n_1,\ldots,n_r}^{(s)} &= \sum_J (\alpha_{s1})_{j_1}\cdots(\alpha_{ss})_{j_s} B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(v^{(1)},\ldots,v^{(s)}\right),
\end{aligned}$$

*if and only if*

$$\begin{aligned}
v_{n_1,\ldots,n_r}^{(1)} &= \sum_J (\beta_{11})_{j_1}\cdots(\beta_{1s})_{j_s} B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(u^{(1)},\ldots,u^{(s)}\right), \\
&\ \ \vdots \\
v_{n_1,\ldots,n_r}^{(s)} &= \sum_J (\beta_{s1})_{j_1}\cdots(\beta_{ss})_{j_s} B_{n_1,\ldots,n_r\ /\ j_1,\ldots,j_s}\left(u^{(1)},\ldots,u^{(s)}\right).
\end{aligned}$$

***Proof.*** The corollary follows by setting, in Theorem 10, $\mathbf{a}=(1,\ldots,1)$ and $F(\mathbf{x})=\left(x_1^{\alpha_{11}}\cdots x_s^{\alpha_{1s}},\ldots,x_1^{\alpha_{s1}}\cdots x_s^{\alpha_{ss}}\right)$. ■

**Remark 4.** If we set $s=1$ in Theorem 10, we obtain Theorem 1, and, if we set $s=r=1$ in Theorem 10, we obtain Theorem 1 given in [7].

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Received 24 February 2011
Revised 6 September 2011