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DISTRIBUTIVE LATTICES OF *t*-*k*-ARCHIMEDEAN SEMIRINGS*

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Abstract

A semiring S in $\mathbb{SL}^+$ is a t-k-Archimedean semiring if for all $a, b \in S$, $b \in \sqrt{Sa} \cap \sqrt{aS}$. Here we introduce the t-k-Archimedean semirings and characterize the semirings which are distributive lattice (chain) of t-k-Archimedean semirings. A semiring S is a distributive lattice of t-k-Archimedean semirings if and only if $\sqrt{B}$ is a k-ideal, and S is a chain of t-k-Archimedean semirings if and only if $\sqrt{B}$ is a completely prime k-ideal, for every k-bi-ideal B of S.

Keywords: k-radical, t-k-Archimedean semiring, completely prime k-ideal, semiprimary k-ideal.

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1. Introduction

In 1942, A.H. Clifford [6] first defined the semilattice decompositions of semigroups. Thus the idea of studying a semigroup through its greatest semilattice decomposition was introduced. The idea consists of decomposing a given semigroup S into subsemigroups (components) which are possibly of considerably simpler structure, through a congruence η on S such that S/η is the greatest semilattice homomorphic image of S and each η-class is a component subsemigroup. Though the idea first appeared in [6] but much attention was given to the semigroups which are union of groups. In 1954, T. Tamura and N. Kimura [13] showed that every commutative semigroup is a semilattice of Archimedean semigroups. This well known result has since been generalized by M. Petrich, M.S.

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Putcha, T. Tamura, N. Kimura, S. Bogdanovic, M. Ciric, F. Kmet and many others [3, 4, 5, 7, 8, 9, 10, 11, 12]. Much attention in this area has been aimed to the semigroups which are decomposable into a semilattice of Archimedean semigroups.

In this article we introduce the t-k-Archimedean semirings and characterize the semirings which are distributive lattices(chain) of t-k-Archimedean semirings. The t-k-Archimedean semirings are the semirings analogue of t-Archimedean semigroups, in some sense. The k-bi-ideals play a crucial role in characterizing such semirings. A necessary and sufficient condition for a semiring S to be a distributive lattice of t-k-Archimedean semirings is that the k-radical of each k-bi-ideal B of S is a k-ideal of S.

The preliminaries and prerequisites we need have been discussed in Section 2. In Section 3, several equivalent characterizations have been made for the semirings which are distributive lattices of t-k-Archimedean semirings, which is the main theorem of this article. In Section 4, the semirings which are chains of t-k-Archimedean semirings has been characterized. A semiring S is a chain of t-k-Archimedean semirings if and only if k-radical of each k-bi-ideal of S is a completely prime k-ideal.

2. Preliminaries

A *semiring* $(S,+,\cdot)$ is an algebra with two binary operations $+$ and $\cdot$ such that both the *additive reduct* $(S,+)$ and the *multiplicative reduct* $(S,\cdot)$ are semigroups and such that the following distributive laws hold:

$$x(y+z)=xy+xz \text{ and } (x+y)z=xz+yz.$$

Thus the semirings can be regarded as a common generalization of both rings and distributive lattices. By $\mathbb{SL}^+$ we denote the variety of all semirings $(S,+,\cdot)$ such that $(S,+)$ is a semilattice, i.e., a commutative and idempotent semigroup. Throughout this paper, unless otherwise stated, S is always a semiring in $\mathbb{SL}^+$.

Let A be a nonempty subset of S. Then the *k-closure* $\overline{A}$ of A in S is defined by

$$\overline{A}=\{x\in S \mid x+a_1=a_2, \text{ for some } a_1,a_2\in A\}.$$

Then we have, $A\subseteq\overline{A}$ and if $(A,+)$ is a subsemigroup of $(S,+)$ then $\overline{A}=\{x\in S \mid x+a\in A, \text{ for some } a\in A\}$, since $(S,+)$ is a semilattice. A is called a k-set if $\overline{A}\subseteq A$. An ideal K of S is called a *k-ideal* of S if it is a k-set. A subsemiring B of S is called a *k-bi-ideal* of S if $BSB\subseteq B$ and B is a k-set. For $a\in S$, the least k-bi-ideal $B_k(a)$ of S [1], which contains a is given by

$$B_k(a)=\left\{u\in S \mid u+a+a^2+asa=a+a^2+asa, \text{ for some } s\in S\right\}.$$

We note that $\overline{aSa} = \{x \in S \mid x + asa = asa$, for some $s \in S\}$ is a k-bi-ideal of S but may not contain a. A nonempty subset A of S is called *completely prime (resp. semiprimary)* if for $x, y \in S$, $xy \in A$ implies $x \in A$ or $y \in A$ (resp. $x^n \in A$ or $y^n \in A$, for some $n \in \mathbb{N}$).

Let F be a subsemiring of S. F is called a left (right) filter of S if:

(i) for any $a, b \in S$, $ab \in F \Rightarrow b \in F$ $(a \in F)$; and
(ii) for any $a \in F$, $b \in S$, $a + b = b \Rightarrow b \in F$. F is a filter of S if it is both a left and a right filter of S. The least filter of S containing a is denoted by $N(a)$. Let $\mathcal{N}$ be the equivalence relation on S defined by

$$\mathcal{N} = \{(x, y) \in S \times S \mid N(x) = N(y)\}.$$

Lemma 1. *Let S be a semiring in $\mathbb{SL}^+$.*

(a) *For $a, b \in S$ the following statements are equivalent:*
  (i) *There are $s_1, s_2, t_1, t_2 \in S$ such that $b + s_1at_1 = s_2at_2$;*
  (ii) *There is $s \in S$ such that $b + sas = sas$.*
(b) *If $a, b, c, d \in S$ such that $c + xa = xa$ and $d + yb = yb$ for some $x, y \in S$, then there is some $z \in S$ such that $c + za = za$ and $d + zb = zb$.*
(c) *If $a, b \in S$ such that $a + bxb = byb$ for some $x, y \in S$, then there is some $z \in S$ such that $a + bzb = bzb$.*

Proof. (a) Since (ii)$\Rightarrow$(i) is clear, we assume (i). For $x = s_1+s_2+t_1+t_2$ one gets $s_1at_1+xax = s_2at_2+xax = xax$, since $(S, +)$ is a semilattice, and then $b+s_1at_1+xax = s_2at_2 + xax$ implies that $b + xax = xax$.

Hence (i) implies (ii).

(b) Clearly, $z = x + y$ is such an element.

(c) Again, $z = x + y$ is such an element. ■

Let A be a non-empty subset of a semiring S. Then the *k-radical* of A in S is given by $\sqrt{A} = \{x \in S \mid (\exists\, n \in \mathbb{N})\; x^n \in \overline{A}\}$. Thus $\sqrt{A} = \{x \in S \mid (\exists\, n \in \mathbb{N})\; x^n + a_1 = a_2$, for some $a_1, a_2 \in A\}$. It is interesting to note that for every $a \in S, \sqrt{aSa} = \sqrt{B_k(a)}$, though $aSa \subseteq B_k(a)$ and the inclusion is likely to be proper.

Let S be a semiring in $\mathbb{SL}^+$. Define a binary relation σ on S by: for $a, b \in S$,

$$a\sigma b \;\Leftrightarrow\; b \in \sqrt{SaS} \;\Leftrightarrow\; b^n \in \overline{SaS} \text{ for some } n \in \mathbb{N}.$$

Then $a^3 \in SaS \subseteq \overline{SaS}$ shows that σ is reflexive. So the transitive closure $\rho = \sigma^*$ is reflexive and transitive, and therefore the symmetric relation $\eta = \rho \cap \rho^{-1}$ is an equivalence relation. This equivalence relation η is the least distributive lattice congruence on S.

Lemma 2 [2]**.** *For any S in $\mathbb{SL}^+$, η is the least distributive lattice congruence on S.*

Definition. A semiring S in $\mathbb{SL}^+$ is called left (right) k-Archimedean if for all $a \in S$, $S = \sqrt{Sa}(\sqrt{aS})$ and t-k-Archimedean semiring if it is both a left k-Archimedean semiring and a right k-Archimedean semiring.

Then by Lemma 1, a semiring S is t-k-Archimedean if and only if for $a, b \in S$ there exist $n \in \mathbb{N}$ and $x \in S$ such that $b^n + xa = xa$ and $b^n + ax = ax$. A semiring S is called a *distributive lattice(chain) of t-k-Archimedean semirings* if there exists a congruence ρ on S such that S/ρ is a distributive lattice(chain) and each ρ-class is a t-k-Archimedean semiring.

3. Distributive lattices of t-k-Archimedean semirings

In this section we describe the semirings S by radical of k-bi-ideal of S. In the following proofs we will use that from $b + c = c$ for $b, c \in S$ in any semiring S in $\mathbb{SL}^+$ it follows that $b^n + c^n = c^n$ for every $n \in \mathbb{N}$. This can be proved by induction. Since the case $n = 1$ is given, we may assume $b^n + c^n = c^n$ for some $n \in \mathbb{N}$. Then $b^{n+1} + c^n b = c^n b$ and by adding c^{n+1} on both sides we get $b^{n+1} + c^n(b + c) = c^n(b + c)$, and hence $b^{n+1} + c^{n+1} = c^{n+1}$.

Lemma 3. *Let S be a semiring in $\mathbb{SL}^+$ such that for all $a, b \in S, ab \in \sqrt{Sa} \cap \sqrt{bS}$. Then*

1. *for all $a, b \in S$, $a \in \overline{bSb} \Rightarrow a \in \sqrt{b^{2^r} S b^{2^r}}$ for all $r \in \mathbb{N}$.*

2. *for all* $a, b \in S$, $a \in \sqrt{bSb}$ *implies that* $\sqrt{aSa} \subseteq \sqrt{bSb}$.
3. *the least distributive lattice congruence* η *on* S *is given by: for* $a, b \in S$,

$$a\eta b \Leftrightarrow b \in \sqrt{aSa} \text{ and } a \in \sqrt{bSb}.$$

Proof. (1) Let $a, b \in S$ such that $a \in \overline{bSb}$. Then there exists $s \in S$ such that $a + bsb = bsb$. By hypothesis, there exist $m \in \mathbb{N}$ and $u \in S$ such that $(b^2s)^m + ub = ub$. Then $a^{m+1} + (bsb)^{m+1} = (bsb)^{m+1}$ gives $a^{m+1} + bsub^2 = bsub^2$. Again, there are $n \in \mathbb{N}$ and $v \in S$ such that $(bsub^2)^n + b^2v = b^2v$. Then we have $a^{(m+1)(n+1)} + b^2vbsub^2 = b^2vbsub^2$ which yields $a \in \sqrt{b^2Sb^2}$. Thus the result is true for $r = 1$. Let $a \in \sqrt{b^{2^k}Sb^{2^k}}$ for some $k \in \mathbb{N}$. Then there are $p \in \mathbb{N}$ and $w \in S$ such that

$$a^p + b^{2^k}wb^{2^k} = b^{2^k}wb^{2^k}$$

Now iterating this implication as above, we get $a \in \sqrt{b^{2^{k+1}}Sb^{2^{k+1}}}$. Therefore, by the method of principle of mathematical induction, we have: for every $r \in \mathbb{N}$, $a \in \sqrt{b^{2^r}Sb^{2^r}}$.

(2) For $a \in \sqrt{bSb}$ there are $n \in \mathbb{N}$ and $s \in S$ such that $a^n + bsb = bsb$. Let $x \in \sqrt{aSa}$. Then there exists $m \in \mathbb{N}$ such that $x^m \in \overline{aSa}$. Suppose $r \in \mathbb{N}$ be such that $2^r > n$. Then by (1), we find $p \in \mathbb{N}$ and $u \in S$ such that $x^p + a^{2^r}ua^{2^r} = a^{2^r}ua^{2^r}$ which implies $x^p + bsba^{2^r-n}ua^{2^r-n}bsb = bsba^{2^r-n}ua^{2^r-n}bsb$, and so $x \in \sqrt{bSb}$. Thus the result.

(3) From Theorem 3.4 [2], we have the least distributive lattice congruence η on S as follows:

$$\eta = \rho \cap \rho^{-1}, \text{ where } \rho = \sigma^* \text{ and } a\sigma b \Leftrightarrow b \in \sqrt{SaS}.$$

Let us define a binary relation ξ on S by: for $a, b \in S$,

$$a\xi b \Leftrightarrow b \in \sqrt{aSa} \text{ and } a \in \sqrt{bSb}.$$

We will show $\xi = \eta$. Clearly $\sqrt{aSa} \subseteq \sqrt{SaS}$. Now let $x \in \sqrt{SaS}$. Then there are $n \in \mathbb{N}$ and $s \in S$ such that $x^n + sas = sas$. Again, $sas \in \sqrt{Ssa} \subseteq \sqrt{Sa}$ implies that $(sas)^m + ta = ta$ for some $m \in \mathbb{N}$ and $t \in S$. Also, there are $p \in \mathbb{N}$ and $u \in S$ such that $(ta)^p + au = au$, which gives $x^{nm(p+1)} + auta = auta$, and so $x \in \sqrt{aSa}$. Thus $\sqrt{SaS} = \sqrt{aSa}$. Now $a\eta b$ implies that there are $c_1, c_2, \ldots, c_n, d_1, d_2, \ldots, d_m \in S$ such that $a\sigma c_1, c_1\sigma c_2, \ldots, c_{n-1}\sigma c_n, c_n\sigma b$ and $b\sigma d_1, d_1\sigma d_2, \ldots, d_{m-1}\sigma d_m, d_m\sigma a$. These give $c_1 \in \sqrt{aSa}, c_2 \in \sqrt{c_1Sc_1}, \ldots, b \in \sqrt{c_nSc_n}$ and $d_1 \in \sqrt{bSb}$, $d_2 \in \sqrt{d_1Sd_1}, \ldots, a \in \sqrt{d_mSd_m}$ so that $b \in \sqrt{aSa}$ and $a \in \sqrt{bSb}$, by (2). Thus $a\xi b$. Again $a\xi b$ implies $b \in \sqrt{aSa}$ and $a \in \sqrt{bSb}$ which yields $a\sigma b$ and $b\sigma a$ to get $a\eta b$. Thus $\xi = \eta$. ■

Remark 4. Let S be a semiring in $\mathbb{SL}^+$ and $a \in S$. Then $\sqrt{aSa} = \sqrt{B_k(a)}$. Thus it follows that if for all $a, b \in S,\ ab \in \sqrt{Sa} \cap \sqrt{bS}$ then the least distributive lattice congruence η on S is given by: for $a,\ b \in S$,

$$a\eta b \Leftrightarrow\ a \in \sqrt{B_k(b)} \text{ and } b \in \sqrt{B_k(a)}.$$

Now we prove the main theorem of this article.

Theorem 5. *The following conditions on a semiring S in $\mathbb{SL}^+$ are equivalent:*

1. *S is a distributive lattice of t-k-Archimedean semirings;*
2. *For all $a, b \in S,\ b \in \overline{SaS}$ implies $b \in \sqrt{Sa} \cap \sqrt{aS}$;*
3. *For all $a, b \in S,\ ab \in \sqrt{Sa} \cap \sqrt{bS}$;*
4. *$\sqrt{B}$ is a k-ideal of S, for every k-bi-ideal B of S;*
5. *$\sqrt{B_k(a)} = \sqrt{aSa}$ is a k-ideal of S, for all $a \in S$;*
6. *$N(a) = \{x \in S \mid a \in \sqrt{Sx} \cap \sqrt{xS}\}$ for all $a \in S$.*

Proof. Scheme of the proof: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1), (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (3), (3) $\Leftrightarrow$ (6).

(1) $\Rightarrow$ (2) : Let ν be a distributive lattice congruence on S such that the ν-classes $T_\alpha; \alpha \in S/\nu$ are t-k-Archimedean semirings. Let $a, b \in S$ be such that $b \in \overline{SaS}$. Then there is $s \in S$ such that $b + sas = sas$, which gives $b^3 + uau = uau$, where $u = bs + sb$. Now $uau\nu au^2\nu au\nu ua$ implies that $uau, au, ua \in T_\alpha$, for some $\alpha \in S/\nu$. Then there exist $m \in \mathbb{N}$ and $v \in T_\alpha$ such that $(uau)^m + auv = auv$ and $(uau)^m + vua = vua$. From these we get $b^{3m} + auv = auv$ and $b^{3m} + vua = vua$, and so $b \in \sqrt{aS} \cap \sqrt{Sa}$.

(2) $\Rightarrow$ (3) : Let $a, b \in S$. Now $(ab)^2 \in \overline{SaS} \cap \overline{SbS}$ implies that there exist $m, n \in \mathbb{N}$ such that $(ab)^{2m} \in \overline{Sa}$ and $(ab)^{2n} \in \overline{bS}$. Thus $ab \in \sqrt{Sa} \cap \sqrt{bS}$.

(3) $\Rightarrow$ (1) : By the Lemma 3, the least distributive lattice congruence η on S is given by: for $a, b \in S$

$$a\eta b \Leftrightarrow a \in \sqrt{bSb} \text{ and } b \in \sqrt{aSa}.$$

Let T be an η-class. Then T is a subsemiring of S, since η is a distributive lattice congruence. Let $a, b \in T$. Then there are $n \in \mathbb{N}$ and $s \in S$ such that $a^n + bsb = bsb$ and $b^n + asa = asa$. Then we have $a^{n+1} + (a + abs)b = (a + abs)b$ and $a^{n+1} + b(sba + a) = b(sba + a)$. Now $a\eta(a + abs)\eta(sba + a)$ implies $a + abs,\ sba + a \in T$, and so $a \in \sqrt{Tb} \cap \sqrt{bT}$. Hence T is a t-k-Archimedean semiring.

(3) $\Rightarrow$ (4) : Let B be a k-bi-ideal of S and let $u, v \in \sqrt{B}, c \in S$. Then there exist $n \in \mathbb{N}$ and $b \in B$ such that $u^n + b = b$ and $v^n + b = b$. By (3), there exist $x, y \in S$ and $m_1, t_1 \in \mathbb{N}$ such that $(uc)^{m_1} + xu = xu,\ (ux)^{t_1} + yu = yu$. Then $(uc)^{m_1(t_1+1)} + x(ux)^{t_1}u = x(ux)^{t_1}u$ implies $(uc)^{m_2} + x_1u^2 = x_1u^2$, where $m_2 = m_1(t_1 + 1)$ and $x_1 = xy$. Also, there exist $s \in S$ and $t_2 \in \mathbb{N}$ such

that $(u^2x_1)^{t_2} + su^2 = su^2$. Iterating similarly we find that for every $r \in \mathbb{N}$, there exists $p \in \mathbb{N}$ such that $(uc)^p + x_r u^{2^r} = x_r u^{2^r}$. Let $r \in \mathbb{N}$ be such that $2^r > n$. Then there exists $q \in \mathbb{N}$ such that $(uc)^q + x_r u^{2^r} = x_r u^{2^r}$. By (3), there exist $t \in \mathbb{N}$ and $z \in S$ such that $(x_r u^{2^r})^t + u^{2^r} z = u^{2^r} z$. Then we have $(uc)^{q(t+1)} + bu^{2^r-n} z x_r u^{2^r-n} b = bu^{2^r-n} z x_r u^{2^r-n} b$. Hence $uc \in \sqrt{B}$. Similarly, $cu \in \sqrt{B}$. Again we have $(u+b)^n + sbs + sb + bs = u^n + sbs + sb + bs$, for some $s \in S$, i.e., $(u+b)^n + b + sbs + sb + bs = b + sbs + sb + bs$. Then for $w = (u+b)s + s(u+b) + u + b$, we have $(u+b)^{n+2} + wbw = wbw$. Also, there are $m \in \mathbb{N}$ and $y \in S$ such that $(wbw)^m + ywb = ywb$. Then we have $(u+b)^{(n+2)(m+1)} + wbwywb = wbwywb$. Again there exist $p \in \mathbb{N}$ and $z \in S$ such that $(wbwywb)^p + bz = bz$. Then we get $(u+b)^{(n+2)(m+1)(p+1)} + bzwbwywb = bzwbwywb$. This implies $(u+b) \in \sqrt{B_k(b)} \subseteq \sqrt{B}$. Again for some $t \in S$ we have $(u+v)^n + tut + tu + ut = v^n + tut + tu + ut$ which implies that $(u+v)^n + t(u+b)t + t(u+b) + (u+b)t + (u+b) = t(u+b)t + t(u+b) + (u+b)t + (u+b)$ which again implies that $(u+v)^{n+2} + s(u+b)s = s(u+b)s$, where $s = (u+v)t + t(u+v) + u + v$. Now $s(u+b)s \in \sqrt{B}$ shows that there are $m \in \mathbb{N}$ and $b \in B$ such that $(s(u+b)s)^m + b = b$ and hence $(u+v)^{(n+2)m} + b = b$. Thus $(u+v) \in \sqrt{B}$, and so $\sqrt{B}$ is an ideal of S. Let $s \in S$, $a \in \sqrt{B}$ such that $s + a = a$. Then there is $b \in B$ and $n \in \mathbb{N}$ such that $a^n + b = b$. Now $s^n + a^n = a^n$ gives $s^n + b = b$ which yields $s \in \sqrt{B}$. Thus $\sqrt{B}$ is a k-ideal of S.

(4) $\Rightarrow$ (5) : Obvious.

(5) $\Rightarrow$ (3) : Let $a, b \in S$. Then $\sqrt{aSa}$ and $\sqrt{bSb}$ are k-ideals of S. Then $ab \in \sqrt{aSa} \cap \sqrt{bSb}$ and hence $ab \in \sqrt{Sa} \cap \sqrt{bS}$.

(3) $\Rightarrow$ (6) : Let $a \in S$ and $F = \{x \in S \mid a \in \sqrt{Sx} \cap \sqrt{xS}\}$ and $y, z \in F$. Then there exist $n \in \mathbb{N}$ and $u \in S$ such that $a^n + uy = uy$, $a^n + uz = uz$ and $a^n + yu = yu$, $a^n + zu = zu$ so that $a^n + u(y+z) = u(y+z)$, and $a^n + (y+z)u = (y+z)u$, which imply $y + z \in F$. By (3), there exist $m \in \mathbb{N}$ and $v \in S$ such that $(yu)^m + vy = vy$ and $(uz)^m + zv = zv$. Now we can write $a^{n(m+1)} + u(yu)^m z = u(yu)^m z$ and $a^{n(m+1)} + y(uz)^m u = y(uz)^m u$, which give $a^{n(m+1)} + uv(yz) = uv(yz)$ and $a^{n(m+1)} + (yz)vu = (yz)vu$ so that $yz \in F$. Thus F is a subsemiring of S. Let $y \in F$ and $c \in S$ such that $y + c = c$. Then there exist $n \in \mathbb{N}$ and $u \in S$ such that $a^n + uy = uy$ and $a^n + yu = yu$, which imply $a^n + uc = uc$ and $a^n + cu = cu$ so that $c \in F$.

Now let $y, z \in S$ such that $yz \in F$. Then there exist $n \in \mathbb{N}$ and $u \in S$ such that $a^n + (uy)z = (uy)z$ and $a^n + y(zu) = y(zu)$. By (3), $(yz)^m + sy = sy$ and $(yz)^m + zs = zs$, for some $m \in \mathbb{N}$ and $s \in S$. Since F is a subsemiring, $(yz)^m \in F$. Then there exist $r \in \mathbb{N}$ and $v \in S$ such that $a^r + v(yz)^m = v(yz)^m$, and $a^r + (yz)^m v = (yz)^m v$. This implies $a^r + (vs)y = (vs)y$, and $a^r + z(sv) = z(sv)$. Hence $y, z \in F$. Thus F is a filter of S containing a. Let T be a filter of S containing a and let $y \in F$. Then $a^n + sy = sy$, for some $n \in \mathbb{N}$ and $s \in S$. Now since T is a filter, $a \in T$ implies $a^n \in T$ and so $a^n + sy = sy$ implies that $y \in T$.

Thus $F = N(a)$. It also follows directly that $\{x \in S \mid a \in \sqrt{Sx} \cap \sqrt{xS}\} = \{x \in S \mid a \in \sqrt{xSx}\}$.

$(6) \Rightarrow (3)$: Let $a, b \in S$. Then $a, b \in N(ab)$, since $N(ab)$ is a filter of S. So by (6), $ab \in \sqrt{Sa} \cap \sqrt{bS}$. ■

4. Chains of t-k-Archimedean semirings

In this section we characterize the semirings which are chains of t-k-Archimedean semirings. Let $(T, +, \cdot)$ be a distributive lattice with the partial order defined by $a \leq b \Leftrightarrow a + b = b$ for all $a, b \in S$. It is well known that $(T, \leq)$ is a chain if and only if $ab = b$ or $ab = a$ for all $a, b \in T$.

Theorem 6. *The following conditions on a semiring S in $\mathbb{SL}^+$ are equivalent:*

1. *S is a chain of t-k-Archimedean semirings;*
2. *S is a distributive lattice of t-k-Archimedean semirings and for all $a, b \in S$,*
$$b \in \sqrt{aSa} \quad or \quad a \in \sqrt{bSb};$$
3. *For all $a, b \in S$, $N(a) = \{x \in S \mid a \in \sqrt{xSx}\}$ and $N(ab) = N(a) \cup N(b)$;*
4. *$\eta = \mathcal{N}$ is the least chain congruence on S such that each of its congruence classes is t-k-Archimedean.*

Proof. $(1) \Rightarrow (2)$: Let S be a chain $\mathcal{C}$ of t-k-Archimedean semirings $S_\alpha (\alpha \in \mathcal{C})$. Then S is a distributive lattice of t-k-Archimedean semirings too. Let $a, b \in S$. Then there exist $\alpha,\ \beta \in \mathcal{C}$ such that $a \in S_\alpha,\ b \in S_\beta$. Since $\mathcal{C}$ is a chain, either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$ then $a,\ ab \in S_\alpha$. Since S_α is t-k-Archimedean, there exist $n \in \mathbb{N}$ and $x \in S_\alpha$ such that $a^n + abxab = abxab$. Since S is a distributive lattice of t-k-Archimedean semirings, there exist $m \in \mathbb{N}$ and $y \in S$ such that $(abxab)^m + bxaby = bxaby$, by Theorem 5. Then we have $a^{n(m+1)} + bxabyabxab = bxabyabxab$, i.e., $a \in \sqrt{bSb}$. If $\alpha\beta = \beta$, then $b, ab \in S_\beta$. Similarly, proceeding as above we have $b \in \sqrt{aSa}$.

$(2) \Rightarrow (3)$: Since S is a distributive lattice of t-k-Archimedean semirings, by Theorem 5, $N(a) = \{x \in S \mid a \in \sqrt{xSx}\}$. Let $a, b \in S$. Then $ab \in N(ab) \Rightarrow a \in N(ab)$ and $b \in N(ab)$. Then $N(a) \subseteq N(ab)$ and $N(b) \subseteq N(ab) \Rightarrow N(a) \cup N(b) \subseteq N(ab)$. By hypothesis, either $a \in \sqrt{bSb}$ or $b \in \sqrt{aSa}$. If $a \in \sqrt{bSb}$, then there exist $m \in \mathbb{N}$ and $x \in S$ such that $a^m + bxb = bxb$. Now $a^{m+1} + abxb = abxb$. Since S is a distributive lattice of t-k-Archimedean semirings, by Theorem 5, there exist $n \in \mathbb{N}$ and $y \in S$ such that $(babx)^n + yba = yba$. Then $a^{m+1} + abxb = abxb$ implies $a^{(m+1)(n+1)} + abxybab = abxybab$ so that $a \in \sqrt{abSab}$, i.e. $ab \in N(a)$, and so $N(ab) \subseteq N(a)$. If $b \in \sqrt{aSa}$, then similarly we have $N(ab) \subseteq N(b)$. Thus $N(ab) \subseteq N(a) \cup N(b)$ and hence $N(ab) = N(a) \cup N(b)$.

$(3) \Rightarrow (4)$: By Theorem 5, S satisfies $ab \in \sqrt{Sa} \cap \sqrt{bS}$. Then by Lemma 3, the least distributive lattice congruence η on S is given by: for all $a, b \in S, a\eta b \Leftrightarrow a \in \sqrt{bSb}$ and $b \in \sqrt{aSa}$. Let $a, b \in S$ be such that $a\eta b$. Then $a \in \sqrt{bSb}$ and $b \in \sqrt{aSa}$ which, by Lemma 3, implies that $\sqrt{aSa} = \sqrt{bSb}$. Then

$$\begin{aligned} x \in N(a) &\Leftrightarrow a \in \sqrt{xSx} \\ &\Leftrightarrow \sqrt{aSa} \subseteq \sqrt{xSx}, \text{ by Lemma 3 and Theorem 5} \\ &\Leftrightarrow \sqrt{bSb} \subseteq \sqrt{xSx} \\ &\Leftrightarrow b \in \sqrt{xSx} \\ &\Leftrightarrow x \in N(b) \end{aligned}$$

shows that $N(a) = N(b)$ and so $a\mathcal{N}b$. Again for $a, b \in S$, $a\mathcal{N}b$ implies that $N(a) = N(b)$. Then $x \in \sqrt{aSa} \Leftrightarrow a \in N(x) \Leftrightarrow N(a) \subseteq N(x) \Leftrightarrow N(b) \subseteq N(x) \Leftrightarrow b \in N(x) \Leftrightarrow x \in \sqrt{bSb}$ shows that $\sqrt{aSa} = \sqrt{bSb}$. Thus $a\eta b$. Hence $\eta = \mathcal{N}$. Let $a, b \in S$. Then $ab \in N(a)$ or $ab \in N(b)$ which implies $N(ab) \subseteq N(a) \subseteq N(ab)$ or $N(ab) \subseteq N(b) \subseteq N(ab)$, i.e. $ab\mathcal{N}a$ or $ab\mathcal{N}b$, and thus $\mathcal{N}$ is a chain congruence. Also by Theorem 5, each η-class is a t-k-Archimedean semiring.

(4) $\Rightarrow$ (1) : The proof is obvious. ■

Theorem 7. *The following conditions on a semiring S in $\mathbb{SL}^+$ are equivalent:*

1. *S is a chain of t-k-Archimedean semirings;*
2. *$\sqrt{B}$ is a completely prime k-ideal of S for every k-bi-ideal B of S;*
3. *$\sqrt{B_k(a)} = \sqrt{aSa}$ is a completely prime k-ideal of S for every $a \in S$;*
4. *$\sqrt{B_k(ab)} = \sqrt{B_k(a)} \cap \sqrt{B_k(b)}$ for all $a, b \in S$ and every k-bi-ideal of S is semiprimary.*

Proof. (1) $\Rightarrow$ (2) : Let S be a chain $\mathcal{C}$ of t-k-Archimedean semirings $\{S_\alpha \mid \alpha \in \mathcal{C}\}$. Consider a k-bi-ideal B of S. Then $\sqrt{B}$ is a k-ideal of S, by Theorem 5. Let $x, y \in S$ such that $xy \in \sqrt{B}$. Then there exists $m \in N$ such that $u = (xy)^m \in \overline{B} = B$. Suppose $\alpha, \beta \in \mathcal{C}$ be such that $x \in S_\alpha$ and $y \in S_\beta$. Since $\mathcal{C}$ is a chain, either $\alpha\beta = \alpha$ or $\alpha\beta = \beta$. If $\alpha\beta = \alpha$, then $x, u \in S_\alpha$. Then $x \in \sqrt{uSu} \subseteq \sqrt{B}$, and so $x \in \sqrt{B}$. If $\alpha\beta = \beta$, then similarly we have $y \in \sqrt{B}$.

(2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (4) : Let $a, b \in S$. Then $\sqrt{B_k(a)}$, $\sqrt{B_k(b)}$ and $\sqrt{B_k(ab)}$ are completely prime k-ideals of S. Let $x \in \sqrt{B_k(ab)}$. Then there exist $m \in \mathbb{N}$ and $u \in S$ such that $x^m + abuab = abuab$. Again there exist $n \in \mathbb{N}$ and $v \in S$ such that $(abuab)^n + va = va$, by Theorem 5. Then we have $x^{m(n+1)} + abuabva = abuabva$, whence $x \in \sqrt{B_k(a)}$. Therefore $\sqrt{B_k(ab)} \subseteq \sqrt{B_k(a)}$. Similarly, $\sqrt{B_k(ab)} \subseteq \sqrt{B_k(b)}$. Thus $\sqrt{B_k(ab)} \subseteq \sqrt{B_k(a)} \cap \sqrt{B_k(b)}$. Let $y \in \sqrt{B_k(a)} \cap \sqrt{B_k(b)}$. Then there exist $n \in \mathbb{N}$ and $s \in S$ such that $y^n + asa = asa$ and $y^n + bsb = bsb$. Again there exist $m \in \mathbb{N}$ and $u \in S$ such that $(asabsb)^m + uasab = uasab$, and we get $y^{2nm} + uasab = uasab$. Again there exist $p \in \mathbb{N}$ and $v \in S$ such that $(uasab)^p + abv = abv$. Then we have $y^{2nm(p+1)} + abvuasab = abuasab$, and so $y \in \sqrt{B_k(ab)}$. Thus $\sqrt{B_k(ab)} = \sqrt{B_k(a)} \cap \sqrt{B_k(b)}$. Let B be a k-bi-ideal of S and $a, b \in S$ be such that $ab \in B$. Then $ab \in \sqrt{B_k(ab)}$ implies that $a \in \sqrt{B_k(ab)}$ or $b \in \sqrt{B_k(ab)}$. Thus $a^n \in B_k(ab) \subseteq B$ or $b^n \in B_k(ab) \subseteq B$, for some $n \in \mathbb{N}$, i.e., $a^n \in B$ or $b^n \in B$ and hence B is semiprimary.

(4) $\Rightarrow$ (1) : Let $a, b \in S$. Then $ab \in \sqrt{B_k(a)} \cap \sqrt{B_k(b)}$. Then there are $m, n \in \mathbb{N}$ and $s \in S$ such that $(ab)^n + asa = asa$ and $(ab)^m + bsb = bsb$, i.e., $ab \in \sqrt{Sa} \cap \sqrt{bS}$. Then by Lemma 3 and Theorem 5, the least distributive lattice congruence η on S is given by : for $a, b \in S, a\eta b \Leftrightarrow \sqrt{aSa} = \sqrt{bSb} \Leftrightarrow \sqrt{B_k(a)} =$

$\sqrt{B_k(b)}$, and each η-class is a t-k-Archimedean semiring. Now there exists $n \in \mathbb{N}$ such that $a^n \in B_k(ab)$ or $b^n \in B_k(ab)$. Then $\sqrt{B_k(ab)} \subseteq \sqrt{B_k(a)} \subseteq \sqrt{B_k(ab)}$ or $\sqrt{B_k(ab)} \subseteq \sqrt{B_k(b)} \subseteq \sqrt{B_k(ab)}$, i.e., $\sqrt{B_k(ab)} = \sqrt{B_k(a)}$ or $\sqrt{B_k(ab)} = \sqrt{B_k(b)}$. Hence $ab\eta a$ or $ab\eta b$. Thus η is a chain congruence and so S is a chain of t-k-Archimedean semirings.

■

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References

[1] A.K. Bhuniya and K. Jana, *Bi-ideals in k-regular and intra k-regular semirings*, accepted for publication in Discuss. Math. General Algebra and Applications **31** (2011), 5–25.

[2] A.K. Bhuniya and T.K. Mondal, *Distributive lattice decompositions of semirings with a semilattice additive reduct*, Semigroup Forum **80** (2010), 293–301. doi:10.1007/s00233-009-9205-6

[3] S. Bogdanovic and M. Ciric, *Semilattice of Archimedean semigroups and completely π-regular semigroups I (survey article)*, Filomat(nis) **7** (1993), 1–40.

[4] S. Bogdanovic and M. Ciric, *Chains of Archimedean semigroups (Semiprimary semigroups)*, Indian J. Pure and Appl. Math. **25** (1994), 229–235.

[5] M. Ciric and S. Bogdanovic, *Semilattice decompositions of semigroups*, Semigroup Forum (1996), 119–132. doi:10.1007/BF02574089

[6] A.H. Clifford, *Semigroups admitting relative inverses*, Annals of Math. **42** (1941), 1037–1049. doi:10.2307/1968781

[7] F. Kmet, *Radicals and their left ideal analogues in a semigroup*, Math. Slovaca **38** (1988), 139–145.

[8] M. Petrich, *The maximal semilattice decomposition of a semigroup*, Math. Zeitschr. **85** (1964), 68–82. doi:10.1007/BF01114879

[9] M.S. Putcha, *Semilattice decomposition of semigroups*, Semigroup Forum **6** (1973), 12–34. doi:10.1007/BF02389104

[10] T. Tamura, *Another proof of a theorem concerning the greatest semilattice decomposition of a semigroup*, Proc. Japan Acad. **40** (1964), 777–780. doi:10.3792/pja/1195522562

[11] T. Tamura, *On Putcha's theorem concerning semilattice of archimedean semigroups*, Semigroup Forum **4** (1972), 83–86. doi:10.1007/BF02570773

[12] T. Tamura, *Note on the greatest semilattice decomposition of semigroups*, Semigroup Forum **4** (1972), 255–261. doi:10.1007/BF02570795

[13] T. Tamura and N. Kimura, *On decomposition of a commutative semigroup*, Kodai Math. Sem. Rep. **4** (1954), 109–112. doi:10.2996/kmj/1138843534

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