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PRE-STRONGLY SOLID VARIETIES OF COMMUTATIVE SEMIGROUPS

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Abstract

Generalized hypersubstitutions are mappings from the set of all fundamental operations into the set of all terms of the same language do not necessarily preserve the arities. Strong hyperidentities are identities which are closed under the generalized hypersubstitutions and a strongly solid variety is a variety which every its identity is a strong hyperidentity. In this paper we give an example of pre-strongly solid varieties of commutative semigroups and determine the least and the greatest pre-strongly solid variety of commutative semigroups.

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1. Introduction

Hyperidentities were invented by Aczel, Belousov and Taylor. The notion of *hyperidentities* and *solid varieties of a given type* as well as *derived algebras of given type* were invented by E. Graczyńska and D. Schweigert in [3]. An identity $t \approx t'$ of terms of any type τ is called a *hyperidentity* for an algebra $\underline{A} = (A; (f_i^A)_{i\in I})$ if $t \approx t'$ holds identically for every choice of n-ary term operation to represent n-ary operation symbols occurring in t and t'. A variety which every its identity is a hyperidentity is called *solid variety*. Hyperidentities can be characterized more precisely using the concept of a hypersubstitution which was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert. A hypersubstitution of type τ is a mapping $\sigma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$ which assigns to every n_i-ary operation symbol f_i an n_i-ary term. The set of all hypersubstitutions of type τ is denoted by $Hyp(\tau)$. For every $\sigma \in Hyp(\tau)$ induces a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ by the following steps:

(i) $\hat{\sigma}[x] := x$, for any variable $x \in X$, and

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, where $\hat{\sigma}[t_j], 1 \leq j \leq n_i$ are already defined.

A binary operation $\circ_h$ on $Hyp(\tau)$ is defined by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma_1} \circ \sigma_2$ for every $\sigma_1, \sigma_2 \in Hyp(\tau)$ where $\circ$ is the natural composition of mappings. Let σ_{id} be the hypersubstitution where $\sigma_{id}(f_i) = f_i(x_1, ..., x_{n_i})$. It turns out that $(Hyp(\tau); \circ_h, \sigma_{id})$ is a monoid with σ_{id} is an identity element.

S. Leeratanavalee and K. Denecke generalized the concepts of hypersubstitutions, hyperidentities and solid varieties to generalized hypersubstitutions, strong hyperidentities and strongly solid varieties [4]. A generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$ from the set of all n_i-ary operation symbols into the set of all terms built up by elements of the alphabet $X := \{x_1, x_2, ...\}$ and operation symbols from $\{f_i | i \in I\}$ which does not necessarily preserve the arity.

We denoted the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. To define a binary operation on $Hyp_G(\tau)$, we defined firstly the concept of generalized superposition of terms $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ by the following steps:

for any term $t \in W_\tau(X)$,

(i) if $t = x_j, 1 \leq j \leq m$, then

$$S^m(x_j, t_1, \ldots, t_m) := t_j,$$

(ii) if $t = x_j, m < j \in I\!N$, then

$$S^m(x_j, t_1, \ldots, t_m) := x_j,$$

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then

$$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m)).$$

Then the generalized hypersubstitution σ can be extended to a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ by the following steps:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

We defined a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Let σ_{id} be the hypersubstitution mapping which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. It turns out that $(Hyp_G(\tau); \circ_G, \sigma_{id})$ is a monoid and the monoid $(Hyp(\tau); \circ_G, \sigma_{id})$ of all arity preserving hypersubstitutions of type τ forms a submonoid of $(Hyp_G(\tau); \circ_G, \sigma_{id})$.

If $\underline{M}$ is a submonoid of $\underline{Hyp_G(\tau)}$ then an identity $t \approx t'$ is called an *M-strong hyperidentity* if $\hat{\sigma}[t] \approx \hat{\sigma}[t']$ are identities for every $\sigma \in M$. A variety V is called *M-strongly solid* if every identity in it is an M-strong hyperidentity. In case of $M = Hyp_G(\tau)$ we will call a *strong hyperidentity* and *strongly solid* respectively.

2. V-proper generalized hypersubstitutions and normal forms

In 2007, S. Leeratanavalee and S. Phatchat generalized the concept of V-proper hypersubstitutions and normal forms of hypersubstitutions introduced by J. Płonka [5] to V-proper generalized hypersubstitutions and normal forms of generalized hypersubstitutions.

Definition 2.1 ([5])**.** Let V be a variety of type τ. A generalized hypersubstitution σ of type τ is called a *V-proper generalized hypersubstitution* if for every identity $s \approx t$ of V, the identity $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ also holds in V. We use $P_G(V)$ for the set of all *V-proper generalized hypersubstitutions* of type τ.

Proposition 2.2 ([5])**.** *For any variety V of type τ, $(P_G(V); \circ_G, \sigma_{id})$ is a submonoid of $(Hyp_G(\tau); \circ_G, \sigma_{id})$.*

Definition 2.3 ([5])**.** Let V be a variety of type τ. Two generalized hypersubstitutions σ_1 and σ_2 of type τ are called a *V-generalized equivalent* if $\sigma_1(f_i) \approx \sigma_2(f_i)$ are identities in V for all $i \in I$. In this case we write $\sigma_1 \sim_{VG} \sigma_2$.

Theorem 2.4 ([5])**.** *Let V be a variety of algebras of type τ,and let $\sigma_1, \sigma_2 \in Hyp_G(\tau)$. Then the following statements are equivalent:*

(i) $\sigma_1 \sim_{VG} \sigma_2$.

(ii) *For all $t \in W_\tau(X)$, the equations $\hat{\sigma}_1[t] \approx \hat{\sigma}_2[t]$ are identities in V.*

(iii) *For all $\underline{A} \in V$, $\sigma_1[\underline{A}] = \sigma_2[\underline{A}]$ where $\sigma_k[\underline{A}] = (A; (\sigma_k(f_i)^A)_{i \in I}); k = 1, 2$.*

Proposition 2.5 ([5])**.** *Let V be a variety of algebras of type τ. Then the following statements hold:*

(i) *For all $\sigma_1, \sigma_2 \in Hyp_G(\tau)$, if $\sigma_1 \sim_{VG} \sigma_2$ then σ_1 is a V-proper generalized hypersubstitution iff σ_2 is a V-proper generalized hypersubstitution.*

(ii) *For all $s, t \in W_\tau(X)$ and for all $\sigma_1, \sigma_2 \in Hyp_G(\tau)$, if $\sigma_1 \sim_{VG} \sigma_2$ then $\hat{\sigma}_1[s] \approx \hat{\sigma}_1[t]$ is an identity in V iff $\hat{\sigma}_2[s] \approx \hat{\sigma}_2[t]$ is an identity in V.*

The relation $\sim_{VG}$ is an equivalence relation on $Hyp_G(\tau)$, but it is not neccessary a congruence relation. We factorize $Hyp_G(\tau)$ by $\sim_{VG}$ and consider the submonoid $\underline{P_G(V)}$ of $\underline{Hyp_G(\tau)}$ is the union of equivalence classes of the relation $\sim_{VG}$. This is also true for a submonoid $\underline{M}$ of $\underline{Hyp_G(\tau)}$ and the relation $\sim_{VG_{|M}}$.

Lemma 2.6 ([5])**.** *Let $\underline{M}$ be a submonoid of $\underline{Hyp_G(\tau)}$ and let V be a variety of type τ. Then the monoid $P_G \cap M$ is the union of all equivalence classes of the restricted relation $\sim_{VG_{|M}}$.*

Definition 2.7 ([5])**.** Let $\underline{M}$ be a monoid of generalized hypersubstitutions of type τ, and let V be a variety of type τ. Let ϕ be a choice function which choosed from M one generalized hypersubstitution from each equivalence class of the relation $\sim_{VG_{|M}}$, and let $N_\phi^M(V)$ be the set of generalized hypersubstitutions which are chosen. Thus $N_\phi^M(V)$ is a set of distinguished generalized hypersubstitutions from M, which we might call *V-normal form generalized hypersubstitutions.* We will say that the variety V is $N_\phi^M(V)$-*strongly solid* if for every identity $s \approx t \in IdV$ and for every generalized hypersubstitution $\sigma \in N_\phi^M(V)$, $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$.

Theorem 2.8 ([5])**.** *Let $\underline{M}$ be a monoid of generalized hypersubstitutions of type τ and let V be a variety of type τ. For any choice function ϕ, V is M-strongly solid if and only if V is $N_\phi^M(V)$-strongly solid.*

3. Pre-strongly solid varieties of semigroups

The concept of pre-solid varieties was introduced by K. Denecke and S.L. Wismath [2]. In 2007, S. Leeratanavalee and S. Phatchat generalized the concept of pre-solid varieties to pre-strongly solid varieties [5]. Firstly, we recall the definitions of a pre-generalized hypersubstitution and a pre-strong hyperidentity. Let us fix a type $\tau = (2)$. So we have only one binary operation symbol, say f. From now on, the generalized hypersubstitution σ which maps f to the term t is denoted by σ_t.

Definition 3.1. A generalized hypersubstitution $\sigma \in Hyp_G(2)$ is called a *pre-generalized hypersubstitution* if $\sigma \in Hyp_G(2) \setminus \{\sigma_{x_1}, \sigma_{x_2}\}$ where σ_{x_1} and σ_{x_2} denoted the generalized hypersubstitutions which map f to x_1 and to x_2, respectively. We denote the set of all pre-generalized hypersubstitutions of type $\tau = (2)$ by $Pre_G(2)$.

The reason to delete the generalized hypersubstitutions σ_{x_1} and σ_{x_2} from $Hyp_G(2)$ is if we apply the generalized hypersubstitution σ_{x_1} or σ_{x_2} on the both sides of the commutative law $x_1x_2 \approx x_2x_1$ we obtain the equation $x_1 \approx x_2$ which satisfied only in a one-element semigroup.

Definition 3.2. An identity $t \approx t'$ is called a *pre-strong hyperidentity* in a variety V if $\hat{\sigma}[t] \approx \hat{\sigma}[t'] \in IdV$ for all $\sigma \in Pre_G(2)$.

A variety V is called a *pre-strongly solid variety* if every identity in V is a pre-strong hyperidentity of V.

For a class K of algebras of type τ and for a set $\sum$ of identities of this type we fix the following notations:

IdK - the set of all identies of K,

$HIdK$ - the set of all hyperidenties of K,

$H_{Pre_G}IdK$ - the set of all pre-strong hyperidenties of K,

$Mod\sum = \{\underline{A} \in Alg(\tau) | \underline{A}$ satisfies$\sum\}$ - the variety defined by $\sum$,

$HMod\sum = \{\underline{A} \in Alg(\tau) | \underline{A}$ hypersatisfies$\sum\}$ - the hyperequational class defined by $\sum$,

$H_{Pre_G}Mod\sum = \{\underline{A} \in Alg(\tau) | \underline{A}$ pre-strong hypersatisfies$\sum\}$ - the pre-strong hyperequational class defined by $\sum$.

Proposition 3.3 ([5]). *$\underline{Pre_G(2)}$ is a submonoid of $\underline{Hyp_G(2)}$.*

Remark 3.4 ([5]). Every strongly solid variety of semigroups is a pre-strongly solid variety.

Remark 3.5 ([5]). Every pre-strongly solid variety of semigroups is a pre-solid variety of semigroups.

Lemma 3.6 ([5]). *The variety $Z := Mod\{x_1x_2 \approx x_3x_4\}$ is the least non-trivial pre-strongly solid variety of semigroups.*

Theorem 3.7 ([5]). *The greatest non-trivial pre-strongly solid variety of semigroups which is not strongly solid is $Z := Mod\{x_1x_2 \approx x_3x_4\}$.*

Theorem 3.8 ([5]). *The variety $V_{big} := Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2, x_1x_2x_3x_4 \approx x_1x_3x_2x_4\}$ is the greatest pre-strongly solid variety of semigroups.*

4. PRE-STRONGLY SOLID VARIETIES OF COMMUTATIVE SEMIGROUPS

Firstly, we recall the definition of a generalized hypersubstitution of type τ is a mapping $\sigma : \{f_i | i \in I\} \longrightarrow W_\tau(X)$ from the set of all n_i-ary operation symbols into the set of all terms built up by elements of the alphabet

$X := \{x_1, x_2, ...\}$ and operation symbols from $\{f_i | i \in I\}$ which does not necessarily preserve the arity. We denote the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. A generalized superposition of terms $S^m : W_\tau(X)^{m+1} \longrightarrow W_\tau(X)$ is defined by the following steps:

for any term $t \in W_\tau(X)$,

(i) if $t = x_j, 1 \leq j \leq m$, then

$$S^m(x_j, t_1, \ldots, t_m) := t_j,$$

(ii) if $t = x_j, m < j \in \mathbb{N}$, then

$$S^m(x_j, t_1, \ldots, t_m) := x_j,$$

(iii) if $t = f_i(s_1, \ldots, s_{n_i})$, then

$$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m)).$$

For every $\sigma \in Hyp_G(\tau)$ induces a mapping $\hat{\sigma} : W_\tau(X) \longrightarrow W_\tau(X)$ by the following steps:

(i) $\hat{\sigma}[x] := x \in X$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$, for any n_i-ary operation symbol f_i where $\hat{\sigma}[t_j]$, $1 \leq j \leq n_i$ are already defined.

In this section, we give an example of pre-strongly solid varieties of commutative semigroups and then determine the least and the greatest pre-strongly solid variety of commuutative semigroups.

Theorem 4.1. *The variety* $V_1 := Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1, x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2, x_1^2 \approx x_2^2\}$ *is a pre-strongly solid variety of commutative semigroups.*

Proof. To show that the variety V_1 is a pre-strongly solid variety of commutative semigroups, we have to show that every identity satisfied in V_1 is a pre-strong hyperidentity of V_1. By using Theorem 2.8 , we can restrict our checking to the following pre-generalized hypersubstitutions σ_t where $t \in \{x_ix_j | i, j \in \mathbb{N}\} \cup \{x_ix_jx_k | \ i \neq j \neq k\} \cup \{x_{i_1}x_{i_2}...x_{i_k} | \ k, i_1, ..., i_k \in \mathbb{N}, k > 3,$ and all of $i_1, ..., i_k$ are distinct$\}$.

If we apply $\sigma_{x_ix_j}$; $i, j \in \mathbb{N}$ on the both sides of the associative law we have the following table.

$i, j \in \mathbb{N}$	$\hat{\sigma}_{x_ix_j}[(x_1x_2)x_3] =$ $S^2(x_ix_j, S^2(x_ix_j, x_1, x_2), x_3)$	$\hat{\sigma}_{x_ix_j}[x_1(x_2x_3)] =$ $S^2(x_ix_j, x_1, S^2(x_ix_j, x_2, x_3))$
$i = j = 1$	$x_1x_1x_1x_1$	x_1x_1
$i = 1, j = 2$	$x_1x_2x_3$	$x_1x_2x_3$
$i = 1, j > 2$	$x_1x_jx_j$	x_1x_j
$i = j = 2$	x_3x_3	$x_3x_3x_3x_3$
$i = 2, j > 2$	x_3x_j	$x_3x_jx_j$
$i, j > 2$	x_ix_j	x_ix_j

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2, x_1^2 \approx x_2^2$ we sides are equal.

If we apply $\sigma_{x_ix_j}$; $i, j \in \mathbb{N}$ on the both sides of the commutative law we have the following tab

$i, j \in \mathbb{N}$	$\hat{\sigma}_{x_ix_j}[x_1x_2] = S^2(x_ix_j, x_1, x_2)$	$\hat{\sigma}_{x_ix_j}[x_2x_1] = S^2(x_ix_j, x_2, x_1)$
$i = j = 1$	x_1x_1	x_2x_2
$i = 1, j = 2$	x_1x_2	x_2x_1
$i = 1, j > 2$	x_1x_j	x_2x_j
$i = j = 2$	x_2x_2	x_1x_1
$i = 2, j > 2$	x_2x_j	x_2x_j
$i, j > 2$	x_ix_j	x_ix_j

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2, x_1^2 \approx x_2^2$ we sides are equal.

If we apply $\sigma_{x_i x_j}$; $i, j \in \mathbb{N}$ on the both sides of the identity $x_1^2 \approx x_2^2$ we have the follo

$i, j \in \mathbb{N}$	$\hat{\sigma}_{x_i x_j}[x_1 x_1] = S^2(x_i x_j, x_1, x_1)$	$\hat{\sigma}_{x_i x_j}[x_2 x_2] = S^2(x_i x_j$
$i = j = 1$	$x_1 x_1$	$x_2 x_2$
$i = 1, j = 2$	$x_1 x_1$	$x_2 x_2$
$i = 1, j > 2$	$x_1 x_j$	$x_2 x_j$
$i = j = 2$	$x_1 x_1$	$x_2 x_2$
$i = 2, j > 2$	$x_1 x_j$	$x_2 x_j$
$i, j > 2$	$x_i x_j$	$x_i x_j$

Using the associative law, the commutative law and identities $x_1^2 x_2 \approx x_1 x_2^2 \approx x_1 x$ sides are equal.

If we apply $\sigma_{x_i x_j}$; $i, j \in \mathbb{N}$ on the both sides of the identity $x_1^2 x_2 \approx x_1 x_2^2 \approx x_1$ table.

$i, j \in \mathbb{N}$	$\hat{\sigma}_{x_i x_j}[(x_1 x_1) x_2] =$ $S^2(x_i x_j, S^2(x_i x_j, x_1, x_1), x_2)$	$\hat{\sigma}_{x_i x_j}[x_1(x_2 x_2)] =$ $S^2(x_i x_j, x_1, S^2(x_i x_j, x_2, x_2))$
$i = j = 1$	$x_1 x_1 x_1 x_1$	$x_1 x_1$
$i = 1, j = 2$	$x_1 x_1 x_2$	$x_1 x_2 x_2$
$i = 1, j > 2$	$x_1 x_j x_j$	$x_1 x_j$
$i = j = 2$	$x_2 x_2$	$x_2 x_2 x_2 x_2$
$i = 2, j > 2$	$x_2 x_j$	$x_2 x_j x_j$
$i, j > 2$	$x_i x_j$	$x_i x_j$

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2, x_1^2 \approx x_2^2$ we
sides are equal.

If we apply $\sigma_{x_ix_jx_k}$; $i \neq j \neq k \in \mathbb{N}$ on the both sides of the associative law we have the followi

$i, j, k \in \mathbb{N}$	$\hat{\sigma}_{x_ix_jx_k}[(x_1x_2)x_3] =$ $S^2(x_ix_jx_k, S^2(x_ix_jx_k, x_1, x_2), x_3)$	$\hat{\sigma}_{x_ix_jx_k}[x_1(x_2x_3)] =$ $S^2(x_ix_jx_k, x_1, S^2(x_ix_jx_k, x_2, x_3)$
$i = 1, j = 2, k > 2$	$x_1x_2x_kx_3x_k$	$x_1x_2x_3x_kx_k$
$i = 1, j, k > 2$	$x_1x_jx_kx_jx_k$	$x_1x_jx_k$
$i = 2, j, k > 2$	$x_3x_jx_k$	$x_3x_jx_kx_jx_k$
$i, j, k > 2$	$x_ix_jx_k$	$x_ix_jx_k$

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2, x_1^2 \approx x_2^2$ we
sides are equal.

If we apply $\sigma_{x_ix_jx_k}$; $i \neq j \neq k \in \mathbb{N}$ on the both sides of the commutative law we have the follo

$i, j, k \in \mathbb{N}$	$\hat{\sigma}_{x_ix_jx_k}[x_1x_2] = S^2(x_ix_jx_k, x_1, x_2)$	$\hat{\sigma}_{x_ix_jx_k}[x_2x_1] = S^2(x_ix_jx_k, x_2, x$
$i = 1, j = 2, k > 2$	$x_1x_2x_k$	$x_2x_1x_k$
$i = 1, j, k > 2$	$x_1x_jx_k$	$x_2x_jx_k$
$i = 2, j, k > 2$	$x_2x_jx_k$	$x_2x_jx_k$
$i, j, k > 2$	$x_ix_jx_k$	$x_ix_jx_k$

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2, x_1^2 \approx x_2^2$ we
sides are equal.

If we apply $\sigma_{x_ix_jx_k}$; $i \neq j \neq k \in \mathbb{N}$ on the both sides of the identity $x_1^2 \approx x_2^2$ we have

$i, j, k \in \mathbb{N}$	$\hat{\sigma}_{x_ix_jx_k}[x_1x_1] = S^2(x_ix_jx_k, x_1, x_1)$	$\hat{\sigma}_{x_ix_jx_k}[x_2x_2] = S^2$
$i = 1, j = 2, k > 2$	$x_1x_2x_k$	x_2x_1x
$i = 1, j, k > 2$	$x_1x_jx_k$	x_2x_jx
$i = 2, j, k > 2$	$x_2x_jx_k$	x_1x_jx
$i, j, k > 2$	$x_ix_jx_k$	x_ix_jx

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x$
sides are equal.

If we apply $\sigma_{x_ix_jx_k}$; $i \neq j \neq k \in \mathbb{N}$ on the both sides of the identity $x_1^2x_2 \approx x$
following table.

$i, j, k \in \mathbb{N}$	$\hat{\sigma}_{x_ix_jx_k}[(x_1x_1)x_2] =$ $S^2(x_ix_jx_k,$ $S^2(x_ix_jx_k, x_1, x_1), x_2)$	$\hat{\sigma}_{x_ix_jx_k}[x_1(x_2x_2)] =$ $S^2(x_ix_jx_k, x_1,$ $S^2(x_ix_jx_k, x_2, x_2))$	$\hat{\sigma}_{x_ix}$ $S^2(x_i$
$i = 1, j = 2, k > 2$	$x_1x_1x_kx_2x_k$	$x_1x_2x_2x_kx_k$	
$i = 1, j, k > 2$	$x_1x_jx_kx_jx_k$	$x_1x_jx_k$	
$i = 2, j, k > 2$	$x_2x_jx_k$	$x_2x_jx_kx_jx_k$	
$i, j, k > 2$	$x_ix_jx_k$	$x_ix_jx_k$	

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x$
sides are equal.

If we apply σ_t where $t = x_{i_1}x_{i_2}...x_{i_k}$ and $k, i_1, ..., i_k \in \mathbb{N}, k > 3$ on the both sides of the associative law we have $\hat{\sigma}_t[(x_1x_2)x_3] = S^2(t, S^2(t, x_1, x_2), x_3)$ and $\hat{\sigma}_t[x_1(x_2x_3)] = S^2(t, x_1, S^2(t, x_2, x_3))$.

(i) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 1$ and $i_m > 2$ for all $m \neq n$, then

$$\hat{\sigma}_t[(x_1x_2)x_3] = x_{i_1}...x_{i_{n-1}}x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_k}x_{i_{n+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_1(x_2x_3)] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_k}.$$

(ii) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 2$ and $i_m > 2$ for all $m \neq n$, then

$$\hat{\sigma}_t[(x_1x_2)x_3] = x_{i_1}...x_{i_{n-1}}x_3x_{i_{n+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_1(x_2x_3)] = x_{i_1}...x_{i_{n-1}}x_{i_1}...x_{i_{n-1}}x_3x_{i_{n+1}}...x_{i_k}x_{i_{n+1}}...x_{i_k}.$$

(iii) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 1$ and there exists a unique $l \in \{1, ..., k\}$ such that $i_l = 2$, $i_m > 2$ for all $m \neq n \neq l$ and $n < l$, then

$$\hat{\sigma}_t[(x_1x_2)x_3]$$
$$= x_{i_1}...x_{i_{n-1}}x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_{l-1}}x_2x_{i_{l+1}}...x_{i_k}x_{i_{n+1}}...x_{i_{l-1}}x_3x_{i_{l+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_1(x_2x_3)]$$
$$= x_{i_1}...x_{i_{n-1}}x_1x_{i_{l-1}}x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_{l-1}}x_3x_{i_{l+1}}...x_{i_k}x_{i_{l+1}}...x_{i_k}.$$

(iv) If $i_m > 2$ for all $m \in \{1, 2, ..., k\}$, then

$$\hat{\sigma}_t[(x_1x_2)x_3] = x_{i_1}...x_{i_k}.$$

$$\hat{\sigma}_t[x_1(x_2x_3)] = x_{i_1}...x_{i_k}.$$

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2, x_1^2 \approx x_2^2$ we have both sides are equal.

If we apply σ_t where $t = x_{i_1}x_{i_2}...x_{i_k}$ and $k, i_1, ..., i_k \in \mathbb{N}, k > 3$ on the both sides of the commutative law we have $\hat{\sigma}_t[x_1x_2] = S^2(t, x_1, x_2)$ and $\hat{\sigma}_t[x_2x_1] = S^2(t, x_2, x_1)$.

(i) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 1$ and $i_m > 2$ for all $m \neq n$, then

$$\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_2x_1] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_k}.$$

(ii) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 2$ and $i_m > 2$ for all $m \neq n$, then

$$\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_2x_1] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_k}.$$

(iii) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 1$ and there exists a unique $l \in \{1, ..., k\}$ such that $i_l = 2$, $i_m > 2$ for all $m \neq n \neq l$ and $n < l$, then

$$\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_{l-1}}x_2x_{i_{l+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_2x_1] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_{l-1}}x_1x_{i_{l+1}}...x_{i_k}.$$

(iv) If $i_m > 2$ for all $m \in \{1, 2, ..., k\}$, then

$$\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_k}.$$

$$\hat{\sigma}_t[x_2x_1] = x_{i_1}...x_{i_k}.$$

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2, x_1^2 \approx x_2^2$ we have both sides are equal.

If we apply σ_t where $t = x_{i_1}x_{i_2}...x_{i_k}$ and $k, i_1, ..., i_k \in \mathbb{N}, k > 3$ on the both sides of the identity $x_1^2 \approx x_2^2$ we have $\hat{\sigma}_t[x_1x_1] = S^2(t, x_1, x_1)$ and $\hat{\sigma}_t[x_2x_2] = S^2(t, x_2, x_2)$.

(i) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 1$ and $i_m > 2$ for all $m \neq n$, then

$$\hat{\sigma}_t[x_1x_1] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_2x_2] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_k}.$$

(ii) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 2$ and $i_m > 2$ for all $m \neq n$, then

$$\hat{\sigma}_t[x_1x_1] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_2x_2] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_k}.$$

(iii) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 1$ and there exists a unique $l \in \{1, ..., k\}$ such that $i_l = 2$, $i_m > 2$ for all $m \neq n \neq l$ and $n < l$, then

$$\hat{\sigma}_t[x_1x_1] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_{l-1}}x_1x_{i_{l+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_2x_2] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_{l-1}}x_2x_{i_{l+1}}...x_{i_k}.$$

(iv) If $i_m > 2$ for all $m \in \{1, 2, ..., k\}$, then

$$\hat{\sigma}_t[x_1x_1] = x_{i_1}...x_{i_k}.$$

$$\hat{\sigma}_t[x_2x_2] = x_{i_1}...x_{i_k}.$$

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2, x_1^2 \approx x_2^2$ we have both sides are equal.

If we apply σ_t where $t = x_{i_1}x_{i_2}...x_{i_k}$ and $k, i_1, ..., i_k \in \mathbb{N}, k > 3$ on the both sides of the identity $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2$ we have $\hat{\sigma}_t[(x_1x_1)x_2] = S^2(t, S^2(t, x_1, x_1), x_2)$ and $\hat{\sigma}_t[x_1(x_2x_2)] = S^2(t, x_1, S^2(t, x_1, x_2))$ and $\hat{\sigma}_t[x_1x_2] = S^2(t, x_1, x_2)$.

(i) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 1$ and $i_m > 2$ for all $m \neq n$, then

$$\hat{\sigma}_t[(x_1x_1)x_2] = x_{i_1}...x_{i_{n-1}}x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_k}x_{i_{n+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_1(x_2x_2)] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_k}.$$

(ii) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 2$ and $i_m > 2$ for all $m \neq n$, then

$$\hat{\sigma}_t[(x_1x_1)x_2] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_1(x_2x_2)] = x_{i_1}...x_{i_{n-1}}x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_k}x_{i_{n+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_k}.$$

(iii) If there exists a unique $n \in \{1, ..., k\}$ such that $i_n = 1$ and there exists a unique $l \in \{1, ..., k\}$ such that $i_l = 2$, $i_m > 2$ for all $m \neq n \neq l$ and $n < l$, then

$$\hat{\sigma}_t[(x_1x_1)x_2]$$
$$= x_{i_1}...x_{i_{n-1}}x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_{l-1}}x_1x_{i_{l+1}}...x_{i_k}x_{i_{n+1}}...x_{i_{l-1}}x_2x_{i_{l+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_1(x_2x_2)]$$
$$= x_{i_1}...x_{i_{n-1}}x_1x_{i_{l-1}}x_{i_1}...x_{i_{n-1}}x_2x_{i_{n+1}}...x_{i_{l-1}}x_2x_{i_{l+1}}...x_{i_k}x_{i_{l+1}}...x_{i_k}.$$

$$\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_{n-1}}x_1x_{i_{n+1}}...x_{i_{l-1}}x_2x_{i_{l+1}}...x_{i_k}.$$

(iv) If $i_m > 2$ for all $m \in \{1, 2, ..., k\}$, then

$\hat{\sigma}_t[(x_1x_1)x_2] = x_{i_1}...x_{i_k}$.

$\hat{\sigma}_t[x_1(x_2x_2)] = x_{i_1}...x_{i_k}$.

$\hat{\sigma}_t[x_1x_2] = x_{i_1}...x_{i_k}$.

Using the associative law, the commutative law and identities $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2, x_1^2 \approx x_2^2$ we have both sides are equal. ■

Theorem 4.2. *The variety $Z := Mod\{x_1x_2 \approx x_3x_4\}$ is the least pre-strongly solid variety of commutative semigroups.*

Theorem 4.3. *The variety $V_2 := Mod\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1, x_1x_2x_3 \approx x_1x_3\}$ is the greatest pre-strongly solid variety of commutative semigroups.*

Proof. The greatest pre-strongly solid variety of commutative semigroups is the class of all commutative semigroups for which the associative law and the commutative law are satisfied as pre-strong hyperidentities, i.e the class $H_{Pre_G}\text{Mod}\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\}$. Applying $\sigma_{x_1x_2}, \sigma_{x_1x_i}, \sigma_{x_ix_1}$ $(i > 2) \in Pre_G$ on the associative law, $\sigma_{x_1x_2}$ gives $(x_1x_2)x_3 \approx x_1(x_2x_3)$, $\sigma_{x_1x_i}$ gives $x_1x_i^2 \approx x_1x_i$, $\sigma_{x_ix_1}$ gives $x_i^2x \approx x_ix$. If we substitute for x_i a new variable x_2, then we have the identities $x_1x_2^2 \approx x_1x_2$, $x_2^2x_1 \approx x_2x_1$. That means $x_1^2x_2 \approx x_1x_2^2 \approx x_1x_2 \in\text{Id}(H_{Pre_G}\text{Mod}\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\})$. Applying $\sigma_{x_1x_2}, \sigma_{x_1x_i}(i > 2)$ on the commutative law, $\sigma_{x_1x_2}$ gives $x_1x_2 \approx x_2x_1$, $\sigma_{x_1x_i}$ gives $x_ix_1 \approx x_ix_2$. Then $x_ix_1x_2 \approx x_ix_2x_2 \approx x_ix_2$, so $x_ix_1x_2 \approx x_ix_2$. If we substitute x_i by x_1, x_1 by x_2 and x_2 by x_3. Then we have $x_1x_2x_3 \approx x_1x_3$. Thus $H_{Pre_G}\text{Mod}\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\}$ satisfies all identities of V_2, i.e $H_{Pre_G}\text{Mod}\{(x_1x_2)x_3 \approx x_1(x_2x_3), x_1x_2 \approx x_2x_1\} \subseteq V_2$. To prove the converse inclusion we have to check the associative law , the commutative law and the rectangular law, i.e. $x_1x_2x_3 \approx x_1x_3$ using all pre-generalized hypersubstitutions. We can restrict our checking to the following pre-generalized hypersubstitutions $\sigma_{x_ix_j}(i, j \in \mathbb{N})$.

If we apply $\sigma_{x_ix_j}$; $i, j \in \mathbb{N}$ on the both sides of the associative law we have the following table.

$i, j \in \mathbb{N}$	$\hat{\sigma}_{x_ix_j}[(x_1x_2)x_3] =$ $S^2(x_ix_j, S^2(x_ix_j, x_1, x_2), x_3)$	$\hat{\sigma}_{x_ix_j}[x_1(x_2x_3)] =$ $S^2(x_ix_j, x_1, S^2(x_ix_j, x_2, x_3))$
$i = j = 1$	$x_1x_1x_1x_1$	x_1x_1
$i = 1, j = 2$	$x_1x_2x_3$	$x_1x_2x_3$
$i = j = 2$	x_3x_3	$x_3x_3x_3x_3$
$i = 1, j > 2$	$x_1x_jx_j$	x_1x_j
$i = 2, j > 2$	x_3x_j	$x_3x_jx_j$
$i, j > 2$	x_ix_j	x_ix_j

Using the associative law, the commutative law and the identity $x_1x_2x_3 \approx x_1x_3$ we have both sides are equal.

If we apply $\sigma_{x_ix_j}$; $i, j \in \mathbb{N}$ on the both sides of the commutative law we have the following table.

$i, j \in \mathbb{N}$	$\hat{\sigma}_{x_ix_j}[x_1x_2] = S^2(x_ix_j, x_1, x_2)$	$\hat{\sigma}_{x_ix_j}[x_2x_1] = S^2(x_ix_j, x_2, x_1)$
$i = j = 1$	x_1x_1	x_2x_2
$i = 1, j = 2$	x_1x_2	x_2x_1
$i = j = 2$	x_2x_2	x_1x_1
$i = 1, j > 2$	x_1x_j	x_2x_j
$i = 2, j > 2$	x_2x_j	x_1x_j
$i, j > 2$	x_ix_j	x_ix_j

Using the associative law, the commutative law and the identity $x_1x_2x_3 \approx x_1x_3$ we have both sides are equal.

If we apply $\sigma_{x_ix_j}$; $i, j \in \mathbb{N}$ on the both sides of the identity $x_1x_2x_3 \approx x_1x_3$ we have the following t

$i, j \in \mathbb{N}$	$\hat{\sigma}_{x_ix_j}[(x_1x_2)x_3] = S^2(x_ix_j, S^2(x_ix_j, x_1, x_2), x_3)$	$\hat{\sigma}_{x_ix_j}[x_1x_3] = S^2(x_ix_j, x_1, x$
$i = j = 1$	$x_1x_1x_1x_1$	x_1x_1
$i = 1, j = 2$	$x_1x_2x_3$	x_1x_3
$i = j = 2$	x_3x_3	x_3x_3
$i = 1, j > 2$	$x_1x_jx_j$	x_1x_j
$i = 2, j > 2$	x_3x_j	x_3x_j
$i, j > 2$	x_ix_j	x_ix_j

Using the associative law, the commutative law and the identity $x_1x_2x_3 \approx x_1x_3$ we have both side

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