Discussiones Mathematicae
General Algebra and Applications 30 (2010) 207–215

CONGRUENCES ON SEMILATTICES WITH SECTION ANTITONE INVOLUTIONS*

Ivan Chajda

Department of Algebra and Geometry
Palacký University Olomouc
Tomkova 40, 779 00 Olomouc, Czech Republic

e-mail: chajda@inf.upol.cz

Abstract

We deal with congruences on semilattices with section antitone involution which rise e.g., as implication reducts of Boolean algebras, MV-algebras or basic algebras and which are included among implication algebras, orthoimplication algebras etc. We characterize congruences by their kernels which coincide with semilattice filters satisfying certain natural conditions. We prove that these algebras are congruence distributive and 3-permutable.

Keywords: semilattice, section, antitone involution, congruence kernel, filter, congruence distributivity, 3-permutability.

2000 Mathematics Subject Classification: 06A12, 06D35, 08A30, 08B10.

Let $(P;\leq)$ be an ordered set. A mapping $x \mapsto x'$ on P is called an *antitone involution* if $x'' = x$ and $x \leq y \Rightarrow y' \leq x'$.

By a *semilattice with section antitone involutions (semilattice with SAI*, for short) is meant a structure $\mathcal{S} = (S;\vee,1,(^a)_{a\in S})$ such that $(S;\vee)$ is a join-semilattice with greatest element 1 and for each $a \in S$ there exists an antitone involution $x \mapsto x^a$ on the interval $[a,1]$ (the so-called *section*, see e.g., [3]).

*This work is supported by the Research Project MSM 6198959214 by Czech Government.

Let us note that semilattices with SAI are rather frequent in algebraic investigations. If e.g., $(B; \vee, \wedge, ', 0, 1)$ is a Boolean algebra and $x^a = x' \vee a$ then $(B; \vee, 1, (^a)_{a\in B})$ is a semilattice with SAI. Similarly, if $(L; \vee, \wedge, ^\perp, 0, 1)$ is an orthomodular lattice and $x^a = x^\perp \vee a$ then $(L; \vee, 1, (^a)_{a\in L})$ is a semilattice with SAI. If $\mathcal{M} = (M; \oplus, \neg, 0)$ is an MV-algebra and we define $1 = \neg 0$, $x \vee y = \neg(\neg x \oplus y) \oplus y$ and $x^a = \neg x \oplus a$ then $(M; \vee, 1, (^a)_{a\in M})$ is a semilattice with SAI. In the same way it can be shown for basic algebras (see e.g., [5]). If $(A; \cdot, 1)$ is a positive BCK-algebra then for $x \vee y = (x \cdot y) \cdot y$ and $x^a = x \cdot a$ we obtain a semilattice with SAI $(A; \vee, 1, (^a)_{a\in A})$ again, see e.g., [5]. Analogously, every implication algebra (see [1]), orthoimplication algebra (see [2]) or weak implication algebra (see [6]) can be converted into a semilattice with SAI.

The aim of this paper is to characterize congruences on semilattices with SAI via their congruence kernels and prove several important congruence identities.

Let $\mathcal{S} = (S; \vee, 1, (^a)_{a\in S})$ be a semilattice with SAI. Although $\mathcal{S}$ is only a partial algebra since the involutions are defined only on sections, it can be easily converted into a total algebra if one define enlarged unary operations $(^a)_{a\in S}$ in the way $x \mapsto (x \vee a)^a$. Since $x \vee a \in [a, 1]$, this is everywhere defined operation which coincides with the original one just on the section $[a, 1]$. Moreover, we can define a new binary operation $x \cdot y = (x \vee y)^y$ which can replace other operations of $\mathcal{S}$ since $x \vee y = (x \cdot y) \cdot y$ and $x^a = x \cdot a$ for $x \in [a, 1]$. However, in some reasonable cases, it is useful to deal with the original structure $\mathcal{S}$ as defined above.

Let $\mathcal{S} = (S; \vee, 1, (^a)_{a\in S})$ be a semilattice with SAI. By a *congruence* on $\mathcal{S}$ is meant an equivalence relation Θ on S having the *substitution property* with respect to all operations of $\mathcal{S}$, i.e. if $\langle a, b\rangle, \langle c, d\rangle \in \Theta$ then $\langle a \vee c, b \vee d\rangle \in \Theta$ and for each $z \leq a$, $z \leq b$ also $\langle a^z, b^z\rangle \in \Theta$. Denote by $\mathrm{Con}\mathcal{S}$ the lattice of all congruences on $\mathcal{S}$ and for $\Theta \in \mathrm{Con}\mathcal{S}$ denote by $[1]_\Theta = \{x \in S; \langle x, 1\rangle \in \Theta\}$, the so-called *kernel of* Θ.

At first we establish connection between congruences and their kernels.

Theorem 1. *Let $\mathcal{S} = (S; \vee, 1, (^a)_{a\in S})$ be a semilattice with SAI and $\Theta \in \mathrm{Con}\mathcal{S}$. Then $\langle x, y\rangle \in \Theta$ if and only if $(x \vee y)^x, (x \vee y)^y \in [1]_\Theta$. If $\Theta, \Phi \in \mathrm{Con}\mathcal{S}$ and $[1]_\Theta = [1]_\Phi$ then $\Theta = \Phi$.*

Proof. If $\langle x, y\rangle \in \Theta$ then $\langle (x \vee y)^x, 1\rangle = \langle (x \vee y)^x, (x \vee x)^x\rangle \in \Theta$ and $\langle (x \vee y)^y, 1\rangle = \langle (x \vee y)^y, (y \vee y)^y\rangle \in \Theta$ thus $(x \vee y)^x, (x \vee y)^y \in [1]_\Theta$.

Conversely, assume $(x \vee y)^y, (x \vee y)^x \in [1]_\Theta$. Then $\langle (x \vee y)^y, 1 \rangle \in \Theta$ and hence

$$\langle x \vee y, y \rangle = \langle (x \vee y)^{yy}, 1^y \rangle \in \Theta$$

and, analogously, $\langle x \vee y, x \rangle \in \Theta$. Due to symmetry and transitivity of Θ, we infer $\langle x, y \rangle \in \Theta$.

Hence, if $\Theta, \Phi \in \mathrm{Con}\mathcal{S}$ and $[1]_\Theta = [1]_\Phi$ then $\langle x, y \rangle \in \Theta$ iff $(x \vee y)^x, (x \vee y)^y \in [1]_\Theta = [1]_\Phi$ iff $\langle x, y \rangle \in \Phi$ thus $\Theta = \Phi$. ■

By Theorem 1, every congruence on $\mathcal{S}$ is uniquely determined by its kernel. Hence, to characterize congruences we need only to characterize their kernels which is our next task.

Let $\mathcal{S} = (S; \vee, 1, ({}^a)_{a \in S})$ be a semilattice with SAI. A subset $F \subseteq S$ is called a *filter* of $\mathcal{S}$ if $1 \in F$ and the following conditions are satisfied

(i) $a \in F$ and $a \leq y$ imply $y \in F$;

(ii) $a \in F$ and $a^x \in F$ imply $x \in F$;

(iii) $a^b \in F$ implies $(a \vee z)^{(b \vee z)} \in F$ for each $z \in S$;

(iv) $a^b \in F$ implies $(b^z)^{(a^z)} \in F$ for each $z \leq b$.

Let us note that if $b \leq a$ then $b \vee z \leq a \vee z$ and if, moreover, $z \leq b$, then $a^z \leq b^z$ thus the conditions (iii) and (iv) are correctly settled.

In what follows, let $\mathcal{S} = (S; \vee, 1, ({}^a)_{a \in S})$ be a semilattice with SAI and F be a subset of S. Define a binary relation Θ_F on S by the rule

$$\langle x, y \rangle \in \Theta_F \quad \text{if and only if} \quad (x \vee y)^x, (x \vee y)^y \in F.$$

Lemma 1. *Let F be a filter of $\mathcal{S}$ and $a \leq b \leq c$.*

(a) *If $b^a \in F$ and $c^b \in F$ then also $c^a \in F$;*

(b) *If $\langle a, b \rangle \in \Theta_F$ and $\langle b, c \rangle \in \Theta_F$ then $\langle a, c \rangle \in \Theta_F$;*

(c) *If $\langle b, c \rangle \in \Theta_F$ then $\langle b^a, c^a \rangle \in \Theta_F$;*

(d) *If $\langle a, b \rangle \in \Theta_F$ and $z \in S$ then $\langle a \vee z, b \vee z \rangle \in \Theta_F$.*

Proof.

(a) Since $c^b \in F$ and $a \leq b \leq c$, by (iv) we have $(b^a)^{(c^a)} \in F$. Since also $b^a \in F$, (ii) implies $c^a \in F$.

(b) If $a \leq b \leq c$ and $\langle a, b\rangle \in \Theta_F$, $\langle b, c\rangle \in \Theta_F$ then, by the definition of Θ_F we have

$$b^a = (a \vee b)^a \in F \quad \text{and} \quad c^b = (b \vee c)^b \in F.$$

By (a) we obtain $c^a \in F$ whence $(a \vee c)^a = c^a \in F$, $(a \vee c)^c = c^c = 1 \in F$ thus $\langle a, c\rangle \in \Theta_F$.

(c) Since $a \leq b \leq c$ we have $c^a \leq b^a$. If $\langle b, c\rangle \in \Theta_F$ then $c^b = (b \vee c)^c \in F$ thus, by (iv), also $(b^a)^{(c^a)} \in F$, i.e. $(b^a \vee c^a)^{(c^a)} = (b^a)^{(c^a)} \in F$ and $(b^a \vee c^a)^{(c^a)} = (c^a)^{(c^a)} = 1 \in F$ whence $\langle b^a, c^a\rangle \in \Theta_F$.

(d) Let $z \in S$ and $\langle a, b\rangle \in \Theta_F$. Then $b^a = (a \vee b)^b \in F$ and, by (iii), also $(b \vee z)^{(a \vee z)} \in F$. Of course, $a \vee z \leq b \vee z$ thus $((a \vee z) \vee (b \vee z))^{(a \vee z)} = (b \vee z)^{(a \vee z)} \in F$ and $((a \vee z) \vee (b \vee z))^{(b \vee z)} = (b \vee z)^{(b \vee z)} = 1 \in F$ whence $\langle a \vee z, b \vee z\rangle \in \Theta_F$.

■

Theorem 2. *Let $\mathcal{S} = (S; \vee, 1, (^a)_{a \in S})$ be a semilattice with SAI. A subset $F \subseteq S$ is a kernel of some congruence of $\mathcal{S}$ if and only if F is a filter of $\mathcal{S}$. If F is a filter of $\mathcal{S}$ then Θ_F is a congruence on $\mathcal{S}$ and $F = [1]_{\Theta_F}$.*

Proof. Assume $F = [1]_\Theta$ for some $\Theta \in \mathrm{Con}\mathcal{S}$. We are going to check (i)–(iv) to prove that F is a filter of $\mathcal{S}$.

(i) If $a \in [1]_\Theta$ and $a \leq y$ then $\langle a, 1\rangle \in \Theta$ and hence $\langle y, 1\rangle = \langle a \vee y, 1 \vee y\rangle \in \Theta$ thus $y \in [1]_\Theta$.

(ii) Let $a \in [1]_\Theta$ and $a^x \in [1]_\Theta$. Then $\langle a^x, 1\rangle \in \Theta$ thus also $\langle a, x\rangle = \langle a^{xx}, 1^x\rangle \in \Theta$. Due to the fact that $\langle a, 1\rangle \in \Theta$, we conclude $\langle x, 1\rangle \in \Theta$ whence $x \in [1]_\Theta$.

(iii) Assume $a^b \in [1]_\Theta$, i.e. $\langle a^b, 1\rangle \in \Theta$. Then $\langle a, b\rangle = \langle a^{bb}, 1^b\rangle \in \Theta$ thus also $\langle a \vee z, b \vee z\rangle \in \Theta$ for each $z \in S$. Hence

$$\langle (a \vee z)^{(b \vee z)}, 1\rangle = \langle (a \vee z)^{(b \vee z)}, (b \vee z)^{(b \vee z)}\rangle \in \Theta$$

thus $(a \vee z)^{(b \vee z)} \in [1]_\Theta$.

(iv) Analogously as above, if $a^b \in F$ then $\langle a, b\rangle \in \Theta$ thus also $\langle a^z, b^z\rangle \in \Theta$ for each $z \leq a$, i.e. $\langle (a^z)^{b^z}, 1\rangle = \langle (a^z)^{b^z}, (b^z)^{b^z}\rangle \in \Theta$ thus $(a^z)^{(b^z)} \in F$.

We have shown that $[1]_\Theta$ is a filter of $\mathcal{S}$.

Conversely, let F be a filter of $\mathcal{S}$. We are going to prove that Θ_F is a congruence on $\mathcal{S}$ whose kernel is F. Since $1 \in F$, Θ_F is reflexive and, by the definition, it is symmetrical. At first we observe that, by the definition of Θ_F,

$$(*) \qquad \langle x, y\rangle \in \Theta_F \quad \text{if and only if} \quad \langle x, x \vee y\rangle, \langle y, x \vee y\rangle \in \Theta_F.$$

Now, we prove transitivity of Θ_F.

Assume $\langle a, b\rangle, \langle b, c\rangle \in \Theta_F$. By $(*)$ we have $\langle a, a \vee b\rangle, \langle b, a \vee b\rangle, \langle b, b \vee c\rangle, \langle c, b \vee c\rangle \in \Theta_F$. Since $b \leq a \vee b$ and $b \leq b \vee c$, we apply (d) of Lemma 1 to get $\langle a \vee b, a \vee b \vee c\rangle \in \Theta_F$ and $\langle b \vee c, a \vee b \vee c\rangle \in \Theta_F$. Applying (b) of Lemma 1 we infer $\langle a, a \vee b \vee c\rangle \in \Theta_F$ and $\langle c, a \vee b \vee c\rangle \in \Theta_F$. By the definition of Θ_F we obtain $(a \vee b \vee c)^a \in F$ and $(a \vee b \vee c)^c \in F$. Since $(a \vee b \vee c)^a \leq (a \vee c)^a$ and $(a \vee b \vee c)^c \leq (a \vee c)^c$, we infer by (i) that also $(a \vee c)^a, (a \vee c)^c \in F$ thus $\langle a, c\rangle \in \Theta_F$.

We prove the substitution property of Θ_F.

Assume $\langle x, y\rangle \in \Theta_F$ and $\langle z, v\rangle \in \Theta_F$. Then $(x \vee y)^x, (x \vee y)^y \in F$ and, by (iii) also

$$((x \vee z) \vee (y \vee z))^{(x \vee z)} = (x \vee y \vee z)^{(x \vee z)} \in F$$

and

$$((x \vee z) \vee (y \vee z))^{(y \vee z)} = (x \vee y \vee z)^{(y \vee z)} \in F$$

thus $\langle x \vee z, y \vee z\rangle \in \Theta_F$. Analogously we prove $\langle y \vee z, y \vee v\rangle \in \Theta_F$ and, due to transitivity of Θ_F, we conclude $\langle x \vee z, y \vee v\rangle \in \Theta_F$.

Assume $\langle x, y\rangle \in \Theta_F$ and $z \leq x, y$. As mentioned above, then $\langle x, x \vee y\rangle \in \Theta_F$ and $\langle y, x \vee y\rangle \in \Theta_F$ thus, by (c) of Lemma 1, also $\langle x^z, (x \vee y)^z\rangle \in \Theta_F$ and $\langle y^z, (x \vee y)^z\rangle \in \Theta_F$. Using symmetry and transitivity, we obtain $\langle x^z, y^z\rangle \in \Theta_F$.

Finally, if $a \in F$ then $(a \vee 1)^a = 1^a = a \in F$ and $(a \vee 1)^1 = 1^1 = 1 \in F$ thus $\langle a, 1\rangle \in \Theta_F$, i.e. $a \in [1]_{\Theta_F}$. If $a \in [1]_{\Theta_F}$ then $\langle a, 1\rangle \in \Theta_F$ and hence $a = 1^a = (1 \vee a)^a \in F$. We conclude $F = [1]_{\Theta_F}$. ■

Recall several concepts from [4]. An algebra $\mathcal{A}$ with a constant 1 is called *congruence distributive at* 1 if

$$[1]_{\Theta\wedge(\Phi\vee\Psi)} = [1]_{(\Theta\wedge\Phi)\vee(\Theta\wedge\Psi)}$$

for all $\Theta, \Phi, \Psi \in \mathrm{Con}\mathcal{A}$. Further, $\mathcal{A}$ is *permutable at* 1 if

$$[1]_{\Theta\cdot\Phi} = [1]_{\Phi\cdot\Theta}$$

for all $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$. It is worth noticing that $\Theta\cdot\Phi$ and $\Phi\cdot\Theta$ need not be congruences of $\mathcal{A}$; they are congruences if and only if they permute, i.e. if $\Theta\cdot\Phi = \Phi\cdot\Theta$.

$\mathcal{A}$ is called *arithmetical at* 1 if it is both congruence permutable at 1 and congruence distributive at 1. The following assertion follows directly from Theorem 8.3.2 from [4].

Proposition. *If $\mathcal{A}$ is an algebra with a constant* 1 *and $\mathcal{A}$ has a binary term function $t(x,y)$ satisfying the identities $t(x,x) = 1 = t(1,x)$ and $t(x,1) = x$ then $\mathcal{A}$ is arithmetical at* 1.

This yields immediately

Corollary 1. *Every semilattice with SAI is arithmetical at* 1.

Proof. One can take $t(x,y) = (x\vee y)^x$. Then $t(x,x) = (x\vee x)^x = x^x = 1$, $t(1,x) = (1\vee x)^1 = 1^1 = 1$ and $t(x,1) = (x\vee 1)^x = 1^x = x$. ■

Since every congruence is uniquelly determined by its kernel, Corollary 1 gets the following

Corollary 2. *Every semilattice $\mathcal{S}$ with SAI is congruence distributive, i.e.* $\mathrm{Con}\mathcal{S}$ *is a distributive lattice.*

Proof. By Corollary 1 we infer

$$[1]_{\Theta\wedge(\Phi\vee\Psi)} = [1]_{(\Theta\wedge\Phi)\vee(\Theta\wedge\Psi)}$$

for all $\Theta, \Phi, \Psi \in \mathrm{Con}\mathcal{S}$ and, by Theorem 1, it yields $\Theta\wedge(\Phi\vee\Psi) = (\Theta\wedge\Phi)\vee(\Theta\wedge\Psi)$. ■

Unfortunately, Corollary 1 does not imply congruence permutability of $\mathcal{S}$ due to the fact that $\Theta \cdot \Phi$ need not be a congruence on $\mathcal{S}$. However, we can prove the following

Theorem 3. *Every semilattice with SAI is congruence* 3*-permutable.*

Proof. By Corollary 2, $\mathcal{S} = (S; \vee, 1, (^a)_{a \in S})$ is congruence permutable at 1 and, due to Lemma 8.1.1 [4], it gets $[1]_{\Theta \vee \Phi} = [1]_{\Theta \cdot \Phi}$ for each $\Theta, \Phi \in \mathrm{Con}\mathcal{S}$. Assume $\langle x, y \rangle \in \Theta \vee \Phi$. By Theorem 1, we have $(x \vee y)^y \in [1]_{\Theta \cdot \Phi}$ and $(x \vee y)^x \in [1]_{\Theta \cdot \Phi}$. Hence, there exist $z, v \in S$ such that

$$\langle (x \vee y)^y, z \rangle \in \Theta, \quad \langle z, 1 \rangle \in \Phi$$

$$\langle (x \vee y)^x, v \rangle \in \Theta, \quad \langle v, 1 \rangle \in \Phi.$$

Then

(i) $\langle x, (v \vee x)^x \rangle = \langle (1 \vee x)^x, (v \vee x)^x \rangle \in \Phi$

(ii) $\langle (z \vee y)^y, y \rangle = \langle (z \vee y)^y, (1 \vee y)^y \rangle \in \Phi$

and

$$\langle x \vee y, (z \vee y)^y \rangle = \langle ((x \vee y)^y \vee y)^y, (z \vee y)^y \rangle \in \Theta$$

$$\langle x \vee y, (v \vee x)^x \rangle = \langle ((x \vee y)^x \vee x)^x, (v \vee x)^x \rangle \in \Theta$$

thus also $\langle (v \vee x)^x, (z \vee y)^y \rangle \in \Theta$. This together with (i) and (ii) yields $\langle x, y \rangle \in \Phi \cdot \Theta \cdot \Phi$. We have shown $\Theta \vee \Phi \subseteq \Phi \cdot \Theta \cdot \Phi$. The converse inclusion is trivial thus $\Theta \vee \Phi = \Phi \cdot \Theta \cdot \Phi$ and hence $\mathcal{S}$ is congruence 3-permutable. ■

If $\mathcal{S}$ has a least element, we can prove a stronger assertion.

Theorem 4. *Let* $\mathcal{S} = (S; \vee, 1, 0, (^a)_{a \in S})$ *be a semilattice with SAI having a least element* 0*. Then* $\mathcal{S}$ *is congruence permutable.*

Proof. Let $\mathcal{S}$ have a least element 0. Since the mapping $x \mapsto x^0$ is an antitone involution in the whole semilattice $(S; \vee)$, it is plain to show that $x \wedge y = (x^0 \vee y^0)^0$ is infimum of x, y thus $(S; \vee, \wedge)$ is a lattice.

Hence, $\wedge$ is a term operation of $\mathcal{S}$ and we can construct the following ternary term operation

$$p(x, y, z) = ((x \vee y)^y \vee z)^z \wedge ((z \vee y)^y \vee x)^x.$$

An easy calculation gets

$$p(x, x, z) = ((x \vee x)^x \vee z)^z \wedge ((z \vee x)^x \vee x)^x$$

$$= 1^z \wedge (z \vee x)^{xx} = z \wedge (z \vee x) = z$$

and, analogously,

$$p(x, z, z) = ((x \vee z)^z \vee z)^z \wedge ((z \vee z)^z \vee x)^x$$

$$= (x \vee z)^{zz} \wedge 1^x = (x \vee z) \wedge x = x.$$

Thus $p(x, y, z)$ is a Maltsev term function on $\mathcal{S}$ and hence $\mathcal{S}$ is congruence permutable. ■

References

[1] J.C. Abbott, *Semi-boolean algebras*, Matem. Vestnik **4** (1967), 177–198.

[2] J.C. Abbott, *Orthoimplication algebras*, Studia Logica **35** (1976), 173–177. doi:10.1007/BF02120879

[3] I. Chajda, *Lattices and semilattices having an antitone involution in every upper interval*, Comment. Math. Univ. Carol. **44** (2003), 577–585.

[4] I. Chajda, G. Eigenthaler and H. Länger, *Congruence Classes in Universal Algebra*, Heldermann Verlag (Lemgo, Germany), 220pp., 2003, ISBN 3-88538-226-1.

[5] I. Chajda, R. Halaš and J. Kühr, *Semilattice Structures*, Heldermann Verlag (Lemgo, Germany), 228pp., 2007, ISBN 978-3-88538-230-0.

[6] I. Chajda, R. Halaš and J. Kühr, *Implication in MV-algebras*, Algebra Universalis **53** (2005), 377–382. doi:10.1007/s00012-004-1862-4

Received 14 December 2009
Revised 21 April 2010