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A NOTE ON GOOD PSEUDO BL-ALGEBRAS

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Abstract

Pseudo BL-algebras are a noncommutative extention of BL-algebras. In this paper we study good pseudo BL-algebras and consider some classes of these algebras.

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1. Introduction

Hájek [9] introduced BL-algebras in 1998. MV-algebras introduced by Chang [1] are contained in the class of BL-algebras. A noncommutative extention of MV-algebras, called pseudo MV-algebras, were introduced by Georgescu and Iorgulescu [6]. A concept of pseudo BL-algebras were firstly introduced by Georgescu and Iorgulescu in 2000 as noncommutative generalization of BL-algebras and pseudo MV-algebras. The basic properties of pseudo BL-algebras were given in [2] and [3]. The pseudo BL-algebras correspond to a pseudo-basic fuzzy logic (see [10] and [11]).

In [8], there were characterized some classes of pseudo BL-algebras. In this paper we give some interesting facts about good pseudo BL-algebras.

We study bipartite good pseudo BL-algebras and some connections between a good pseudo BL-algebra A and the set $M(A)$ of elements $a \in A$ such that $a = (a^-)^\sim = (a^\sim)^-$.

2. Preliminaries

Definition 2.1. An algebra $(A, \vee, \wedge, \odot, \to, \rightsquigarrow, 0, 1)$ of type (2,2,2,2,2,0,0) is called *a pseudo BL-algebra* if it satisfies the following axioms for any $a, b, c \in A$:

(C1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice,

(C2) $(A, \odot, 1)$ is a monoid,

(C3) $a \odot b \leq c \Leftrightarrow a \leq b \to c \Leftrightarrow b \leq a \rightsquigarrow c$,

(C4) $a \wedge b = (a \to b) \odot a = a \odot (a \rightsquigarrow b)$,

(C5) $(a \to b) \vee (b \to a) = (a \rightsquigarrow b) \vee (b \rightsquigarrow a) = 1$.

Throughout this paper A will denote a pseudo BL-algebra. For any $a \in A$ and $n = 0, 1, \dots$, we put $a^0 = 1$ and $a^{n+1} = a^n \odot a$.

Proposition 2.2 ([2]). *The following properties hold in A for all $a, b, c \in A$:*

(i) $a \leq b \Leftrightarrow a \to b = 1$,

(ii) $b \leq a \to b$ *and* $b \leq a \rightsquigarrow b$,

(iii) $a \odot b \leq a$ *and* $a \odot b \leq b$,

(iv) $a \to (b \to c) = a \odot b \to c$ *and* $a \rightsquigarrow (b \rightsquigarrow c) = b \odot a \rightsquigarrow c$,

(v) $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$ *and* $(b \vee c) \odot a = (b \odot a) \vee (c \odot a)$,

(vi) $a \leq b \Leftrightarrow a \odot c \leq b \odot c$.

We define $a^- := a \to 0$ and $a^\sim := a \rightsquigarrow 0$. We have

Proposition 2.3 ([2]). *The following properties hold in A for all $a, b, c \in A$:*

(i) $a \leq (a^-)^\sim$ *and* $a \leq (a^\sim)^-$,

(ii) $a^- \odot a = a \odot a^\sim = 0$,

(iii) $(a \odot b)^- = a \to b^-$ *and* $(a \odot b)^\sim = b \rightsquigarrow a^\sim$,

(iv) $a \rightsquigarrow b \leq b^\sim \to a^\sim$ *and* $a \to b \leq b^- \rightsquigarrow a^-$,

(v) $(a \vee b)^- = a^- \wedge b^-$ *and* $(a \vee b)^\sim = a^\sim \wedge b^\sim$,

(vi) $(a \wedge b)^- = a^- \vee b^-$ *and* $(a \wedge b)^\sim = a^\sim \vee b^\sim$,

(vii) $((a^-)^\sim)^- = a^-$ *and* $((a^\sim)^-)^\sim = a^\sim$,

(viii) $a \to b^\sim = b \rightsquigarrow a^-$,

(ix) $a \leq b$ *implies* $b^- \leq a^-$ *and* $b^\sim \leq a^\sim$.

Definition 2.4. A nonempty subset F of A is called *a filter* if it satisfies the following two conditions:

(F1) If $a \in F$ and $a \leq b$, then $b \in F$,

(F2) If $a, b \in F$, then $a \odot b \in F$.

A filter F is called *proper* if $F \neq A$. A proper filter F is called *maximal* or *an ultrafilter* if F is not contained in any other proper filter.

Let $\operatorname{Max} A$ denote the set of all ultrafilters of A. Denote $\mathcal{M}(A) = \bigcap \operatorname{Max} A$. For every filter F of A we define sets

$$F^*_\sim = \{a \in A : a \leq x^\sim \text{ for some } x \in F\},$$

$$F^*_- = \{a \in A : a \leq x^- \text{ for some } x \in F\}.$$

Proposition 2.5 ([8])**.**

(a) $F^*_\sim = \{a \in A : a^- \in F\}$,

(b) $F^*_- = \{a \in A : a^\sim \in F\}$.

Definition 2.6. A is called:

(1) *bipartite* if $A = F \cup F^*_\sim = F \cup F^*_-$ *for some ultrafilter* F *of* A.

(2) *strongly bipartite* if $A = F \cup F^*_\sim = F \cup F^*_-$ *for all* $F \in \operatorname{Max} A$.

Proposition 2.7 ([13]). *Let F be a proper filter of A. Then the following conditions are equivalent:*

(i) $A = F \cup F^*_\sim = F \cup F^*_-$,

(ii) $F^*_- = F^*_\sim = A - F$,

(iii) $\forall a \in A\ (a \in F \textit{ or } (a^- \in F \textit{ and } a^\sim \in F))$.

Let $S(A) := \{a \vee a^\sim : a \in A\} \cup \{a \vee a^- : a \in A\}$.

Proposition 2.8 ([8]). $S(A) = \{a \in A : a \geq a^\sim \text{ or } a \geq a^-\}$.

Proposition 2.9 ([13]). $\mathcal{M}(A) \subseteq S(A)$.

Proposition 2.10 ([13]). *The following conditions are equivalent:*

(i) *A is strongly bipartite,*

(ii) $\forall_{F \in \operatorname{Max} A}\ A = F \cup F^*_\sim = F \cup F^*_-$,

(iii) $\forall_{F \in \operatorname{Max} A}\ S(A) \subseteq F$,

(iv) $S(A) = \mathcal{M}(A)$.

In the sequel, we need to recall same facts about pseudo MV-algebras, which are the noncommutative generalizations of MV-algebras.

Definition 2.11. A *pseudo MV-algebra* is an algebra $(M; \oplus, ^-, ^\sim, 0, 1)$ of type $(2, 1, 1, 0, 0)$, which satisfies the following conditions for all $a, b, c \in M$:

(A1) $a \oplus (b \oplus c) = (a \oplus b) \oplus c$,

(A2) $a \oplus 0 = 0 \oplus a = a$,

(A3) $a \oplus 1 = 1 \oplus a = 1$,

(A4) $1^{\sim} = 0$; $1^{-} = 0$,

(A5) $(a^{\sim} \oplus b^{\sim})^{-} = (a^{-} \oplus b^{-})^{\sim}$,

(A6) $a \oplus a^{\sim} \cdot b = b \oplus b^{\sim} \cdot a = a \cdot b^{-} \oplus b = b \cdot a^{-} \oplus a$,

(A7) $a \cdot (a^{-} \oplus b) = (a \oplus b^{\sim}) \cdot a$,

(A8) $(a^{-})^{\sim} = a$.

where $a \cdot b = (b^{-} \oplus a^{-})^{\sim}$ and the operation $\cdot$ has a priority to the operation $\oplus$.

Recall that in a pseudo MV-algebra M the following conditions hold:

(i) $(a^{\sim})^{-} = a$,

(ii) $a \cdot b = (b^{\sim} \oplus a^{\sim})^{-}$,

(iii) $0^{-} = 1$.

Definition 2.12. The nonempty subset $I \subseteq M$ is called *an ideal* of a pseudo MV-algebra M if the following conditions hold for all $a, b \in M$:

(I1) If $a \in I$, $b \in M$ and $b \leq a$, then $b \in I$;

(I2) If $a, b \in I$, then $a \oplus b \in I$.

Definition 2.13. An ideal I of M is called *proper* if $I \neq M$. A proper ideal I of M is called *maximal* if I is not contained in any other proper ideal of M.

The set of all maximal ideals of a pseudo MV-algebra M is denoted by $\mathbf{Max}M$ and the intersection of all maximal ideals of M by Rad$M$.

Set $T(M) = \{a \wedge a^{-} : a \in M\}$. We have

Proposition 2.14 ([5]) **.** Rad$M \subseteq T(M)$.

Let I be an ideal of a pseudo MV-algebra M. We set

$$I^- = \{a^- : a \in I\},$$

$$I^\sim = \{a^\sim : a \in I\}.$$

A pseudo MV-algebra M is called *bipartite* if there exists a maximal ideal I of M such that $M = I \cup I^- = I \cup I^\sim$. If $M = I \cup I^- = I \cup I^\sim$ for all $I \in$**Max**M, then M is called *strongly bipartite*.

Proposition 2.15 ([4]). *The following conditions are equivalent for pseudo MV-algebra M :*

(i) *M is strongly bipartite,*

(ii) *for all $I \in$**Max**M, $M = I \cup I^- = I \cup I^\sim$,*

(iii) *$T(M) =$*Rad*M.*

3. Good pseudo BL-algebras

Definition 3.1. A *good* pseudo BL-algebra is a pseudo BL-algebra which satisfies the following identity:

$$(a^-)^\sim = (a^\sim)^-.$$

From this place to the end of this paper, A will denote a good pseudo BL-algebra.

We consider the subset

$$M(A) = \{a \in A : a = (a^-)^\sim = (a^\sim)^-\}$$

of A.

For any $a, b, \in A$, we define

$$a \oplus b := (b^- \odot a^-)^\sim.$$

Proposition 3.2 ([8]). *The following properties hold in A :*

(i) $0, 1 \in M(A)$,

(ii) *$a^- \in M(A)$ and $a^\sim \in M(A)$ for any $a \in A$,*

(iii) *If* $a, b \in M(A)$, *then* $a \oplus b = b^{\sim} \to a = a^{-} \rightsquigarrow b$,

(iv) *If* $a, b \in M(A)$, *then* $a \oplus b^{-} = b \to a, a \oplus b^{\sim} = a^{-} \rightsquigarrow b^{\sim}, a^{-} \oplus b = b^{\sim} \to a^{-}$ *and* $a^{\sim} \oplus b = a \rightsquigarrow b$.

Proposition 3.3 ([8])**.** *The structure* $(M(A), \oplus, ^{\sim}, ^{-}, 0, 1)$ *is a pseudo MV-algebra. The order on* A *agrees with the one of* $M(A)$*, defined by* $a \leq_{M(A)} b$ *iff* $a^{\sim} \oplus b = 1$.

Following [8] we define two maps: $\varphi_1 : A \to M(A)$ by $\varphi_1(a) = a^{-}$ and $\varphi_2 : A \to M(A)$ by $\varphi_2(a) = a^{\sim}$.

Let $X \subseteq A$. Write $X^{-} = \varphi_1(X)$ and $X^{\sim} = \varphi_2(X)$. It is obvious that

$$X^{-} = \{a^{-} : a \in X\},$$

$$X^{\sim} = \{a^{\sim} : a \in X\}.$$

Set

$$X_{\sim} = \{a : a^{\sim} \in X\},$$

$$X_{-} = \{a : a^{-} \in X\}.$$

If $X \subseteq M(A)$, then $\varphi_1^{-1}(X) = X_{-}$ and $\varphi_2^{-1}(X) = X_{\sim}$.

Following [8] we have

Proposition 3.4. *If* F *is a filter of* A *and* I *is an ideal of* $M(A)$*, then:*

(i) F^{-} *and* $F^{\sim}$ *are ideals of* $M(A)$*;*

(ii) I_{-} *and* $I_{\sim}$ *are filters of* A;

(iii) *if* I *is proper, then* I_{-} *and* $I_{\sim}$ *are proper filters of* A;

(iv) *if* F *is proper, then* F^{-} *and* $F^{\sim}$ *are proper ideals of* $M(A)$*;*

(v) $F \subseteq (F^{-})_{-}$ *and* $F \subseteq (F^{\sim})_{\sim}$;

(vi) *if* F *is an ultrafilter, then* $(F^{-})_{-} = (F^{\sim})_{\sim} = F$;

(vii) $(I_{\sim})^{\sim} = (I_{-})^{-} = I$;

(viii) *if* I *is maximal, then* I_{-} *and* $I_{\sim}$ *are ultrafilters of* A*;*

(ix) *if* F *is an ultrafilter, then* $F^{-}, F^{\sim}$ *are maximal ideals of* $M(A)$.

Proposition 3.5. *Let* F *be a filter of* A*. Then* $F^{-} = F^{*}_{-}$ *and* $F^{\sim} = F^{*}_{\sim}$.

Proof. Let $b \in F^{-}$. Then $b = a^{-}$, where $a \in F$. Obviously, $b^{\sim} = (a^{-})^{\sim}$. Since $a \leq (a^{-})^{\sim}$, $a \in F$ and F is a filter, we have $b^{\sim} \in F$ and hence $b \in F^{*}_{-}$.

Conversely, let $b \in F^{*}_{-}$. Then $b^{\sim} \in F$. So we have $(b^{\sim})^{-} \in F^{-}$. Since $b \leq (b^{\sim})^{-}$, $(b^{\sim})^{-} \in F^{-}$ and F^{-} is an ideal, we have $b \in F^{-}$.

Similarly we can show that $F^{\sim} = F^{*}_{\sim}$. ■

From Propositions 2.5 and 3.5 we obtain

Corrolary 3.6. *Let* F *be a filter of* A*. Then* $F^{-} = F_{\sim}$ *and* $F^{\sim} = F_{-}$.

Proposition 3.7. *Let* I *be an ideal of* $M(A)$*. Then* $I^{-} = M(A) \cap I_{\sim}$ *and* $I^{\sim} = M(A) \cap I_{-}$.

Proof. Let $b \in I^{-}$. Then $b = a^{-}$, where $a \in I$. Hence $b^{\sim} = (a^{-})^{\sim}$. Since $I \subseteq M(A)$ and $a \in I$, we have $b^{\sim} = a$. Therefore $b^{\sim} \in I$. Consequently $b \in I_{\sim}$. By Proposition 3.2 (ii), $b = a^{-} \in M(A)$. We obtain that $b \in M(A) \cap I_{\sim}$.

Conversely, let $b \in M(A) \cap I_{\sim}$. Then $b \in M(A)$ and $b \in I_{\sim}$, i.e., $b \in M(A)$ and $b^{\sim} \in I$. Hence $b = (b^{\sim})^{-} \in I^{-}$.

Similarly we can prove that $I^{\sim} = M(A) \cap I_{-}$ ■

Proposition 3.8. $(\mathrm{Rad} M(A))_{-} = (\mathrm{Rad} M(A))_{\sim} = \mathcal{M}(A)$.

Proof. Let us notice that:

$$a \notin \mathcal{M}(A) \Leftrightarrow a \notin \bigcap_{F \in \mathrm{Max}\, A} F \Leftrightarrow \exists_{F \in \mathrm{Max}\, A}\ a \notin F \Leftrightarrow$$

$$\Leftrightarrow \exists_{F \in \mathrm{Max}\, A}\ a \notin (F^{\sim})_{\sim} \Leftrightarrow \exists_{F \in \mathrm{Max}\, A}\ a^{\sim} \notin F^{\sim} \Leftrightarrow$$

$$\Leftrightarrow \exists_{I = F^{\sim} \in \mathbf{Max}(M(A))}\ a^{\sim} \notin I \Leftrightarrow a^{\sim} \notin \mathrm{Rad} M(A) \Leftrightarrow$$

$$\Leftrightarrow a \notin (\mathrm{Rad} M(A))_{\sim}.$$

Similarly, we can prove that $(\mathrm{Rad} M(A))_{-} = \mathcal{M}(A)$. ■

Proposition 3.9. $\mathrm{Rad} M(A) = (\mathcal{M}(A))^{-} = (\mathcal{M}(A))^{\sim}$.

Proof. $\mathrm{Rad} M(A)$ is an ideal. By Proposition 3.4 (vii) $\mathrm{Rad} M(A) = ((\mathrm{Rad} M(A))_{-})^{-}$. From Proposition 3.8 we obtain $\mathrm{Rad} M(A) = (\mathcal{M}(A))^{-}$. Similarly, $\mathrm{Rad} M(A) = (\mathcal{M}(A))^{\sim}$. ■

Corrolary 3.10.

(i) $\left((\mathcal{M}(A))^{-}\right)_{-} = ((\mathcal{M}(A))^{\sim})_{\sim} = \mathcal{M}(A)$,

(ii) $((\mathrm{Rad} M(A))_{-})^{-} = ((\mathrm{Rad} M(A))_{\sim})^{\sim} = \mathrm{Rad} M(A)$.

Proof. By Propositions 3.8 and 3.9 $\left((\mathcal{M}(A))^{-}\right)_{-} = (\mathrm{Rad} M(A))_{-} = \mathcal{M}(A)$ and $((\mathcal{M}(A))^{\sim})_{\sim} = (\mathrm{Rad} M(A))_{\sim} = \mathcal{M}(A)$.

(ii) Follows from Proposition 3.4 (vii). ■

Proposition 3.11. *If $M(A)$ is bipartite by I, then $I_{-} = I_{\sim}$.*

Proof. By assumption, $M(A) = I \cup I^{\sim} = I \cup I^{-}$. Hence $I^{-} = M(A) - I = I^{\sim}$.

Let $a \in I_{\sim}$, then $a^{\sim} \in I$, which implies $(a^{\sim})^{-} = (a^{-})^{\sim} \in I^{-} = I^{\sim}$. Hence $(a^{-})^{\sim} = b^{\sim}$ for some $b \in I$. Since $b \in M(A)$, we conclude that $b = (b^{\sim})^{-} = [(a^{-})^{\sim}]^{-} = a^{-}$. Therefore $a^{-} \in I$. Thus $a \in I_{-}$. We have $I_{\sim} \subseteq I_{-}$. Similarly we can show that $I_{-} \subseteq I_{\sim}$. Consequently, $I_{-} = I_{\sim}$. ■

Proposition 3.12. *If A is bipartite by F, then $F^{-} = F^{\sim}$.*

Proof. Let F be an ultrafilter such that $A = F \cup F^*_- = F \cup F^*_\sim$. By Proposition 2.7, $F^*_\sim = F^*_- = A - F$. Then from Proposition 3.5 we have $F^- = F^\sim$. ■

Theorem 3.13. *A good pseudo BL-algebra A is bipartite iff $M(A)$ is a bipartite pseudo MV-algebra.*

Proof. Let A be bipartite, i.e. there exists an ultrafilter F such that $A = F \cup F^*_- = F \cup F^*_\sim$. Then we have $M(A) = (F \cap M(A)) \cup F^-$.

By Propositions 3.4 and 3.7, $F \cap M(A) = (F^-)_- \cap M(A) = (F^-)^\sim$.

So we obtain, $M(A) = (F^-)^\sim \cup F^-$ and by Proposition 3.4 (ix), F^- is a maximal ideal of $M(A)$. From Propositions 3.4, 3.7 and 3.12 we have $F \cap M(A) = (F^\sim)_\sim \cap M(A) = (F^\sim)^- = (F^-)^-$.Then we have $M(A) = (F \cap M(A)) \cup F^- = (F^-)^- \cup F^-$, thus $M(A)$ is bipartite.

Conversely, let $M(A) = I \cup I^\sim = I \cup I^-$, where I is a maximal ideal of $M(A)$. Now we prove that

$$\forall_{a\in A}[a \in I_- \text{ or } (a^\sim \in I_- \text{ and } a^- \in I_-)] \tag{1}$$

holds. Suppose $a \notin I_- = I_\sim$ (see Proposition 3.11) we have $a^\sim \notin I$. Hence $a^\sim \in I^\sim$. Then $a^\sim \in I_-$, by Proposition 3.7. Thus (1) satisfied. I_- is proper due to Proposition 3.4 (iii). Applying Proposition 2.7 we get $A = I_- \cup (I_-)^*_\sim = I_- \cup (I_-)^*_-$ where, by Proposition 3.4 (viii), I_- is an ultrafilter of A. ■

Corrolary 3.14.

(i) *If $M(A)$ is a strongly bipartite pseudo MV-algebra, then $I_- = I_\sim$ for any maximal ideal I of $M(A)$.*

(ii) *If A is strongly bipartite pseudo BL-algebra, then $F^- = F^\sim$ for any ultrafilter F of A.*

Proof. By Propositions 3.11 and 3.12. ■

Theorem 3.15. *A good pseudo BL-algebra A is strongly bipartite iff $M(A)$ is a strongly bipartite pseudo MV-algebra.*

Proof. Let A be a strongly bipartite pseudo BL-algebra and suppose that $M(A)$ is not strongly bipartite. Then there exists a maximal ideal I of $M(A)$ such that $M(A) \neq I \cup I^-$ or $M(A) \neq I \cup I^\sim$. Without less of generality we can assume that there is $a_0 \in M(A) - (I \cup I^-)$. Let $F = I_-$. By Proposition 3.4 (viii), F is an ultrafilter of A. From Proposition 3.4 (viii) and Corrolary 3.14 we have $I = (I_-)^- = (I_-)^\sim$. Observe that

$$a \in M(A) - I \Rightarrow a^- \notin I_-. \tag{2}$$

Indeed, suppose that $a \in M(A) - I$ and $a^- \in I_-$. Then $a = (a^-)^\sim \in (I_-)^\sim = I$, a contradiction. Thus (2) holds. Since $a_0 \in M(A) - I$, we conclude that $a_0^- \notin I_-$. It is easy to see that $a_0^\sim \notin I$. Applying (2) yields $a_0 = (a_0^\sim)^- \notin I_-$. Consequently, $a_0 \notin F$ and $a_0^- \notin F$. By Propositions 2.7 and 2.10, A is not strongly bipartite. A contradiction.

Conversely, let $M(A)$ be a strongly bipartite pseudo MV-algebra and A is not bipartite. Then there exists an ultrafilter F of A such that

$$\exists_{a \in A}\ [a \notin F \text{ and } (a^- \notin F \text{ or } a^\sim \notin F)].$$

Suppose that $b, b^- \notin F$. Let $I = F^-$. Then I is a maximal ideal of $M(A)$, by Proposition 3.4 (ix). From Proposition 3.2 we see that $b^- \in M(A)$. Observe that $b^- \notin I$. Indeed, $b \notin F = (F^-)^-$ and hence $b^- \notin F^- = I$. Since $I_- = (F^-)_- = F$ (see Proposition 3.4) and $b^- \notin F$, we have $b^- \notin I_-$ and hence $b^- \notin M(A) \cap I_- = I^\sim$. Thus $b^- \in M(A) - (I \cup I^\sim)$ Therefore $M(A) \neq I \cup I^\sim$. It is a contradiction. ∎

Corrolary 3.16. *Let A be strongly bipartite. Then:*

(a) $T(M(A))_- = (T(M(A))_\sim = S(A)$,

(b) $(S(A))^- = (S(A))^\sim = T(M(A))$.

Proof. (a) By Theorem 3.15, $M(A)$ is a strongly bipartite pseudo MV-algebra and hence $T(M(A)) =$Rad$M(A)$ (see Proposition 2.15). Applying Propositions 3.8 and 2.10 we obtain

$$(T(M(A)))_- = (\mathrm{Rad}M(A))_- = \mathcal{M}(A) = S(A).$$

Similarly, $(T(M(A)))_{\sim} = S(A)$.

(b) From the proof of (a) and by Proposition 3.4 (vii) we have

$$(S(A))^{-} = ((\mathrm{Rad}M(A))_{-})^{-} = \mathrm{Rad}M(A) = T(M(A))$$

and similarly, $(S(A))^{\sim} = T(M(A))$. ■

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