Discussiones Mathematicae
General Algebra and Applications 30 (2010) 91–118

ON MONADIC QUANTALE ALGEBRAS: BASIC PROPERTIES AND REPRESENTATION THEOREMS

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Abstract

Motivated by the concept of quantifier (in the sense of P. Halmos) on different algebraic structures (Boolean algebras, Heyting algebras, MV-algebras, orthomodular lattices, bounded distributive lattices) and the resulting notion of monadic algebra, the paper introduces the concept of a monadic quantale algebra, considers its properties and provides several representation theorems for the new structures.

Key words: m-semilattice, $\bigvee$-lattice, quantale, quantale module, topological system, tropological system, quantale algebra, quantaloid, quantale algebroid, quantifier, monadic quantale algebra, Girard quantale, Q-equivalence relation, Ω-valued set, GL-monoid, commutative integral cl-monoid.

2000 Mathematics Subject Classification: 18B99, 06F07, 03G25.

1. Introduction

In [23] P. Halmos introduced the notion of *quantifier* (more precisely, *existential quantifier*) on a Boolean algebra A as a map $A \xrightarrow{\exists} A$ such that for every $a, b \in A$,

(1) $\exists \bot = \bot$ ($\bot$ is the smallest element of A),

(2) $a \leqslant \exists a$,

(3) $\exists(a \wedge \exists b) = \exists a \wedge \exists b$.

As a result the concepts of *monadic algebra* (a Boolean algebra A together with a quantifier $\exists$ on A) and *monadic homomorphism* (a Boolean homomorphism $(A, \exists) \xrightarrow{f} (A', \exists')$ such that $f \circ \exists = \exists' \circ f$) appeared. The theory of monadic Boolean algebras is an algebraic treatment of the logic of propositional functions of one argument, with Boolean operations and a single (existential) quantifier.

The new concept induced many researchers to study its properties. In particular, there exist (at least) three representation theorems for quantifiers. The first one was originally mentioned by C. Davis in 1954 for *S5 operators* (which are essentially quantifiers) [14, Theorem 2.1] and restated for quantifiers by O. Varsavsky in 1956 [52].

Theorem 1. *In the Boolean algebra $\mathcal{P}(X)$ of all subsets of a given set X, there exists a one-to-one correspondence between quantifiers on $\mathcal{P}(X)$ and equivalence relations on X.*

The proof of the theorem is based on the well-known bijective correspondence between binary relations R on a set X and $\bigcup$-preserving maps $\mathcal{P}(X) \xrightarrow{f} \mathcal{P}(X)$, given by the rules:

$$R \mapsto f_R \text{ with } f_R(S) = \{x \in X \mid xRs \text{ for some } s \in S\}, \tag{1}$$

$$f \mapsto R_f \text{ with } xR_f y \text{ iff } x \in f(\{y\}). \tag{2}$$

The second representation was given by P. Halmos himself in [23, Theorem 5]. It was motivated by the fact that every quantifier is a closure (monotone, expansive, idempotent [55]) operator. Moreover, the range of a quantifier on A is a *relatively complete subalgebra* of A, i.e., it has meets of all subsets of the form $\{b \in \exists^{\rightarrow}(A) \mid a \leqslant b\}$, where a is an arbitrary element of A and $\exists^{\rightarrow}(A) = \{\exists c \mid c \in A\}$.

Theorem 2. *Given a Boolean algebra A, there exists a one-to-one correspondence between quantifiers on A and relatively complete subalgebras of A.*

The proof uses the standard technique of closure operators. In particular, given a relatively complete subalgebra S of A, the respective quantifier is defined by $\exists\, a = \bigwedge\{s \in S \,|\, a \leqslant s\}$.

The last and the strongest representation was also provided by P. Halmos [23, Theorem 12]. It is based on the concept of *functional monadic algebra*, i.e., the pair $(A, \exists)$, where A is a Boolean subalgebra of some powerset B^X (B is a Boolean algebra and X is a set) such that:

(1) for every p in A, the join $\bigvee p^{\to}(X)$ and the meet $\bigwedge p^{\to}(X)$ exist in B;

(2) the constant maps $\exists\, p$ and $\forall\, p$, defined by $\exists\, p(x) = \bigvee p^{\to}(X)$ and $\forall\, p(x) = \bigwedge p^{\to}(X)$, belong to A;

and the quantifier $\exists$ is given by $\exists\, p = \bigvee p^{\to}(X)$.

Theorem 3. *Every monadic Boolean algebra is isomorphic to a functional monadic Boolean algebra.*

The proof is based on Stone's representation theorem for Boolean algebras claiming that there exists a one-to-one correspondence between Boolean algebras A and Boolean (totally disconnected Hausdorff) spaces X such that each algebra A is isomorphic to the algebra of all clopen (closed and open) subsets of the corresponding space X. Theorem 3, in effect, asserts that functional algebras, with which the theory of monadic algebras began, exhaust all possible cases.

There exist various generalizations of the concept of P. Halmos. To mention a few of them recall the notions of *monadic Heyting algebra* of A. Monteiro and O. Varsavsky [32], *monadic MV-algebra* of J. D. Rutledge [45], *monadic orthomodular lattice* of M. F. Janowitz [26] and *Q-distributive lattice* of R. Cignoli [11]. In each of the above-mentioned cases the authors (and their collaborators) tried to provide representation theorems for their structures generalizing one of the aforesaid theorems for monadic Boolean algebras [5, 6, 7, 12, 38, 40]. An up to date state of the field is contained in [13], where J. Cīrulis considers quantifiers on multiplicative semilattices with the aim "to find out how weak a lattice-structure may be in order that a reasonable theory of existential quantifiers on it still could be developed".

Recall that a *multiplicative semilattice*, or an *m-semilattice*, is a $\vee$-semilattice A equipped with a binary operation $\otimes$ such that for every a, b, cı!nA, $a \otimes (b \vee c) = (a \vee b) \otimes (a \vee c)$ and $(a \vee b) \otimes c = (a \vee c) \otimes (b \vee c)$. Every distributive lattice (in particular, every Heyting algebra) is an m-semilattice. If $(A, \oplus, \odot, \neg, 0, 1)$ is an MV-algebra, then its reduct $(A, \vee, \odot)$, where $\vee$ is defined by $x \vee y = x \oplus (\neg x \odot y)$, is an m-semilattice. The main results of [13] include generalizations of Theorems 1, 2 for the case of m-semilattices, leaving Theorem 3 for the intended continuation of the paper (a suitable topological representation *à la* Stone was missing). It is the aim of this manuscript to illustrate the approach of J. Cīrulis with one more example.

Notice that every quantale [29, 42] is an m-semilattice. Moreover, the one-to-one correspondence between $Rel(X)$ (relations) and $End(\mathcal{P}(X))$ ($\bigvee$-lattice endomorphisms) mentioned after Theorem 1 is a quantale isomorphism [34]. On the other hand, Proposition 3.1.2 in [42] provides a duality between quantic nuclei on a given quantale (closure operators j with the property $j(a) \otimes j(b) \leqslant j(a \otimes b)$) and its quantic quotients (essentially images of nuclei). Unfortunately, we are still unaware of a suitable topological representation for quantales (probably the non-commutative approach of [8, 19, 35] could help).

In [50] we introduced the category Q-**Alg** of algebras over a given unital commutative quantale Q (shortly Q-algebras), motivated by the problem of finding a common framework for both Chang-Goguen and Lowen (stratified) approaches to fuzzy topology [10, 21, 30]. In [47] we showed that a suitable generalization of the structure (called *quantale algebroid*) serves the purpose nicely in case the algebras, underlying fuzzy topologies, are partial. In this paper we introduce the notion of a *monadic quantale algebra*, consider its basic properties and generalize Theorems 1, 2 to the new setting.

The necessary categorical background can be found in [3, 24, 31]. For algebraic notions we recommend [29, 42, 44]. Although we tried to make the paper as self-contained as possible it is expected that the reader is acquainted with basic concepts of category theory, e.g., with that of a categorical equivalence.

2. Algebraic preliminaries

In this section we recall basic algebraic concepts used in the paper. Ultimately, we arrive at the concept of *quantale algebroid*, which provides a convenient categorical framework for one of our representation theorems.

The notion is motivated by the concept of a *quantaloid*, introduced by K.I. Rosenthal in [43] as a natural generalization of quantale. Our concept generalizes the notion of *quantale algebra* introduced in [50]. Let us start by recalling the definition of quantales and their homomorphisms [29, 42].

Definition 4. A *quantale* is a triple $(Q, \leqslant, \otimes)$ such that:

(1) $(Q, \leqslant)$ is a $\bigvee$-lattice (i.e., a partially ordered set having arbitrary joins);

(2) $(Q, \otimes)$ is a semigroup;

(3) $q \otimes (\bigvee S) = \bigvee_{s \in S} (q \otimes s)$ and $(\bigvee S) \otimes q = \bigvee_{s \in S} (s \otimes q)$ for every $q \in Q$ and every $S \subseteq Q$.

Given quantales Q and Q', a map $Q \xrightarrow{f} Q'$ is a *quantale homomorphism* provided that it preserves $\otimes$ and $\bigvee$. **Quant** is the category of quantales and their homomorphisms.

Every quantale, being a complete lattice, has a largest element "$\top$" and has a smallest element "$\bot$". The concept was introduced by C.J. Mulvey in [33] to provide a possible setting for constructive foundations of quantum mechanics and to study the spectra of non-commutative C^*-algebras. A bunch of examples of quantales can be found in, e.g., [2, 29, 37, 39, 42, 44]. To clarify the above-mentioned quantale isomorphism (1)-(2) between $Rel(X)$ and $End(\mathcal{P}(X))$ we will mention just two.

Example 5. Given a set X, the set $Rel(X)$ of binary relations on X is a quantale, where $\bigvee$ are unions and $\otimes$ is given by the composition of relations, i.e., $S \otimes T = \{(x, y) \in X \times X \mid (x, z) \in T \text{ and } (z, y) \in S \text{ for some } z \in X\}$.

Example 6. Given a $\bigvee$-lattice L, the set $End(L)$ of $\bigvee$-preserving maps on L is a quantale, with point-wise $\bigvee$ and $\otimes$ given by the composition of maps.

Easy calculations show that the maps (1)–(2) are quantale isomorphisms. This has been generalized in [34] to deal with the so-called *linear relations*.

The next definition lists some special types of quantales which will be encountered throughout the paper.

Definition 7. A quantale Q is called:

(1) *unital* provided that there exists an element $\imath \in Q$ such that $(Q, \otimes, \imath)$ is a monoid;

(2) *strictly two-sided* provided that Q is unital and $\imath = \top$;

(3) *commutative* provided that $q \otimes s = s \otimes q$ for every $q, s \in Q$.

The quantales of Examples 5, 6 are unital (the diagonal relation and the identity map provide the respective units), but neither strictly two-sided nor commutative. Moreover, the isomorphism between them is unit-preserving. Also notice that every frame (a complete lattice L such that $a \wedge (\bigvee S) = \bigvee_{s \in S}(a \wedge s)$ for every $a \in L$ and every $S \subseteq L$ [27]) is a strictly two-sided, commutative quantale.

On the next step we need the notion of *module over a quantale* (cf. [36, 44, 49]; we especially recommend the very comprehensive survey of [29]).

Definition 8. Given a unital quantale Q, a *unital left Q-module* is a pair $(A, *)$, where A is a $\bigvee$-lattice and $Q \times A \xrightarrow{*} A$ is a map such that:

(1) $q * (\bigvee S) = \bigvee\limits_{s \in S} (q * s)$ for every $q \in Q$ and every $S \subseteq A$,

(2) $(\bigvee T) * a = \bigvee\limits_{t \in T} (t * a)$ for every $a \in A$ and every $T \subseteq Q$,

(3) $q_1 * (q_2 * a) = (q_1 \otimes q_2) * a$ for every q_1, $q_2 \in Q$ and every $a \in A$,

(4) $\imath * a = a$ for every $a \in A$.

Given Q-modules A and B, a map $A \xrightarrow{f} B$ is a *unital left Q-module homomorphism* provided that f preserves $\bigvee$ and $f(q * a) = q * f(a)$ for every $a \in A$ and every $q \in Q$. Q-**Mod** is the category of unital left Q-modules an d their homomorphisms.

For shortness sake from now on "Q-module" means "unital left Q-module". The first lattice analogy of ring module appeared in [28] in connection with analysis of descent theory. The idea of quantale module appeared in [2] as the key notion for treatment of process semantics, generalizing the already existing concept of *topological system* [53] based on the logic of finite observations (see also [39] for another generalization coined as *tropological system*). Recently there appeared the notion of *variable-basis* topological system [15, 48] motivated by the problems from fuzzy topology [16]. Coming back to modules, one can construct a functor from the category of (unital quantale)-valued topological systems to the dual of the category of quantale modules over arbitrary unital quantales. The functor is injective on objects, but (unfortunately) not faithful. A few interesting consequences arise here which will be the topic of our forthcoming papers.

Some properties of the category Q-**Mod** are considered in [49]. For later use we recall from there the construction of free Q-modules (also called *function modules* in [29]).

Example 9. Given a unital quantale Q and a set X, Q^X is the *Q-powerset* of X. Equipped with the point-wise structure, Q^X is a free Q-module over X.

In connection with Example 9 we use the following notations. An arbitrary element of Q^X is denoted by S (by analogy with the so-called crisp powerset $\mathcal{P}(X)$). The constant member of Q^X having value q is denoted by $\underline{q}$. Given $x \in X$, we define a map $\{x\} \in Q^X$ (a generalization of a point) by $\{x\}(y) = 1$ if $y = x$; otherwise, $\{x\}(y) = \bot$.
In the following we recall the notion of quantale algebra [50]. It is motivated by the concept of algebra over a ring [4].

Definition 10. Given a unital commutative quantale Q, a *Q-algebra* is a Q-module $(A, *)$ such that:

(1) $(A, \leqslant, \otimes)$ is a quantale,

(2) $q * (a \otimes b) = (q * a) \otimes b = a \otimes (q * b)$ for every $a, b \in A$ and every $q \in Q$.

Given Q-algebras A and B, $A \xrightarrow{f} B$ is a *Q-algebra homomorphism* provided that f is both a quantale and a Q-module homomorphism. Q-**Alg** is the category of Q-algebras and their homomorphisms.

On the last step we recall the notion of quantale algebroid [47]. The concept generalizes the notion of *quantaloid* of K.I. Rosenthal defined as a category, whose hom-sets are $\bigvee$-lattices, with composition in the category preserving $\bigvee$ in both variables. In the language of enriched category theory this says that quantaloids are precisely the categories enriched in the category **CSLat**$(\bigvee)$ of $\bigvee$-lattices and $\bigvee$-preserving maps. Replacing **CSLat**$(\bigvee)$

by the category Q-**Mod** one gets the following notion (notice that given a category $\mathbf{C}$, $\mathcal{O}(\mathbf{C})$ stands for the class of its objects).

Definition 11. Given a unital commutative quantale Q, a Q-algebroid is a category $\mathbf{A}$ such that:

(1) for $A, B \in \mathcal{O}(\mathbf{A})$, the hom-set $\mathbf{A}(A, B)$ is a Q-module;

(2) composition of morphisms in $\mathbf{A}$ preserves $\bigvee$ and $*$ in both variables.

Given Q-algebroids $\mathbf{A}$, $\mathbf{B}$, a functor $\mathbf{A} \xrightarrow{F} \mathbf{B}$ is a *Q-algebroid homomorphisms* provided that it induces on hom-sets a Q-module homomorphism. Q-**Abrds** is the (quasi)category of quantale algebroids and their homomorphism s.

For convenience sake from now on we do not distinguish between (quasi)categories and categories. Since we are using the standard technique of [3], where the appropriate set-theoretic foundation is provided, no set-theoretic problems resulting from Russel-like paradoxes could arise. The possible restriction to small categories will unnecessary narrow our topic of study.

In [46, 47] we showed that the structure of quantale algebroid provides a common framework for both Chang-Goguen and Lowen (stratified) approaches to fuzzy topology. Moreover, it was shown that the concept is (essentially) a fuzzification of the notion of quantaloid, thereby providing a good motivation for studying it on its own. Notice that a Q-algebroid with one object is just a unital Q-algebra and therefore quantale algebroids can be thought of as quantale algebras "with many objects". Given a unital commutative quantale Q, the category Q-**Mod** is a Q-algebroid. On the other hand, the category Q-**Alg** is not even a quantaloid (introduction of quantale operation collapses everything).

3. Monadic quantale algebras

This section introduces the main object of our study, namely, quantifiers on quantale algebras that ultimately results in the notion of *monadic quantale algebra.*

Definition 12. Given a Q-algebra A, a map $A \xrightarrow{\exists} A$ is called a *left existential quantifier* (LEQ) on A provided that for every $a, b \in A$ and every $q \in Q$,

(1) $a \leqslant \exists\, a$,

(2) $\exists(a \vee b) = \exists\, a \vee \exists\, b$,

(3) $\exists(a \otimes \exists\, b) = \exists\, a \otimes \exists\, b$,

(4) $\exists(q * \exists\, a) = q * \exists\, a$.

Since we are working with complete latices, the following definition is justified.

Definition 13. A LEQ $\exists$ on a Q-algebra A is called *completely additive* provided that $\exists(\bigvee S) = \bigvee_{s \in S} \exists\, s$ for every $S \subseteq A$.

From now on every LEQ is supposed to be completely additive. To illustrate the notion, we provide some examples of quantifiers. All of them are motivated by those for Boolean algebras [23] and therefore have the same names.

Example 14. Given a Q-algebra A, the identity map on A is a LEQ called *discrete*.

Example 15. Given a strictly two-sided Q-algebra A such that $q * \top \in \{\bot, \top\}$ for every $q \in Q$, the map $A \xrightarrow{\exists} A$ defined by $\exists\, a = \bot$ if $a = \bot$; otherwise, $\exists\, a = \top$, is a LEQ called *simple*.

Notice that the condition on $*$ in Example 15 is required to prove the last item of Definition 12.

Example 16. Given a Q-algebra A and a set X, the map $A^X \xrightarrow{\exists} A^X$ (notice that A^X is a Q-algebra with the point-wise structure) defined by $\exists\, S(x) = \bigvee S^{\rightarrow}(X)$ is a LEQ called *functional*.

Example 17. Given a Q-algebra A and a set X, let G be the group of all one-to-one transformations of X. Every $g \in G$ provides a map $A^X \xrightarrow{g_A} A^X$ given by $g_A(\alpha) = \alpha \circ g$. Take a map $G \xrightarrow{\mu} Q$ such that (notice that μ is *not* a Q-subgroup of G in the sense of [41]):

(1) $\mathrm{1} \leqslant \mu(g)$ for $g \in G$,

(2) $\mu(g_1) \otimes \mu(g_2) \leqslant \mu(g_1 \circ g_2)$ for $g_1, g_2 \in G$.

Define $A^X \xrightarrow{\exists} A^X$ by $\exists\,\alpha = \bigvee_{g\in G}\mu(g) * g_A(\alpha)$ and get a LEQ on A^X (the proof uses straightforward computations, i.e., to show that $\exists$ is expansive notice that given $\alpha \in A^X$, $\alpha = 1 * \alpha \leqslant \mu(1_X) * (1_X)_A(\alpha) \leqslant \bigvee_{g\in G}\mu(g) * g_A(\alpha) = \exists\,\alpha$).

With the notion of quantifier in mind, we introduce the concept of monadic quantale algebra.

Definition 18. Given a unital commutative quantale Q, a *monadic Q-algebra* is a pair $(A,\exists)$, where A is a Q-algebra and $\exists$ is a LEQ on A. Given monadic Q-algebras $(A,\exists)$ and $(A',\exists')$, a map $(A,\exists) \xrightarrow{f} (A',\exists')$ is a *monadic Q-algebra homomorphism* provided that $A \xrightarrow{f} A'$ is a Q-algebra homomorphism such that $f\circ\exists = \exists'\circ f$. $\boldsymbol{MQ}$-**Alg** is the category of monadic Q-algebras and their homomorph isms, with the underlying functor to the ground category **Set** of sets denoted by U.

4. Algebraic representation theorem

In this section we provide an analogue of Theorem 2 for monadic quantale algebras. Notice that given a Q-algebra A, every $a \in A$ yields the adjunctions (in the sense of partially ordered sets)
$A \underset{\cdot\otimes a}{\overset{a\to_l\cdot}{\rightleftarrows}} A$ with $a \to_l b = \bigvee\{x \in A \,|\, x \otimes a \leqslant b\}$, and $A \underset{a\otimes\cdot}{\overset{a\to_r\cdot}{\rightleftarrows}} A$ with $a \to_r b = \bigvee\{x \in A \,|\, a \otimes x \leqslant b\}$. The map $A \xrightarrow{(-)^\star} A$ defined by $a^\star = a \to_l \bot$ is a *(left) pseudocomplementation* on A [9], since $a \leqslant b^\star$ iff $a \otimes b = \bot$ for every $a, b \in A$.

Lemma 19. *If $\exists$ is a LEQ on a Q-algebra A, then for every $a, b \in A$ and every $S \subseteq A$,*

(1) $(\exists\circ\exists)a = \exists\,a$,

(2) $\exists(\exists\,a \otimes \exists\,b) = \exists\,a \otimes \exists\,b$,

(3) $\exists(\bigwedge_{s\in S}\exists\,s) = \bigwedge_{s\in S}\exists\,s$,

(4) $\exists(\exists\,a \to_l \exists\,b) = \exists\,a \to_l \exists\,b$,

(5) $\exists\,(\exists\,a)^\star = (\exists\,a)^\star$.

Proof.

(1) $(\exists \circ \exists)a = \exists(1 * \exists a) = 1 * \exists a = \exists a$ by item (4) of Definition 12.

(2) $\exists(\exists a \otimes \exists b) = (\exists \circ \exists)a \otimes \exists b = \exists a \otimes \exists b$ by item (3) of Definition 12.

(3) Follows from the fact that $\exists$ is a closure operator on A.

(4) $\exists(\exists a \rightarrow_l \exists b) \otimes \exists a = \exists((\exists a \rightarrow_l \exists b) \otimes \exists a) \leqslant (\exists \circ \exists)(b) = \exists b$ and therefore $\exists(\exists a \rightarrow_l \exists b) \leqslant \exists a \rightarrow_l \exists b$. The converse inequality follows from item (1) of Definition 12.

(5) Follows from (4) and the fact that $\exists \bot = \bot$. ■

Corollary 20. *Given a LEQ $\exists$ on a Q-algebra A, define $A_{\exists} = \{a \in A \mid \exists a = a\}$. Then $A_{\exists} = \exists^{\rightarrow}(A)$ is a subalgebra of A closed under $\bigwedge$ and $\rightarrow_l$.*

Corollary 20 provides an analogue of Theorem 4 in [23]. Notice that relative completeness is superseded by completeness. The corollary gives rise to some useful new notions.

Definition 21. A Q-**Alg**-morphism $B \xrightarrow{m} A$ is a *MIP-morphism* provided that m preserves $\bigwedge$ and $\rightarrow_l$.

Notice that "MIP" comes from "meet- and left implication-preserving". The next lemma provides the motivation for the concept.

Lemma 22. *If $B \xrightarrow{m} A$ is a MIP-morphism, then m has a lower adjoint $A \xrightarrow{n} B$. Moreover, the map $\exists = m \circ n$ is a LEQ on A.*

Proof. As an example we show the last two items of Definition 12.

(3) $a \otimes \exists b \leqslant \exists a \otimes \exists b$ implies $\exists(a \otimes \exists b) \leqslant \exists(\exists a \otimes \exists b) = (m \circ n)((m \circ n)(a) \otimes (m \circ n)(b)) = (m \circ n \circ m)(n(a) \otimes n(b)) = m(n(a) \otimes n(b)) = (m \circ n)(a) \otimes (m \circ n)(b) = \exists a \otimes \exists b$. On the other hand, $a \otimes \exists b \leqslant \exists(a \otimes \exists b)$ implies $a \leqslant \exists b \rightarrow_l \exists(a \otimes \exists b) = (m \circ n)(b) \rightarrow_l (m \circ n)(a \otimes \exists b) = m(n(b) \rightarrow_l n(a \otimes \exists b))$, implies $(m \circ n)(a) \leqslant (m \circ n \circ m)(n(b) \rightarrow_l n(a \otimes \exists b)) = m(n(b) \rightarrow_l n(a \otimes \exists b)) = (m \circ n)(b) \rightarrow_l (m \circ n)(a \otimes \exists b)$, implies $\exists a \leqslant \exists b \rightarrow_l \exists(a \otimes \exists b)$, implies $\exists a \otimes \exists b \leqslant \exists(a \otimes \exists b)$.

(4) $\exists(q*\exists a) = (m\circ n)(q*(m\circ n)(a)) = (m\circ n\circ m)(q*n(a)) = m(q*n(a)) = q*(m\circ n)(a) = q*\exists a$. ■

Definition 23. A subalgebra B of a Q-algebra A is a *MIC-subalgebra* provided that B is closed under $\bigwedge$ and $\rightarrow_l$.

Notice that "MIC" comes from "meet- and left implication-closed". By Corollary 20, every monadic Q-algebra $(A,\exists)$ provides a MIC-subalgebra of A, namely, the range of $\exists$. Moreover, there exists a simple relation between MIC-subalgebras and MIP-morphisms.

Lemma 24. *If B is a MIC-subalgebra of a Q-algebra A, then the inclusion map $B \overset{i}{\hookrightarrow} A$ is a MIP-monomorphism with the lower adjoint $A \xrightarrow{n} B$ given by $n(a) = \bigwedge(\uparrow a \bigcap B)$ where $\uparrow a = \{c \in A \mid a \leqslant c\}$.*

The next two definitions provide useful categorical tools for establishing a relation between MIP-morphisms, MIC-subalgebras and LEQ.

Definition 25. Given a unital commutative quantale Q, $\boldsymbol{S}Q$-**Alg** is the category, the objects of which are pairs (A,B), where A is a Q-algebra and B is a MIC-subalgebra of A. Morphisms $(A,B) \xrightarrow{f} (A',B')$ are Q-**Alg**-morphisms $A \xrightarrow{f} A'$ such that $f^{\rightarrow}(B) \subseteq B'$ and $f|_B^{B'} \circ n = n' \circ f$, where n (resp. n') is the lower adjoint of the inclusion $B \overset{i}{\hookrightarrow} A$ (resp. $B' \overset{i'}{\hookrightarrow} A'$). The forgetful functor $\boldsymbol{S}Q$-**Alg** $\xrightarrow{V}$ Q-**Alg** is given by $V((A,B) \xrightarrow{f} (A',B')) = A \xrightarrow{f} A'$.

Notice that "$\boldsymbol{S}$" in $\boldsymbol{S}Q$-**Alg** comes from "subalgebra". The next definition provides an extension of the category $\boldsymbol{S}Q$-**Alg**.

Definition 26. Given a unital commutative quantale Q, $(Q\text{-}\mathbf{Alg})^{(\mathbf{2})}$ is the category, the objects of which are (MIP-)Q-**Alg**-monomorphisms $B \xrightarrow{m} A$ (denoted by (B,m,A)). Morphisms $(B,m,A) \xrightarrow{(g,f)} (B',m',A')$ are $(Q\text{-}\mathbf{Alg} \times Q\text{-}\mathbf{Alg})$-morphisms $(B,A) \xrightarrow{(g,f)} (B',A')$ such that $f \circ m = m' \circ g$ and $g \circ n = n' \circ f$, where n (resp. n') is the lower adjoint of m (resp. m'). The forgetful functor $(Q\text{-}\mathbf{Alg})^{(\mathbf{2})} \xrightarrow{W} Q$-**Alg** is g iven by $W((B,m,A) \xrightarrow{(g,f)} (B',m',A')) = A \xrightarrow{f} A'$. □

By Lemma 24 the functor $\boldsymbol{S}Q\text{-}\mathbf{Alg} \xrightarrow{E} (Q\text{-}\mathbf{Alg})^{(\mathbf{2})}$ defined by $E((A,B) \xrightarrow{f} (A',B')) = (B,i,A) \xrightarrow{(f|_B^{B'},f)} (B',i',A')$ is a full embedding of $\boldsymbol{S}Q\text{-}\mathbf{Alg}$ into $(Q\text{-}\mathbf{Alg})^{(\mathbf{2})}$ that justifies our word "extension" before the definition.

The following result provides an analogue of Theorem 2 (the second representation theorem mentioned in the Introduction) in case of monadic quantale algebras (notice that morphisms are also taken into account).

Theorem 27. *There exist the functors* $\boldsymbol{M}Q\text{-}\mathbf{Alg} \xrightarrow{F} \boldsymbol{S}Q\text{-}\mathbf{Alg}$*,* $F((A,\exists) \xrightarrow{f} (A',\exists')) = (A,\exists^{\rightarrow}(A)) \xrightarrow{f} (A',\exists'^{\rightarrow}(A'))$ *and* $\boldsymbol{S}Q\text{-}\mathbf{Alg} \xrightarrow{G} \boldsymbol{M}Q\text{-}\mathbf{Alg}$*,* $G((A,B) \xrightarrow{f} (A',B')) = (A, i\circ n) \xrightarrow{f} (A', i'\circ n')$ *such that* $G\circ F = 1_{\boldsymbol{M}Q\text{-}\mathbf{Alg}}$*,* $F\circ G = 1_{\boldsymbol{S}Q\text{-}\mathbf{Alg}}$ *and* $V\circ F = U$*. In particular,* $(\boldsymbol{M}Q\text{-}\mathbf{Alg}, U) \xrightarrow{F} (\boldsymbol{S}Q\text{-}\mathbf{Alg}, V)$ *is a concrete isomorphism.*

Proof. The proof of the Theorem is based on Lemmas 22, 24 and Corollary 20. For example, to show $G\circ F = 1_{\boldsymbol{M}Q\text{-}\mathbf{Alg}}$ notice that the equality clearly holds on morphisms and on objects it follows that $G\circ F(A,\exists) = (A, i\circ n)$ with $i\circ n(a) = \bigwedge(\uparrow a \bigcap \exists^{\rightarrow}(A)) = \exists\, a$. ■

The next theorem is motivated (actually goes in line with) by [17, Theorem 6].

Theorem 28. *There exist the functors* $\boldsymbol{M}Q\text{-}\mathbf{Alg} \xrightarrow{R} (Q\text{-}\mathbf{Alg})^{(\mathbf{2})}$ *given by the formula* $R((A,\exists) \xrightarrow{f} (A',\exists')) = (\exists^{\rightarrow}(A), i, A) \xrightarrow{(f|_{\exists^{\rightarrow}(A)}^{\exists'^{\rightarrow}(A')},f)} (\exists'^{\rightarrow}(A'), i', A')$ *as well as* $(Q\text{-}\mathbf{Alg})^{(\mathbf{2})} \xrightarrow{T} \boldsymbol{M}Q\text{-}\mathbf{Alg}$ *given by* $T((B,m,A) \xrightarrow{(g,f)} (B',m',A')) = (A, m\circ n) \xrightarrow{f} (A', m'\circ n')$ *such that* $T\circ R = 1_{\boldsymbol{M}Q\text{-}\mathbf{Alg}}$*,* $R\circ T \cong 1_{(Q\text{-}\mathbf{Alg})^{(\mathbf{2})}}$ *and* $W\circ R = U$*,* $U\circ T = W$*. In particular,* $(\boldsymbol{M}Q\text{-}\mathbf{Alg}, U) \xrightarrow{R} ((Q\text{-}\mathbf{Alg})^{(\mathbf{2})}, W)$ *is a concrete equivalence.*

Proof. The proof is similar to that of Theorem 27, additionally using the fact [50] that monomorphisms in Q-**Alg** are injective. ■

Theorems 27, 28 give rise to some useful consequences. Notice that every Q-algebra A provides the following fibres in the above-mentioned categories:

(1) $Fb_{\boldsymbol{M}Q\text{-}\mathbf{Alg}}(A) = \{(A,\exists) \,|\, (A,\exists) \in \boldsymbol{M}Q\text{-}\mathbf{Alg}\}$ which has the natural order $(A,\exists) \leqslant (A',\exists')$ iff $\exists\, a \leqslant \exists' a$ for every $a \in A$;

(2) $Fb_{\boldsymbol{S}Q\text{-}\mathbf{Alg}}(A) = \{(A,B) \mid (A,B) \in \boldsymbol{S}Q\text{-}\mathbf{Alg}\}$ which can be ordered by $(A,B) \leqslant (A',B')$ iff $B \subseteq B'$;

(3) $Fb_{(Q\text{-}\mathbf{Alg})^{(2)}}(A) = \{(B,m,A) \mid (B,m,A) \in (Q\text{-}\mathbf{Alg})^{(2)}\}$ which can be ordered by the following (rather technical) procedure: introduce a pre-order by $(B,m,A) \leqslant (B',m',B')$ iff there exists a Q-**Alg**-morphism $B \xrightarrow{f} B'$ such that $m' \circ f = m$; factorize it by equivalence relation $(B,m,A) \sim (B',m',A')$ iff $(B,m,A) \leqslant (B',m',A')$ and $(B',m',A') \leqslant (B,m,A)$; denote the ensuing partially ordered class by $\overline{Fb_{(Q\text{-}\mathbf{Alg})^{(2)}}(A)}$.

The next theorem provides a duality between LEQ, MIC-subalgebras and MIP-monomorphisms (notice that given a poset P, P^{op} stands for the poset dual to P).

Theorem 29. *For every Q-algebra A, the functors $\boldsymbol{M}Q\text{-}\mathbf{Alg} \xrightarrow{F} \boldsymbol{S}Q\text{-}\mathbf{Alg}$ and $\boldsymbol{M}Q\text{-}\mathbf{Alg} \xrightarrow{R} \overline{(Q\text{-}\mathbf{Alg})^{(2)}}$ of Theorems 27, 28 provide $(Fb_{\boldsymbol{S}Q\text{-}\mathbf{Alg}}(A))^{op} \cong Fb_{\boldsymbol{M}Q\text{-}\mathbf{Alg}}(A) \cong (\overline{Fb_{\boldsymbol{M}Q\text{-}\mathbf{Alg}}(A)})^{op}$.*

4.1. Existential quantifiers and closure operators

In [23, Theorem 3] P. Halmos establishes a relation between quantifiers on Boolean algebras and closure operators [55]. In [40] the author shows that quantifiers on orthomodular lattices are closely related to localic nuclei [27] (every quantifier is a nucleus but not conversely). In the following we provide an analogue of these results in our setting. We begin by modifying the notion of closure operator to suit our needs.

Definition 30. Given a Q-algebra A, a map $A \xrightarrow{\Phi} A$ is called a *closure operator* (CO) on A provided that for every $a, b \in A$ and every $q \in Q$,

(1) $a \leqslant \Phi\, a$,

(2) $(\Phi \circ \Phi)a = \Phi\, a$,

(3) $\Phi\, a \leqslant \Phi\, b$ provided that $a \leqslant b$,

(4) $\Phi(a \otimes b) \leqslant \Phi\, a \otimes \Phi\, b$,

(5) $\Phi(q * a) \leqslant q * \Phi\, a$.

Notice that in general CO is neither a quantic nucleus in the sense of [42, Definition 3.1.1] nor a modules nucleus in the sense of [29, Definition 2.3.3]. The particular choice of the properties of CO is motivated by our wish to obtain a result similar to [23, Theorem 3].

Definition 31. A CO Φ on a Q-algebra A is called *completely additive* provided that $\Phi(\bigvee S) = \bigvee_{s \in S} \Phi\, s$ for every $S \subseteq A$.

The next lemma establishes a relation between completely additive CO and LEQ on a given Q-algebra A.

Lemma 32. *For a completely additive CO Φ on a Q-algebra A equivalent are:*

(1) *Φ is a LEQ on A;*

(2) *$\Phi^{\rightarrow}(A)$ is a subalgebra of A closed under $\rightarrow_l$;*

(3) *$\Phi(\Phi\, a \rightarrow_l \Phi\, b) = \Phi\, a \rightarrow_l \Phi\, b$ for $a, b \in A$.*

Proof. (1)$\Rightarrow$(2) follows from Corollary 20. (2)$\Rightarrow$(3) follows from the fact that $\Phi^{\rightarrow}(A) = \{a \in A \mid \Phi\, a = a\}$. To show (3)$\Rightarrow$(1) it will be enough to establish the last two items of Definition 12.

(3) $\Phi(q * \Phi\, a) \leqslant q * (\Phi \circ \Phi)a = q * \Phi\, a$ by items (2), (5) of Definition 30.

(4) $\Phi(\Phi\, a \otimes \Phi\, b) \leqslant (\Phi \circ \Phi)a \otimes (\Phi \circ \Phi)b = \Phi\, a \otimes \Phi\, b$ by items (2), (4) of Definitions 30 and therefore $\Phi(\Phi\, a \otimes \Phi\, b) = \Phi\, a \otimes \Phi\, b$. Thus, $a \otimes \Phi\, b \leqslant \Phi\, a \otimes \Phi\, b$ gives $\Phi(a \otimes \Phi\, b) \leqslant \Phi(\Phi\, a \otimes \Phi\, b) = \Phi\, a \otimes \Phi\, b$. On the other hand, $a \otimes \Phi\, b \leqslant \Phi(a \otimes \Phi\, b)$ gives $a \leqslant \Phi\, b \rightarrow_l \Phi(a \otimes \Phi\, b)$ and $\Phi\, a \leqslant \Phi(\Phi\, b \rightarrow_l \Phi(a \otimes \Phi\, b)) = \Phi\, b \rightarrow_l \Phi(a \otimes \Phi\, b)$ (item (3) of this lemma), i.e., $\Phi\, a \otimes \Phi\, b \leqslant \Phi(a \otimes \Phi\, b)$.

■

4.2. Universal quantifiers

Several papers on quantifiers, e.g., [7, 17, 18, 23, 32] consider both their existential and universal analogues (recall, e.g., the definition of functional monadic Boolean algebra from Introduction, where the universal quantifier could be defined by $\forall\, p = \bigwedge p^{\rightarrow}(X)$). Following the general move, we introduce universal quantifiers on Q-algebras. There are several possibilities to achieve the goal. The most popular are either to introduce a pair of maps with suitable properties with respect to each other, calling one of them existential and the other universal quantifier [7, 32] or derive one quantifier from the other using a suitable involution-like operation [17, 18, 23]. We will use the second approach exploiting the notion of *Girard quantale* [42, Definition 6.1.2]. The crucial property of such quantales is the exi! stence of a *cyclic dualizing element*.

Definition 33. Given a quantale A and an element $d \in A$,

(1) d is called a *dualizing element* provided that $(a \rightarrow_l d) \rightarrow_r d = a = (a \rightarrow_r d) \rightarrow_l d$ for every $a \in A$;

(2) d is called *cyclic* provided that $a \rightarrow_l d = a \rightarrow_r d$ for every $a \in A$.

Definition 34. A quantale A is called a *Girard quantale* provided it has a cyclic dualizing element d.

Notice that every complete Boolean algebra is a Girard quantale with dualizing element $\bot$. Moreover, $Rel(X)$ is a Girard quantale with $d = (X \times X) \backslash \Delta_X$. The interval $[0, 1]$ of reals with the usual order and with multiplication $a \otimes b = max\{0, a+b-1\}$ form the so-called *Łukasiewicz quantale* which is a Girard one with $d = 0$. The focus of applications of Girard quantales lies mostly in linear logic [20].

Let A be a Q-algebra which is also a Girard quantale with a cyclic dualizing element d. Given $a \in A$, define $a^{\bot} = a \rightarrow_l d = a \rightarrow_r d$. It follows that $(-)^{\bot}$ is an *antitone involution* on A since $a^{\bot\bot} = a$ for every $a \in A$ and $(-)^{\bot}$ is order-reversing. Given $a, b \in A$ and $q \in Q$, define $a \oplus b = (a^{\bot} \otimes b^{\bot})^{\bot}$ and $q \circledast a = (q * a^{\bot})^{\bot}$. Properties of $(-)^{\bot}$ imply that $(A^{op}, \oplus, \circledast, d)$ is a unital Q-algebra.

Definition 35. Given a Q-algebra A which is a Girard quantale, a map $A \xrightarrow{\forall} A$ is called a *left universal quantifier* (LUQ) on A provided that for every $a, b \in A$ and every $q \in Q$,

(1) $\forall a \leqslant a$,

(2) $\forall(a \wedge b) = \forall a \wedge \forall b$,

(3) $\forall(a \oplus \forall b) = \forall a \oplus \forall b$,

(4) $\forall(q \circledast \forall a) = q \circledast \forall a$.

Definition 36. A LUQ $\forall$ on a Q-algebra A which is a Girard quantale is called *completely multiplicative* provided that $\forall(\bigwedge S) = \bigwedge_{s \in S} \forall s$ for every $S \subseteq A$.

Given a monadic Q-algebra $(A, \exists)$, where A is a Girard quantale, define a map $A \xrightarrow{\forall} A$ by $\forall a = (\exists a^{\perp})^{\perp}$.

Lemma 37. *$\forall$ is a completely multiplicative LUQ on A. If $\exists d = d$, then the following conditions hold:*

(1) $\exists (\exists a)^{\perp} = (\exists a)^{\perp}$ *for* $a \in A$,

(2) $\forall (\forall a)^{\perp} = (\forall a)^{\perp}$ *for* $a \in A$,

(3) $\exists \circ \forall = \forall$ *and* $\forall \circ \exists = \exists$,

(4) $a \leqslant \forall a$ *iff* $\exists a \leqslant b$ *for* $a, b \in A$.

Proof. The proof consists of straightforward computations. For example, to show (1) notice that, $\exists (\exists a)^{\perp} = \exists(\exists \rightarrow_l d) = \exists(\exists a \rightarrow_l \exists d) = \exists a \rightarrow_l \exists d = \exists a \rightarrow_l d = (\exists a)^{\perp}$ by item (4) of Lemma 19. To show (3) notice that $(\exists \circ \forall)a = \exists (\exists a^{\perp})^{\perp} = (\exists a^{\perp})^{\perp} = \forall a$ and $(\forall \circ \exists)a = (\exists (\exists a)^{\perp})^{\perp} = (\exists a)^{\perp\perp} = \exists a$ by (1). ■

5. Useful categorical tools

This section provides the necessary categorical framework for representing monadic quantale algebras through generalized equivalence relations, or, in other words, a way to form an analogue of Theorem 1 from the introduction in our setting. In the following we show a method of getting new categories from old. It is based on the notion of *arrow category* considered in almost every treatise on category theory, e.g., in [3, Example 3K(b)]. The procedure is illustrated by the case of Q-algebroids (Definition 11). The method can be readily applied to quantaloids or just categories.

The first procedure could be coined as *arrow extension* (notice that given a category $\mathbf{C}$, $\mathcal{M}(\mathbf{C})$ stands for the class of its objects).

Lemma 38. *Every Q-algebroid homomorphism $\mathbf{A} \xrightarrow{H} \mathbf{B}$ has the arrow extension $\mathbf{A}^{\star} \xrightarrow{H^{\star}} \mathbf{B}^{\star}$ where:*

(1) $\mathbf{A}^\star$ *is the* Q*-algebroid of pairs* (A,∇) *such that* $A \xrightarrow{\nabla} A \in \mathcal{M}(\mathbf{A})$, *and* $\mathbf{A}$*-morphisms* $(A,\nabla) \xrightarrow{f} (A',\nabla')$ *such that* $f\circ\nabla = \nabla'\circ f$ ($\mathbf{B}^\star$ *is defined similarly);*

(2) $\mathbf{A}^\star \xrightarrow{H^\star} \mathbf{B}^\star$ *is the* Q*-algebroid homomorphism defined by* $H^\star((A,\nabla) \xrightarrow{f} (A',\nabla')) = (HA, H\nabla) \xrightarrow{Hf} (HA', H\nabla')$.

If $\mathbf{B} \xrightarrow{G} \mathbf{A}$ *is an inverse of* H, *then* $G^\star$ *is an inverse of* $H^\star$. *Moreover, there exists a full embedding* $\mathbf{A} \xrightarrow{E} \mathbf{A}^\star$ *defined by* $E(A \xrightarrow{f} B) = (A, 1_A) \xrightarrow{f} (B, 1_B)$.

Proof. The proof is based on the fact that the required Q-algebroid operations on $\mathbf{A}^\star$ (resp. $\mathbf{B}^\star$) are implied by $\mathbf{A}$ (resp. $\mathbf{B}$), and well-known properties of arrow categories. ∎

The second procedure could be denoted as *subclass restriction.* It requires some additional assumptions. Let $\mathbf{A} \xrightarrow{H} \mathbf{B}$ be a Q-algebroid isomorphism with the inverse G. Suppose $O_\mathbf{A}, M_\mathbf{A} \subseteq \mathcal{M}(\mathbf{A})$ and $O_\mathbf{B}, M_\mathbf{B} \subseteq \mathcal{M}(\mathbf{B})$ have the restrictions $O_\mathbf{A} \underset{G}{\overset{H}{\rightleftarrows}} O_\mathbf{B}$ and $M_\mathbf{A} \underset{G}{\overset{H}{\rightleftarrows}} M_\mathbf{B}$. Define $\mathbf{A}^\star_{OM}$ to be a "substructure" of $\mathbf{A}^\star$ with objects (A,∇), $\nabla \in O_\mathbf{A}$ and morphisms $(A,\nabla) \xrightarrow{f} (A',\nabla')$, $f \in M_\mathbf{A}$. By analogy with $\mathbf{A}^\star_{OM}$ define $\mathbf{B}^\star_{OM}$ on the basis of $O_\mathbf{B}$, $M_\mathbf{B}$.

Lemma 39. *If* $\mathbf{B}^\star_{OM}$ *is a subalgebroid of* $\mathbf{B}^\star$, *then* $\mathbf{A}^\star_{OM}$ *is a subalgebroid of* $\mathbf{A}^\star$ *and there exist the restriction* $\mathbf{A}^\star_{OM} \underset{G^\star_{OM}}{\overset{H^\star_{OM}}{\rightleftarrows}} \mathbf{B}^\star_{OM}$, *where* $H^\star_{OM}$ *and* $G^\star_{OM}$ *are the restrictions of* $H^\star$ *and* $G^\star$ *respectively.*

Proof. The proof is based on the procedure of moving all required categorical properties from $\mathbf{B}^\star_{OM}$ to $\mathbf{A}^\star_{OM}$. For example, to show that $\mathbf{A}^\star_{OM}$ has identities one proceeds as follows. Take any $\mathbf{A}^\star_{OM}$-object (A,∇). By the assumption, $(HA, H\nabla)$ is a $\mathbf{B}^\star_{OM}$-object and therefore it has the identity 1_{HA} in $\mathbf{B}^\star_{OM}$. Since $1_A = 1_{GHA} = G(1_{HA})$, 1_A is in $M_\mathbf{A}$ and therefore (A,∇) has the identity in $\mathbf{A}^\star_{OM}$. ∎

Notice that notwithstanding the fact that both lemmas are rather simple, the author was not able to find their suitable analogues in the literature and therefore decided to prove them himself.

6. Relational representation theorem

In this section we show an analogue of Theorem 1 for the case of monadic quantale algebras. Similarly to the algebraic case (Theorem 27) we will be interested not only in objects but also in morphisms and therefore will provide a categorical version of Theorem 1. First of all we have to find a suitable substitute for the quantale $Rel(X)$ of binary relations on a set X.

Definition 40. Given a unital quantale Q, Q-**SetRel** is the category, with objects sets and morphisms Q-valued relations (or just Q-relations for short) $X \xrightarrow{R} Y$, i.e., maps $X \times Y \xrightarrow{R} Q$. Given Q-relations $X \xrightarrow{R_1} Y$ and $Y \xrightarrow{R_2} Z$, the composition $R_2 \circ R_1$ is defined by $R_2 \circ R_1(x,z) = \bigvee_{y \in Y} R_1(x,y) \otimes R_2(y,z)$. The identity on a set X is the relation $X \xrightarrow{\Delta_X} X$ given by $\Delta_X(x,x') = \imath$ if $x = x'$; otherwise, $\Delta_X(x,x') = \bot$.

Notice that we do not assume any kind of commutativity here. In [49, Proposition 4.12] we showed that Q-**SetRel** is isomorphic to the Kleisli category of the Q-valued powerset monad on **Set**. The following provides a restatement of the result suitable for our needs. Let $\boldsymbol{F}Q$-**Mod** be the full subcategory of Q-**Mod** with objects all Q^X for an arbitrary set X (recall from Example 9 that Q^X is the free Q-module over X; from here the "$\boldsymbol{F}$" in the name of the category).

Lemma 41. *There exist the functors* $Q\text{-}\mathbf{SetRel} \xrightarrow{H} \boldsymbol{F}Q\text{-}\mathbf{Mod}$, $H(X \xrightarrow{R} Y) = Q^X \xrightarrow{f_R} Q^Y$, $(f_R(S))(y) = \bigvee_{x \in X} S(x) \otimes R(x,y)$ *and* $\boldsymbol{F}Q\text{-}\mathbf{Mod} \xrightarrow{G} Q\text{-}\mathbf{SetRel}$, $G(Q^X \xrightarrow{f} Q^Y) = X \xrightarrow{R_f} Y$, $R_f(x,y) = (f(\{x\}))(y)$ *such that* $H \circ G = 1_{\boldsymbol{F}Q\text{-}\mathbf{Mod}}$ *and* $G \circ H = 1_{Q\text{-}\mathbf{SetRel}}$.

Proof. The proof consists of straightforward computations, i.e., to show that f_R is $\bigvee$-preserving notice that $(f_R(\bigvee_{i \in I} S_i))(y) = \bigvee_{x \in X}(\bigvee_{i \in I} S_i)(x) \otimes R(x,y) = \bigvee_{i \in I}(\bigvee_{x \in X} S_i(x) \otimes R(x,y)) = (\bigvee\limits_{i \in I}(f_R(S_i)))(y)$. To show $G \circ H = 1_{Q\text{-}\mathbf{SetRel}}$ notice that the identity clearly holds on objects. Given a Q-**SetRel** morphism $X \xrightarrow{R} Y$, $((G \circ H)(R))(x,y) = ((H(R))(\{x\}))(y) = \bigvee_{x' \in X} \{x\}(x') \otimes R(x',y) = R(x,y)$. ■

Before moving forward let us recall from [44, Definition 2.5.1] the notion of *involutive quantaloid*.

Definition 42. A quantaloid $\mathbf{Q}$ is called *involutive* provided that it has an isomorphism $\mathbf{Q}^{op} \xrightarrow{(-)^{\circ}} \mathbf{Q}$ which is the identity on objects and for every $\mathbf{Q}$-morphisms $A \xrightarrow{f} B$, $B \xrightarrow{g} C$, $D \xrightarrow{f_i} E$ with $i \in I$,

(1) $(g \circ f)^{\circ} = f^{\circ} \circ g^{\circ}$,

(2) $(1_A)^{\circ} = 1_A$,

(3) $(f^{\circ})^{\circ} = f$,

(4) $(\bigvee_{i \in I} f_i)^{\circ} = \bigvee_{i \in I} f_i^{\circ}$.

The following lemma uses the procedure of arrow extension from the previous section.

Lemma 43. *Both Q-$\mathbf{SetRel}$ and $\boldsymbol{F}Q$-$\mathbf{Mod}$ are quantaloids. If Q is commutative, then*

(1) *both Q-$\mathbf{SetRel}$ and $\boldsymbol{F}Q$-$\mathbf{Mod}$ are Q-algebroids;*

(2) *Q-$\mathbf{SetRel}$ has an involution given by $R^{\circ}(x, y) = R(y, x)$;*

(3) *Q-$\mathbf{SetRel} \underset{G}{\overset{H}{\rightleftarrows}} \boldsymbol{F}Q$-$\mathbf{Mod}$ are Q-algebroid isomorphisms;*

(4) *Q-$\mathbf{SetRel}^{\star} \underset{G^{\star}}{\overset{H^{\star}}{\rightleftarrows}} \boldsymbol{F}Q$-$\mathbf{Mod}^{\star}$ are Q-algebroid isomorphisms.*

Proof. (4) follows from (3) by Lemma 38, and other claims are easy calculations, i.e., to show that Q-$\mathbf{SetRel} \xrightarrow{H} \boldsymbol{F}Q$-$\mathbf{Mod}$ is a quantaloid homomorphism notice that for a family of Q-relations $X \xrightarrow{R_i} Y$ for $i \in I$, $((H(\bigvee_{i \in I} R_i))(S))(y) = \bigvee_{x \in X} S(x) \otimes (\bigvee_{i \in I} R_i)(x, y) = \bigvee_{i \in I}(\bigvee_{x \in X} S(x) \otimes R_i(x, y)) = ((\bigvee_{i \in I} H(R_i))(S))(y)$. ■

To apply the procedure of subclass restriction from the previous section to Q-$\mathbf{SetRel}^{\star} \underset{G^{\star}}{\overset{H^{\star}}{\rightleftarrows}} \boldsymbol{F}Q$-$\mathbf{Mod}^{\star}$ we have first to fix suitable subclasses of morphisms. The next lemma, which provides a characterization of quantifiers in terms of module homomorphisms, gives a hint on the possible candidates.

Lemma 44. *For a Q-algebra A and a map $A \xrightarrow{\exists} A \in \mathcal{M}(Q\text{-}\mathbf{Mod})$, the following are equivalent:*

(1) $\exists$ *is a LEQ on A;*

(2) $a \leqslant \exists\, a$, $(\exists \circ \exists)a \leqslant \exists\, a$ *and* $\exists(a \otimes \exists\, b) = \exists\, a \otimes \exists\, b$ *for every* $a, b \in A$.

Proof. Clearly (1) implies (2). To show the converse one should verify only item (4) of Definition 12 and that follows from the assumption that $\exists$ is a module homomorphism. ■

With Lemma 44 in mind we introduce the next definition which helps to define the required classes $O_{Q\text{-}\mathbf{SetRel}}$ (resp. $O_{\boldsymbol{F}Q\text{-}\mathbf{Mod}}$) and $M_{Q\text{-}\mathbf{SetRel}}$ (resp. $M_{\boldsymbol{F}Q\text{-}\mathbf{Mod}}$) used in Lemma 39.

Definition 45. Define the following classes, where in each $\mathtt{R}_i$ (resp. $\mathtt{H}_i$) X varies through all objects of **Set**:

$$\mathtt{R}_1 = \{\rho \in Q\text{-}\mathbf{SetRel}(X, X) \mid \Delta_X \leqslant \rho\},$$
$$\mathtt{H}_1 = \{\nabla \in \boldsymbol{F}Q\text{-}\mathbf{Mod}(Q^X, Q^X) \mid 1_{Q^X} \leqslant \nabla\};$$

$$\mathtt{R}_2 = \{\rho \in Q\text{-}\mathbf{SetRel}(X, X) \mid \rho \circ \rho \leqslant \rho\},$$
$$\mathtt{H}_2 = \{\nabla \in \boldsymbol{F}Q\text{-}\mathbf{Mod}(Q^X, Q^X) \mid \nabla \circ \nabla \leqslant \nabla\};$$

$$\mathtt{R}_3 = \{\rho \in Q\text{-}\mathbf{SetRel}(X, X) \mid q \otimes \rho(x, y) \otimes \rho(y, z) = \rho(y, z) \otimes q \otimes \rho(x, z) \text{ for every } q \in Q\},$$
$$\mathtt{H}_3 = \{\nabla \in \boldsymbol{F}Q\text{-}\mathbf{Mod}(Q^X, Q^X) \mid \nabla(S \otimes \nabla S') = \nabla S \otimes \nabla S'\};$$

$$\mathtt{R}_4 = \{R \in Q\text{-}\mathbf{SetRel}(X, Y) \mid S(x) \otimes R(x, y) = R(x, y) \otimes \Big(\bigvee_{x' \in X} S(x') \otimes R(x', y)\Big) \text{ for } S \in Q^X\},$$
$$\mathtt{H}_4 = \{f \in \boldsymbol{F}Q\text{-}\mathbf{Mod}(Q^X, Q^Y) \mid f(S \otimes S') = f(S) \otimes f(S')\};$$

$$\mathtt{R}_5 = \{R \in Q\text{-}\mathbf{SetRel}(X, Y) \mid \bigvee_{x \in X} R(x, y) = 1\},$$
$$\mathtt{H}_5 = \{f \in \boldsymbol{F}Q\text{-}\mathbf{Mod}(Q^X, Q^Y) \mid f(\underline{1}) = \underline{1}\}.$$

Lemma 46. *There exists the restriction* $\mathtt{R}_i \underset{G}{\overset{H}{\rightleftarrows}} \mathtt{H}_i$ *for* $1 \leqslant i \leqslant 5$.

Proof. The proof is straightforward, i.e., for $i = 1$ and $\rho \in \mathtt{R}_1$, $\varphi \in \mathtt{H}_1$, $1_{Q^X} = H(\Delta_X) \leqslant H(\rho)$ and $\Delta_X = G(1_{Q^X}) \leqslant G(\varphi)$. For $i = 5$ and $R \in \mathtt{R}_5$, $f \in \mathtt{H}_5$, $((H(R))(\underline{1}))(y) = \bigvee_{x\in X} \underline{1}(x) \otimes R(x,y) = \bigvee_{x\in X} R(x,y) = 1$ and $\bigvee_{x\in X}(G(f))(x,y) = \bigvee_{x\in X}(f(\{x\}))(y) = (f(\bigvee_{x\in X}\{x\}))(y) = (f(\underline{1}))(y) = \underline{1}(y) = 1$. ■

Define $O_{Q\text{-}\mathbf{SetRel}} = \mathtt{R}_1 \bigcap \mathtt{R}_2 \bigcap \mathtt{R}_3$, $M_{Q\text{-}\mathbf{SetRel}} = \mathtt{R}_4 \bigcap \mathtt{R}_5$ and $O_{\boldsymbol{F}Q\text{-}\mathbf{Mod}} = \mathtt{H}_1 \bigcap \mathtt{H}_2 \bigcap \mathtt{H}_3$, $M_{\boldsymbol{F}Q\text{-}\mathbf{Mod}} = \mathtt{H}_4 \bigcap \mathtt{H}_5$. Lemmas 39 and 46 provide the restriction $Q\text{-}\mathbf{SetRel}^{\star}_{OM} \underset{G^{\star}_{OM}}{\overset{H^{\star}_{OM}}{\rightleftarrows}} \boldsymbol{F}Q\text{-}\mathbf{Mod}^{\star}_{OM}$. Denote $Q\text{-}\mathbf{SetRel}^{\star}_{OM}$ by Q-**Equiv** and call it the category of *Q-equivalence relations*. The motivation for the term "equivalence" is provided by the following considerations. If Q is strictly two-sided (Definition 7) and $(X, \rho) \in \mathcal{O}(Q\text{-}\mathbf{Equiv})$, then $\rho^{\circ} \leqslant \rho$ (recall item (2) of Lemma 43) since $\rho(x', x) = \rho(x', x) \otimes \top \geqslant \rho(x', x) \otimes \rho(x, x') = \rho(x, x') \otimes \rho(x', x') = \rho(x, x')$ by two-sidedness, $\mathtt{R}_3$ and $\mathtt{R}_1$, i.e., ρ satisfies the following properties:

(1) $\Delta_X \leqslant \rho$ (reflexivity),

(2) $\rho = \rho^{\circ}$ (symmetry),

(3) $\rho \circ \rho \leqslant \rho$ (transitivity).

It follows that ρ is a Q-equivalence relation in the sense of L. A. Zadeh [56].

Definition 47. Given a unital Q-algebra A, $\boldsymbol{M}_A Q$-**Alg** is the subcategory of $\boldsymbol{M}Q$-**Alg** with objects all pairs $(A^X, \exists)$ such that $\exists \in \mathcal{M}(A\text{-}\mathbf{Mod})$ (the module operation induced by A is denoted by $\circledast$) and morphisms all unit-preserving $(A^X, \exists) \xrightarrow{f} (A^{X'}, \exists')$ such that $f \in \mathcal{M}(A\text{-}\mathbf{Mod})$.

Lemma 48. *Given a unital Q-algebra A and some sets X and Y, it follows that* $A\text{-}\mathbf{Mod}(A^X, A^Y) \subseteq Q\text{-}\mathbf{Mod}(A^X, A^Y)$.

Proof. For $f \in A\text{-}\mathbf{Mod}(A^X, A^Y)$, $q \in Q$ and $S \in A^X$, $f(q * S) = f(q * (\bigvee_{x\in X} S(x) \circledast \{x\})) = f(\bigvee_{x\in X}(q * S(x)) \circledast \{x\}) = \bigvee_{x\in X}(q * S(x)) \circledast f(\{x\}) = q * (\bigvee_{x\in X} S(x) \circledast f(\{x\})) = q * f(\bigvee_{x\in X} S(x) \circledast \{x\}) = q * f(S)$. ■

By Lemmas 48, 44 there exists an isomorphism $\boldsymbol{F}A\text{-}\mathbf{Mod}^{\star}_{OM} \xrightarrow{K} \boldsymbol{M}_A Q\text{-}\mathbf{Alg}$ given by $K((A^X,\nabla) \xrightarrow{f} (A^{X'},\nabla')) = (A^X,\nabla) \xrightarrow{f} (A^{X'},\nabla')$. The next result then follows immediately, providing a categorical analogue of Theorem 1 from Introduction.

Theorem 49. *Every unital Q-algebra A has the isomorphism* $A\text{-}\mathbf{Equiv} \xrightarrow{K\circ H^{\star}_{OM}} \boldsymbol{M}_A Q\text{-}\mathbf{Alg}$.

7. Conclusion: some open problems

In the paper we introduced the notion of monadic quantale algebra and proved some representation theorems for the new structure. It will be the topic of our further research to provide an analogue of Theorem 3 from the introduction in our setting. The main problem is the (already mentioned) lack of a suitable topological representation for quantale algebras. On the other hand, there exists another technique used in [7, Theorem 3.6] for monadic Heyting algebras. It is still an open question whether it is possible to modify the approach for our setting. A slightly less appalling task would be the problem of free monadic quantale algebras over sets (cf. [1]). However, in this last section we would like to draw the attention of the reader to another problem suggested by the notion of Q-equivalence relation introduced in the previous section and closely related to fuzzy set theory.

Stimulated by the internal structure of topoi, in [22, Section 11.9] R. Goldblatt considers a generalized concept of a set as consisting of a collection of (partial) elements, with some Heyting-algebra-valued measure of the degree of equality of these elements. The notion admits an abstract axiomatic development in the following way.

Definition 50 (R. Goldblatt)**.** Given a complete Heyting algebra Ω, Ω-**Set** is the category, the objects of which are pairs $(X, [\cdot \approx \cdot]_X)$ (denoted by $\mathbb{X}$; called *Ω-valued sets*), where X is a set and $X \times X \xrightarrow{[\cdot\approx\cdot]_X} \Omega$ is a map (called *Ω-valued equality*) such that:

(1) $[x \approx y]_X \leqslant [y \approx x]_X$ (symmetry),

(2) $[x \approx y]_X \wedge [y \approx z]_X \leqslant [x \approx z]_X$ (transitivity).

Morphisms are Ω-valued relations $\mathbb{X} \xrightarrow{R} \mathbb{Y}$ such that:

(1) $[x \approx x']_X \wedge R(x,y) \leqslant R(x',y)$ (extensionality),

(2) $R(x,y) \wedge [y \approx y']_Y \leqslant R(x,y')$ (extensionality),

(3) $R(x,y) \wedge R(x,y') \leqslant [y \approx y']_Y$ (functionality),

(4) $[x \approx x]_X = \bigvee\{R(x,y) \mid y \in Y\}$ (totality).

Given $\mathbb{X} \xrightarrow{R_1} \mathbb{Y}$ and $\mathbb{Y} \xrightarrow{R_2} \mathbb{Z}$, the composition $R_2 \circ R_1$ is defined by $R_2 \circ R_1(x,z) = \bigvee\limits_{y \in Y} R_1(x,y) \wedge R_2(y,z)$. The identity on $\mathbb{X}$ is given by $[\cdot \approx \cdot]_X$ itself. d

Notice that every $a \in \Omega$ gives rise to the adjunction $\Omega \underset{\cdot\wedge a}{\overset{a\to\cdot}{\rightleftarrows}} \Omega$, where $a \to b = \bigvee\{c \in \Omega \mid c \wedge a \leqslant b\}$. Given an Ω-valued set $\mathbb{X}$, define a map $X \times X \xrightarrow{[\cdot\sim\cdot]_X} \Omega$ by $[x \sim y]_X = ([x \approx x]_X \vee [y \approx y]_X) \to [x \approx y]_X$. It is not difficult to see that $[\cdot \sim \cdot]_X$ has the following properties:

(1) $[x \sim x]_X = \top$,

(2) $[x \sim y]_X \leqslant [y \sim x]_X$,

(3) $[x \sim y]_X \wedge [y \sim z]_X \leqslant [x \sim z]_X$;

and therefore $[\cdot \sim \cdot]_X$ is called an Ω-*valued equivalence*.

The aforesaid ideas were taken up by the fuzzy community and developed further in, e.g., [25, 51, 54] replacing complete Heyting algebras with *GL-monoids* (commutative, strictly two-sided, divisible quantales) or *commutative integral cl-monoids* (commutative, strictly two-sided quantales). Stimulated by the results obtained in this paper we could try to replace Ω with some Q-algebra A. In such a case the following problems arise (all of them will be addressed to in our future research).

Problem 51. How should one define the category A-**Set** of A-valued sets in order to get A-equivalence relations?

Problem 52. Is it possible to develop the theory of A-valued sets by analogy with that of their Ω-valued counterparts? In particular, how should one define the concept of A-sheaf?

Acknowledgements

This research was partially supported by ESF Project of the University of Latvia No. 2009/0223/1DP/1.1.1.2.0/09/APIA/VIAA/008. The author is grateful to the Department of Mathematics "Ennio De Giorgi" of the University of Salento in Lecce, Italy (especially to Prof. C. Guido) for the opportunity of spending a month at the university during which period the manuscript was prepared.

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Received 20 April 2009
Revised 18 July 2009