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SEMI-OPEN SETS IN BICLOSURE SPACES

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Abstract

The aim of this paper is to introduce and study semi-open sets in biclosure spaces. We define semi-continuous maps and semi-irresolute maps and investigate their behavior. Moreover, we introduce pre-semi-open maps in biclosure spaces and study some of their properties.

Keywords: closure operator, biclosure space, semi-open set, semi-continuous map, semi-irresolute map, pre semi-open map.

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1. Introduction

In 1963, bitopological spaces were introduced by J.C. Kelly [10] as triples (X, τ_1, τ_2) where X is a set and τ_1 and τ_2 are topologies defined on X. After that, a larger number of papers have been written to generalize the topological concept to a bitopological setting, see for instance, [1, 7] and [8]. Closure spaces were introduced by E. Čech in [3] and then studied by many mathematicians, see e.g. [4, 5, 6] and [12]. The concept of biclosure spaces was introduced and studied in [2]. In 1966, N. Levine [11] introduced semi-open sets and semi-continuous maps in a topological space. If (X, τ) is a topological space and $A \subseteq X$, then A is semi-open if there exists $G \in \tau$ such that $G \subseteq A \subseteq \bar{G}$ where $\bar{G}$ denotes the closure of G in (X, τ). The concepts of semi-open sets and semi-continuous maps in closure spaces were introduced in [9]. In this paper, we introduce semi-open sets in biclosure spaces and investigate some of their fundamental properties.

Then we use semi-open sets to define semi-open maps, semi-continuous maps, semi-irresolute maps and pre-semi-open maps. We obtain certain properties of semi-openness, semi-continuity, semi-irresoluteness and pre-semi-openness in biclosure spaces.

2. PRELIMINARIES

In this section, we recall some basic definitions concerning closure spaces and biclosure spaces.

A map $u : P(X) \to P(X)$ defined on the power set $P(X)$ of a set X is called a *closure operator* on X and the pair (X, u) is called a *closure space* if the following axioms are satisfied:

(A1) $u\emptyset = \emptyset$,

(A2) $A \subseteq uA$ for every $A \subseteq X$,

(A3) $A \subseteq B \Rightarrow uA \subseteq uB$ for all $A, B \subseteq X$.

A closure operator u on a set X is called *idempotent* if $A \subseteq X \Rightarrow uuA = uA$. A subset $A \subseteq X$ is *closed* in the closure space (X, u) if $uA = A$ and it is *open* if its complement in X is closed. The empty set and the whole space are both open and closed.

A closure space (Y, v) is said to be a *subspace* of (X, u) if $Y \subseteq X$ and $vA = uA \cap Y$ for each subset $A \subseteq Y$.
A subset A of a closure space (X, u) is called *semi-open* if there exists an open set G in (X, u) such that $G \subseteq A \subseteq uG$. A subset $A \subseteq X$ is called *semi-closed* if its complement is semi-open.

If (X, u) and (Y, v) are closure spaces, then a map $f : (X, u) \to (Y, v)$ is called:

(i) *open* (respectively, *closed*) if the image of each open(respectively, closed) set in (X, u) is open (respectively, closed) in (Y, v).

(ii) *continuous* if $f(uA) \subseteq vf(A)$ for every subset $A \subseteq X$. One can see that, if f is continuous, then the inverse image under f of each open set in (Y, v) is open in (X, u).

A *biclosure space* is a triple (X, u_1, u_2) where X is a set and u_1, u_2 are two closure operators on X. A subset A of a biclosure space (X, u_1, u_2) is called *closed* if $u_1u_2A = A$. The complement of closed set is called *open*.

Let (X, u_1, u_2) be a biclosure space. A biclosure space (Y, v_1, v_2) is called a *subspace* of (X, u_1, u_2) if $Y \subseteq X$ and $v_i A = u_i A \cap Y$ for all $i \in \{1, 2\}$ and every subset A of Y.

Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $i \in \{1, 2\}$. Then a map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called:

(i) *i-open* (respectively, *i-closed*) if the map $f : (X, u_i) \to (Y, v_i)$ is open (respectively, closed).

(ii) *open* (respectively, *closed*) if f is i-open (respectively, i-closed) for all $i \in \{1, 2\}$.

(iii) *biopen* (respectively, *biclosed*) if the map $f : (X, u_1) \to (Y, v_2)$ is open (respectively, closed).

(iv) *i-continuous* if the map $f : (X, u_i) \to (Y, v_i)$ is continuous for all $i \in \{1, 2\}$.

(v) *continuous* if f is *i-continuous* for all $i \in \{1, 2\}$.

(vi) *bi-continuous* if the map $f : (X, u_1) \to (Y, v_2)$ is continuous.

Remark 2.1. Let A be a subset of a biclosure space (X, u_1, u_2).

(i) A is open in (X, u_1, u_2) if and only if A is open in both (X, u_1) and (X, u_2)

(ii) If A is an open set in (X, u_1, u_2), then $u_1 u_2(X - A) = u_2 u_1(X - A)$.

The converse of the statement (ii) in Remark 2.1 need not be true as can be seen from the following example.

Example 2.2. Let $X = \{1, 2, 3\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{1\} = \{1\}$, $u_1\{2\} = \{2\}$, $u_1\{3\} = \{3\}$, $u_1\{1, 3\} = \{1, 3\}$ and $u_1\{1, 2\} = u_1\{2, 3\} = u_1 X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{1\} = \{1, 3\}$, $u_2\{2\} = \{2\}$, $u_2\{3\} = \{3\}$ and $u_2\{1, 2\} = u_2\{1, 3\} = u_2\{2, 3\} = u_2 X = X$. We can see that $u_1 u_2(X - \{1\}) = u_2 u_1(X - \{1\}) = X$ but $\{1\}$ is not open in (X, u_1, u_2).

Proposition 2.3. *Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of open sets in a biclosure space (X, u_1, u_2). Then $\cup_{\alpha\in J}A_\alpha$ is an open set.*

Proof. Let A_α be open in (X, u_1, u_2) for each $\alpha \in J$, then $X - A_\alpha$ is closed for all $\alpha \in J$. Since $\cap_{\alpha\in J}(X - A_\alpha) \subseteq X - A_\alpha$ for all $\alpha \in J$, $u_1u_2 \cap_{\alpha\in J} (X - A_\alpha) \subseteq u_1u_2(X - A_\alpha)$ for each $\alpha \in J$. But $X - A_\alpha = u_1u_2(X - A_\alpha)$ for all $\alpha \in J$, hence $u_1u_2 \cap_{\alpha\in J} (X - A_\alpha) \subseteq X - A_\alpha$ for each $\alpha \in J$. Consequently, $u_1u_2\cap_{\alpha\in J}(X-A_\alpha) \subseteq \cap_{\alpha\in J}(X-A_\alpha) \subseteq u_1u_2\cap_{\alpha\in J}(X-A_\alpha)$, i.e. $u_1u_2 \cap_{\alpha\in J} (X - A_\alpha) = \cap_{\alpha\in J}(X - A_\alpha)$. Thus, $\cap_{\alpha\in J}(X - A_\alpha) = X - \cup_{\alpha\in J}A_\alpha$ is closed in (X, u_1, u_2). Therefore, $\cup_{\alpha\in J}A_\alpha$ is open. ■

The intersection of two open sets in a biclosure space (X, u_1, u_2) need not be an open set as can be seen from Example 2.2 where $\{1, 2\}$ and $\{1, 3\}$ are open in (X, u_1, u_2) but $\{1, 2\} \cap \{1, 3\}$ is not open.

Proposition 2.4. *If $\{A_\alpha\}_{\alpha\in J}$ is a collection of subsets in a biclosure space (X, u_1, u_2), then $u_1u_2 \cap_{\alpha\in J} A_\alpha \subseteq \cap_{\alpha\in J}u_1u_2A_\alpha$.*

By Example 2.2, $u_1u_2\{1, 2\} \cap u_1u_2\{1, 3\}$ is not contained in $u_1u_2(\{1, 2\} \cap \{1, 3\})$, i.e. the inclusion of Proposition 2.4 cannot be replaced by equality in general.

Proposition 2.5. *If $\{A_\alpha\}_{\alpha\in J}$ is a collection of closed subsets in a biclosure space (X, u_1, u_2), then $u_1u_2 \cap_{\alpha\in J} A_\alpha = \cap_{\alpha\in J}u_1u_2A_\alpha$.*

Proof. Let A_α be closed in (X, u_1, u_2) for all $\alpha \in J$. Then $X - A_\alpha$ is open and $A_\alpha = u_1u_2A_\alpha$ for each $\alpha \in J$. By Proposition 2.3, $\cup_{\alpha\in J}(X - A_\alpha)$ is open. But $\cup_{\alpha\in J}(X - A_\alpha) = X - \cap_{\alpha\in J}A_\alpha$, hence $\cap_{\alpha\in J}A_\alpha$ is closed in (X, u_1, u_2). Therefore, $u_1u_2 \cap_{\alpha\in J} A_\alpha = \cap_{\alpha\in J}A_\alpha = \cap_{\alpha\in J}u_1u_2A_\alpha$. ■

The converse of Proposition 2.5 is not true in general as shown in the following example.

Example 2.6. Let $X = \{1, 2, 3\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{2\} = u_1\{3\} = u_1\{2, 3\} = \{2, 3\}$ and $u_1\{1\} = u_1\{1, 2\} = u_1\{1, 3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{1\} = u_2\{2\} = u_2\{1, 2\} = \{1, 2\}$ and $u_2\{3\} = u_2\{1, 3\} = u_2\{2, 3\} = u_2X = X$. It is easy to see that $u_1u_2(\{1, 2\}\cap\{1, 3\}) = u_1u_2\{1, 2\}\cap u_1u_2\{1, 3\}$ but neither $\{1, 2\}$ nor $\{1, 3\}$ is closed in (X, u_1, u_2).

Proposition 2.7. *Let* (X, u_1, u_2) *be a biclosure space. If* G *is a subset of* X*, then* $u_1u_2G - G$ *has no nonempty open subset of* (X, u_1, u_2)*.*

Proof. Let G be a subset of X and H be a nonempty open subset of (X, u_1, u_2) such that $H \subseteq u_1u_2G-G$. Since H is nonempty, there is $x \in H \subseteq u_1u_2G - G$, i.e. $x \notin X - H$. Thus, u_1u_2G is not contained in $X - H$. Since $H \subseteq u_1u_2G-G$, $G \subseteq u_1u_2G-H \subseteq X-H$. It follows that $u_1u_2G \subseteq u_1u_2(X-H)$. But H is open in (X, u_1, u_2), $u_1u_2(X - H) = X - H$. Consequently, $u_1u_2G \subseteq X - H$, which is a contradiction. Therefore, $u_1u_2G - G$ contains no nonempty open set of (X, u_1, u_2). ■

Remark 2.8. The following statement is equivalent to Proposition 2.7:

Let (X, u_1, u_2) be a biclosure space and G be a subset of X. If H is an open subset of (X, u_1, u_2) with $H \subseteq u_1u_2G - G$, then H is an empty set.

Moreover, if the subset H is an open subset of (X, u_1) but not open in (X, u_2), then H need not be empty. And if the subset H is an open subset of (X, u_2) but not open in (X, u_1), then H need not be empty. By Example 2.6, $\{2\}$ is a subset of X such that $\{1\}$ and $\{3\}$ are nonempty subsets of $u_1u_2\{2\} - \{2\}$. We can see that $\{1\}$ is open in (X, u_1) but not open in (X, u_2), and $\{3\}$ is an open subset of (X, u_2) but not open in (X, u_1).

Proposition 2.9. *If* (Y, v_1, v_2) *is a biclosure subspace of* (X, u_1, u_2)*, then for every open subset* G *of* (X, u_1, u_2)*,* $G \cap Y$ *is an open set in* (Y, v_1, v_2)*.*

Proof. Let G be an open set in (X, u_1, u_2). By Remark 2.1 (i), G is open in both (X, u_1) and (X, u_2). Thus, $v_i(Y - (G\cap Y)) = u_i(Y - (G\cap Y))\cap Y \subseteq u_i(X-G)\cap Y = (X-G)\cap Y = Y-(G\cap Y)$ for each $i \in \{1, 2\}$. Consequently, $G \cap Y$ is open in both (Y, v_1) and (Y, v_2). Therefore, $G \cap Y$ is open in (Y, v_1, v_2). ■

Remark 2.10. By Proposition 2.9, if $E \subseteq Y$ and $E = G\cap Y$ for some open subset G of (X, u_1, u_2), then E is an open set in (Y, v_1, v_2). The converse is not true as can be seen from the following example.

Example 2.11. Let $X = \{1, 2, 3\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{1\} = \{1, 3\}$, $u_1\{2\} = u_1\{2, 3\} = \{2, 3\}$, $u_1\{3\} = \{3\}$ and $u_1\{1, 2\} = u_1\{1, 3\} = u_1X = X$. Define a closure operator u_2 on X by

$u_2\emptyset = \emptyset$, $u_2\{1\} = \{1,2\}$, $u_2\{2\} = \{2,3\}$, $u_2\{3\} = \{3\}$ and $u_2\{1,2\} = u_2\{1,3\} = u_2\{2,3\} = u_2X = X$. Thus, there are only three open subset of (X, u_1, u_2), namely $\emptyset$, $\{1,2\}$ and X. Let $Y = \{1,2\}$ and (Y, v_1, v_2) be a biclosure subspace of (X, u_1, u_2). Then $v_1\emptyset = \emptyset$, $v_1\{1\} = \{1\}$, $v_1\{2\} = \{2\}$, $v_1Y = Y$, $v_2\emptyset = \emptyset$, $v_2\{2\} = \{2\}$ and $v_2\{1\} = v_2Y = Y$. We can see that $\{1\}$ is an open subset of (Y, v_1, v_2) but there is no any open subset G of (X, u_1, u_2) such that $\{1\} = G \cap Y$.

Proposition 2.12. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces, let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps.*

(i) *If f is 1-open and g is biopen, then $g \circ f$ is biopen.*

(ii) *If f is biopen and g is 2-open, then $g \circ f$ is biopen.*

Proof.

(i) Let G be an open set in (X, u_1). Since f is 1-open, $f(G)$ is open in (Y, v_1). As g is biopen, $g(f(G)) = g \circ f(G)$ is open in (Z, w_2). Thus, $g \circ f$ is biopen.

(ii) Let G be an open set in (X, u_1). Since f is biopen, $f(G)$ is open in (Y, v_2). And since g is 2-open, $g(f(G)) = g \circ f(G)$ is open in (Z, w_2). Thus, $g \circ f$ is biopen.

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The composition of two biopen maps need not be a biopen map as can be seen from the following example.

Example 2.13. Let $X = Y = Z = \{1,2\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{2\} = \{2\}$, and $u_1\{1\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$ and $u_2\{1\} = u_2\{2\} = u_2X = X$. Define a closure operator v_1 on Y by $v_1\emptyset = \emptyset$, $v_1\{1\} = \{1\}$ and $v_1\{2\} = v_1Y = Y$ and define a closure operator v_2 on Y by $v_2\emptyset = \emptyset$, $v_2\{1\} = \{1\}$, $v_2\{2\} = \{2\}$ and $v_2Y = Y$. Define a closure operator w_1 on Z by $w_1\emptyset = \emptyset$ and $w_1\{1\} = w_1\{2\} = w_1Z = Z$ and define a closure operator w_2 on Z by $w_2\emptyset = \emptyset$, $w_2\{1\} = \{1\}$ and $w_2\{2\} = w_2Z = Z$. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be identity maps. We can see that f and g are biopen. But $g \circ f$ is not biopen because $\{1\}$ is open in (X, u_1) but $g \circ f(\{1\})$ is not open in (Z, w_2).

Proposition 2.14. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps.*

(i) *If $g \circ f$ is biopen and f is a 1-continuous surjection, then g is biopen.*

(ii) *If $g \circ f$ is biopen and g is a 2-continuous injection, then f is biopen.*

Proof.

(i) Let H be an open set in (Y, v_1). Since f is 1-continuous, $f^{-1}(H)$ is open in (X, u_1). But $g \circ f$ is biopen, hence $g \circ f(f^{-1}(H))$ is open in (Z, w_2). As f is a surjection, $g \circ f(f^{-1}(H)) = g(H)$. Therefore, g is biopen.

(ii) Let G be an open set in (X, u_1). Since $g \circ f$ is biopen, $g \circ f(G)$ is open in (Z, w_2). But g is 2-continuous, hence $g^{-1}(g \circ f(G))$ is open in (Y, v_2). As g is an injection, $g^{-1}(g \circ f(G)) = f(G)$. Therefore, f is biopen. ■

Proposition 2.15. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. If f is open, then $f(G)$ is open in (Y, v_1, v_2) for every open subset G of (X, u_1, u_2).*

Proof. Let G be an open subset of (X, u_1, u_2). By Remark 2.1 (i), G is open in both (X, u_1) and (X, u_2). Since f is open, f is both 1-open and 2-open. Hence, $f(G)$ is open in both (Y, v_1) and (Y, v_2). Consequently, $f(G)$ is open in (Y, v_1, v_2) by Remark 2.1 (i). ■

The converse of Proposition 2.15 is not true in general as can be seen from the following example.

Example 2.16. Let $X = \{1, 2\} = Y$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{2\} = \{2\}$ and $u_1\{1\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$ and $u_2\{1\} = u_2\{2\} = u_2X = X$. Define a closure operator v_1 on Y by $v_1\emptyset = \emptyset$ and $v_1\{1\} = v_1\{2\} = v_1Y = Y$ and define a closure operator v_2 on Y by $v_2\emptyset = \emptyset$, $v_2\{1\} = \{1\}$ and $v_2\{2\} = v_2Y = Y$. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be an identity map. It is easy to see that $f(G)$ is open in (Y, v_1, v_2) for every open subset G of (X, u_1, u_2). But f is not 1-open because $f(\{1\})$ is not open in (Y, v_1) while $\{1\}$ is open in (X, u_1). Consequently, f is not open.

Proposition 2.17. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. If f is continuous, then $f^{-1}(H)$ is open in (X, u_1, u_2) for every open subset H of (Y, v_1, v_2).*

Proof. Let H be an open subset of (Y, v_1, v_2). By Remark 2.1 (i), H is open in both (Y, v_1) and (Y, v_2). Since f is continuous, f is both 1-continuous and 2-continuous. It follows that $f^{-1}(H)$ is open in both (X, u_1) and (X, u_2). Therefore, $f^{-1}(H)$ is open in (X, u_1, u_2) by Remark 2.1 (i). ■

The converse of Proposition 2.17 need not be true in general as can be seen from the following example.

Example 2.18. In Example 2.16, $f^{-1}(H)$ is open in (X, u_1, u_2) for every open subset H of (Y, v_1, v_2). But the map f is not 2-continuous because $f^{-1}\{2\}$ is not open in (X, u_2) while $\{2\}$ is open in (Y, v_2). Consequently, f is not continuous.

Definition 2.19. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$, where (X, u_1, u_2) and (Y, v_1, v_2) are biclosure spaces, is called a *homeomorphism* if f is bijective, continuous and open.

Proposition 2.20. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. If f is a bijective continuous map, then the following statements are equivalent:*

(i) *f is a homeomorphism,*

(ii) *f is a closed map,*

(iii) *f is an open map.*

Proof. (i) $\to$ (ii) Since f is a homeomorphism, i.e. f is open and bijective. It follows that f is both 1-open and 2-open. Let $i \in \{1, 2\}$ and let F_i be a closed subset of (X, u_i). Then $f(X - F_i) = Y - f(F_i)$ is open in (Y, v_i). Hence, $f(F_i)$ is closed in (Y, v_i). Thus, f is both 1-closed and 2-closed. Therefore, f is closed.

(ii)$\to$(iii) Let $i \in \{1, 2\}$ and let G_i be an open subset of (X, u_i). Then $X - G_i$ is closed in (X, u_i). By the assumption, f is both closed and bijective. It follows that f is both 1-closed and 2-closed. Consequently, $f(X - G_i) = Y - f(G_i)$ is closed in (Y, v_i). Hence, $f(G_i)$ is open in (Y, v_i). Thus, f is both 1-open and 2-open. Therefore, f is open.

(iii)$\to$(i) By the assumption, f is a homeomorphism. ■

3. Semi-open sets in biclosure spaces

In this section, we introduce a new type of open sets in biclosure spaces and study some of their properties.

Definition 3.1. A subset A of a biclosure space (X, u_1, u_2) is called *semi-open*, if there exists an open subset G of (X, u_1) such that $G \subseteq A \subseteq u_2G$. The complement of a semi-open set in X is called *semi-closed*.

Clearly, if (X, u_1, u_2) is a biclosure space and A is open (respectively, closed) in (X, u_1), then A is semi-open (respectively, semi-closed) in (X, u_1, u_2). The converse is not true as can be seen from the following example.

Example 3.2. Let $X = \{1, 2, 3\}$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{1\} = u_1\{3\} = u_1\{1, 3\} = \{1, 3\}$, $u_1\{2\} = \{2, 3\}$ and $u_1\{1, 2\} = u_1\{2, 3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{3\} = \{3\}$ and $u_2\{1\} = u_2\{2\} = u_2\{1, 2\} = u_2\{1, 3\}, u_2\{2, 3\} = u_2X = X$. It follows that $\{2, 3\}$ is semi-open in (X, u_1, u_2) but $\{2, 3\}$ is open in neither (X, u_1) nor (X, u_2). Moreover, $\{1\}$ is semi-closed in (X, u_1, u_2) but $\{1\}$ is closed in neither (X, u_1) nor (X, u_2).

Proposition 3.3. *Let (X, u_1, u_2) be a biclosure space and let $A \subseteq X$. Then A is semi-closed if and only if there exists a closed subset F of (X, u_1) such that $X - u_2(X - F) \subseteq A \subseteq F$.*

Proof. Let A be semi-closed in (X, u_1, u_2). Then there exists an open set G in (X, u_1) such that $G \subseteq X - A \subseteq u_2G$. Thus, there exists a closed subset F of (X, u_1) such that $G = X - F$ and $X - F \subseteq X - A \subseteq u_2(X - F)$. Therefore, $X - u_2(X - F) \subseteq A \subseteq F$.

Conversely, by the assumption, there is a closed subset F of (X, u_1) such that $X - u_2(X - F) \subseteq A \subseteq F$. Thus, there exists an open set G in (X, u_1) such that $F = X - G$ and $X - u_2G \subseteq A \subseteq X - G$. It follows that $G \subseteq X - A \subseteq u_2G$. Therefore, A is semi-closed in (X, u_1, u_2). ■

Proposition 3.4. *Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of semi-open sets in a biclosure space (X, u_1, u_2). Then $\cup_{\alpha \in J} A_\alpha$ is a semi-open set in (X, u_1, u_2).*

Proof. Let A_α be semi-open in (X, u_1, u_2) for all $\alpha \in J$. Hence, for each $\alpha \in J$, we have an open set G_α in (X, u_1) such that $G_\alpha \subseteq A_\alpha \subseteq u_2G_\alpha$.

Thus, $\cup_{\alpha\in J}G_\alpha \subseteq \cup_{\alpha\in J}A_\alpha \subseteq \cup_{\alpha\in J}u_2G_\alpha$. Since $G_\alpha \subseteq \cup_{\alpha\in J}G_\alpha$ for each $\alpha \in J$, $u_2G_\alpha \subseteq u_2 \cup_{\alpha\in J} G_\alpha$ for all $\alpha \in J$. Thus, $\cup_{\alpha\in J}u_2G_\alpha \subseteq u_2 \cup_{\alpha\in J} G_\alpha$. Consequently, $\cup_{\alpha\in J}G_\alpha \subseteq \cup_{\alpha\in J}A_\alpha \subseteq u_2 \cup_{\alpha\in J} G_\alpha$. As G_α is open in (X, u_1) for all $\alpha \in J$, $u_1 \cap_{\alpha\in J} (X - G_\alpha) \subseteq u_1(X - G_\alpha) = X - G_\alpha$ for each $\alpha \in J$. Thus, $u_1 \cap_{\alpha\in J} (X - G_\alpha) \subseteq \cap_{\alpha\in J}(X - G_\alpha)$. It follows that $\cap_{\alpha\in J}(X - G_\alpha)$ is closed in (X, u_1), i.e. $\cup_{\alpha\in J}G_\alpha$ is open in (X, u_1). Therefore, $\cup_{\alpha\in J}A_\alpha$ is semi-open in (X, u_1, u_2). ■

If $\{A_\alpha\}_{\alpha\in J}$ is a collection of semi-open sets in a biclosure space (X, u_1, u_2), then $\cap_{\alpha\in J}A_\alpha$ need not be a semi-open set in (X, u_1, u_2) as shown in the following example.

Example 3.5. In the biclosure space (X, u_1, u_2) from Example 2.2, we can see that $\{1, 2\}$ and $\{1, 3\}$ are semi-open but $\{1, 2\} \cap \{1, 3\}$ is not semi-open.

Proposition 3.6. *Let $\{A_\alpha\}_{\alpha\in J}$ be a collection of semi-closed sets in a biclosure space (X, u_1, u_2). Then $\cap_{\alpha\in J}A_\alpha$ is semi-closed.*

Proof. Clearly, the complement of $\cap_{\alpha\in J}A_\alpha$ is $\cup_{\alpha\in J}(X - A_\alpha)$. Since A_α is semi-closed in (X, u_1, u_2) for each $\alpha \in J$, $X - A_\alpha$ is semi-open for all $\alpha \in J$. But $\cup_{\alpha\in J}(X - A_\alpha)$ is a semi-open set by Proposition 3.4. Therefore, $\cap_{\alpha\in J}A_\alpha$ is semi-closed in (X, u_1, u_2). ■

If $\{A_\alpha\}_{\alpha\in J}$ is a collection of semi-closed sets in a biclosure space (X, u_1, u_2), then $\cup_{\alpha\in J}A_\alpha$ need not be a semi-closed set as shown in the following example.

Example 3.7. In the biclosure space (X, u_1, u_2) from Example 2.2, we can see that $\{2\}$ and $\{3\}$ are semi-closed but $\{2\} \cup \{3\}$ is not semi-closed.

Proposition 3.8. *Let (X, u_1, u_2) be a biclosure space and u_2 be idempotent. If A is semi-open in (X, u_1, u_2) and $A \subseteq B \subseteq u_2A$, then B is semi-open.*

Proof. Let A be semi-open in (X, u_1, u_2). Then there exists an open set G in (X, u_1) such that $G \subseteq A \subseteq u_2G$, hence $u_2A \subseteq u_2u_2G$. Since u_2 is idempotent, $u_2A \subseteq u_2G$. Thus, $G \subseteq A \subseteq B \subseteq u_2A \subseteq u_2G$. Therefore, B is semi-open. ■

Proposition 3.9. *Let (Y, v_1, v_2) be a biclosure subspace of (X, u_1, u_2) and $A \subseteq Y$. If A is semi-open in (X, u_1, u_2), then A is semi-open in (Y, v_1, v_2).*

Proof. Let A be semi-open in (X, u_1, u_2). Then there exists an open set G in (X, u_1) such that $G \subseteq A \subseteq u_2 G$. It follows that $A \cap Y \subseteq u_2 G \cap Y$. But $A = A \cap Y$, hence $G \subseteq A = A \cap Y \subseteq u_2 G \cap Y = v_2 G$. Since G is open in (X, u_1), $v_1(Y - G) = u_1(Y - G) \cap Y \subseteq u_1(X - G) \cap Y = (X - G) \cap Y = Y - G$. Thus, $Y - G$ is closed in (Y, v_1), i.e. G is open in (Y, v_1). Therefore, A is semi-open in (Y, v_1, v_2). ■

The converse of Proposition 3.9 need not be true as can be seen from the following example.

Example 3.10. In the biclosure spaces (X, u_1, u_2) and (Y, v_1, v_2) from Example 2.11, we can see that $\{2\} \subseteq Y$ and $\{2\}$ is semi-open in (Y, v_1, v_2) but $\{2\}$ is not semi-open in (X, u_1, u_2).

Definition 3.11. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *semi-open* (respectively, *semi-closed*) if $f(A)$ is semi-open (respectively, semi-closed) in (Y, v_1, v_2) for every open (respectively, closed) subset A of (X, u_1, u_2).

Clearly, if f is open (respectively, closed), then f is semi-open (respectively, semi-closed). The converse need not be true in general as can be seen from the following example.

Example 3.12. Let $X = \{1, 2\} = Y$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{1\} = \{1\}$ and $u_1\{2\} = u_1 X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{1\} = \{1\}$, $u_2\{2\} = \{2\}$ and $u_2 X = X$. Define a closure operator v_1 on Y by $v_1\emptyset = \emptyset$, $v_1\{1\} = \{1\}$ and $v_1\{2\} = v_1 Y = Y$ and define a closure operator v_2 on Y by $v_2\emptyset = \emptyset$ and $v_2\{1\} = v_2\{2\} = v_2 Y = Y$. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be an identity map. It is easy to see that f is semi-open but not open because $f(\{2\})$ is not open in (Y, v_1, v_2) while $\{2\}$ is open in (X, u_1, u_2). Moreover, we can see that f is semi-closed but not closed because $f(\{1\})$ is not closed in (Y, v_1, v_2) while $\{1\}$ is closed in (X, u_1, u_2).

Proposition 3.13. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. Then $g \circ f$ is semi-open if f is open and g is semi-open.*

Proof. Let G be an open subset of (X, u_1, u_2) and let f be open. By Proposition 2.15, $f(G)$ is open in (Y, v_1, v_2). As g is semi-open, $g(f(G)) = g \circ f(G)$ is semi-open in (Z, w_1, w_2). Therefore, $g \circ f$ is semi-open. ■

Proposition 3.14. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. If $g \circ f$ is semi-open and f is a continuous surjection, then g is semi-open.*

Proof. Let H be an open set in (Y, v_1, v_2) and let f be continuous. By Proposition 2.17, $f^{-1}(H)$ is open in (X, u_1, u_2). Since $g \circ f$ is semi-open, $g \circ f(f^{-1}(H))$ is semi-open in (Z, w_1, w_2). But f is a surjection, hence $g \circ f(f^{-1}(H)) = g(H)$. Thus, $g(H)$ is semi-open in (Z, w_1, w_2). Therefore, g is semi-open. ∎

4. Semi-continuous maps in biclosure spaces

In this section, we study the concept of semi-continuous maps obtained by using semi-open sets.

Definition 4.1. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *semi-continuous* if $f^{-1}(G)$ is a semi-open subset of (X, u_1, u_2) for every open subset G of (Y, v_1, v_2).

Clearly, if f is continuous, then f is semi-continuous. The converse need not be true as can be seen from the following example.

Example 4.2. Let $X = \{1, 2\} = Y$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{1\} = \{1\}$, $u_1\{2\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$ and $u_2\{1\} = u_2\{2\} = u_2X = X$. Define a closure operator v_1 on Y by $v_1\emptyset = \emptyset$, $v_1\{1\} = \{1\}$, $v_1\{2\} = \{2\}$, $v_1Y = Y$ and define a closure operator v_2 on Y by $v_2\emptyset = \emptyset$, $v_2\{1\} = \{1\}$ and $v_2\{2\} = v_2Y = Y$. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be an identity map. It is easy to see that f is semi-continuous but not continuous because $f^{-1}(\{2\})$ is not open in (X, u_1, u_2) while $\{2\}$ is open in (Y, v_1, v_2).

Proposition 4.3. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is semi-continuous if and only if $f^{-1}(F)$ is a semi-closed subset of (X, u_1, u_2) for every closed subset F of (Y, v_1, v_2).*

Proposition 4.4. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. If g is continuous and f is semi-continuous, then $g \circ f$ is semi-continuous.*

Proof. Let H be an open subset of (Z, w_1, w_2) and let g be continuous. By Proposition 2.17, $g^{-1}(H)$ is open in (Y, v_1, v_2). As f is semi-continuous, $f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H)$ is semi-open in (X, u_1, u_2). Therefore, $g \circ f$ is semi-continuous. ■

Definition 4.5. A biclosure space (X, u_1, u_2) is said to be a *T_s-space* if every semi-open set in (X, u_1, u_2) is open in (X, u_1, u_2). Clearly, the closure space (X, u_1, u_2) in Example 3.12 is a T_s-space.

Proposition 4.6. *Let (X, u_1, u_2) and (Z, w_1, w_2) be biclosure spaces and (Y, v_1, v_2) be a T_s-space and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. If f and g are semi-continuous, then $g \circ f$ is semi-continuous.*

Proof. Let H be open in (Z, w_1, w_2). Since g is semi-continuous, $g^{-1}(H)$ is semi-open in (Y, v_1, v_2). But (Y, v_1, v_2) is a T_s-space, hence $g^{-1}(H)$ is open in (Y, v_1, v_2). As f is semi-continuous, $f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H)$ is semi-open in (X, u_1, u_2). Therefore, $g \circ f$ is semi-continuous. ■

Proposition 4.7. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces, and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps.*

(i) *If f is a semi-open surjection and $g \circ f$ is continuous, then g is semi-continuous.*

(ii) *If g is a semi-continuous injection and $g \circ f$ is open, then f is semi-open.*

(iii) *If g is an open injection and $g \circ f$ is semi-continuous, then f is semi-continuous.*

Proof.

(i) Let H be an open subset of (Z, w_1, w_2) and let $g \circ f$ be continuous. By Proposition 2.17, $(g \circ f)^{-1}(H)$ is open in (X, u_1, u_2). Since f is a semi-open map, $f((g \circ f)^{-1}(H)) = f(f^{-1}(g^{-1}(H)))$ is semi-open in (Y, v_1, v_2). But f is a surjection, thus $f(f^{-1}(g^{-1}(H))) = g^{-1}(H)$. Therefore, g is semi-continuous.

(ii) Let G be an open subset of (X, u_1, u_2) and let $g \circ f$ be open. By Proposition 2.15, $g \circ f(G)$ is open in (Z, w_1, w_2). Since g is semi-continuous,

$g^{-1}(g \circ f(G))$ is semi-open in (Y, v_1, v_2). But g is an injection, hence $g^{-1}(g \circ f(G)) = f(G)$. Therefore, f is semi-open.

(iii) Let H be an open subset of (Y, v_1, v_2) and let g is open. By Proposition 2.15, $g(H)$ is open in (Z, w_1, w_2). Since $g \circ f$ is semi-continuous, $(g \circ f)^{-1}(g(H))$ is semi-open in (X, u_1, u_2). But g is an injection, it follows that $(g \circ f)^{-1}(g(H)) = f^{-1}(g^{-1}(g(H))) = f^{-1}(H)$. Therefore, f is semi-continuous. ■

5. Semi-irresolute maps in biclosure spaces

In this section, we introduce semi-irresolute maps in biclosure spaces obtained by using semi-open sets. We then study some of their basic properties.

Definition 5.1. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *semi-irresolute* if $f^{-1}(G)$ is semi-open in (X, u_1, u_2) for every semi-open set G in (Y, v_1, v_2).

It is easy to show that the composition of two semi-irresolute maps of biclosure spaces is again a semi-irresolute map.

Remark 5.2. If a map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is semi-irresolute, then f is semi-continuous. The converse need not be true as shown in the following example.

Example 5.3. Let $X = \{1, 2\} = Y$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$ and $u_1\{1\} = u_1\{2\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$ and $u_2\{1\} = u_2\{2\} = u_2X = X$. Define a closure operator v_1 on Y by $v_1\emptyset = \emptyset$, $v_1\{1\} = \{1\}$, $v_1\{2\} = v_1Y = Y$ and define a closure operator v_2 on Y by $v_2\emptyset = \emptyset$ and $v_2\{1\} = v_2\{2\} = v_2Y = Y$. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be an identity map. It is easy to see that there are only two open sets in (Y, v_1, v_2), namely $\emptyset$ and Y, and their inverse images are semi-open in (X, u_1, u_2). Thus, f is semi-continuous. But f is not semi-irresolute because $f^{-1}(\{2\})$ is not semi-open in (X, u_1, u_2) while $\{2\}$ is semi-open in (Y, v_1, v_2).

Proposition 5.4. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. Then f is semi-irresolute if and only if $f^{-1}(B)$ is semi-closed in (X, u_1, u_2), whenever B is semi-closed in (Y, v_1, v_2).*

Proposition 5.5. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be an open, semi-irresolute and surjective map. Then (Y, v_1, v_2) is a T_s-space if (X, u_1, u_2) is a T_s-space.*

Proof. Let (X, u_1, u_2) be a T_s-space and let B be a semi-open subset of (Y, v_1, v_2). Since f is semi-irresolute, $f^{-1}(B)$ is semi-open in (X, u_1, u_2). As (X, u_1, u_2) is a T_s-space, $f^{-1}(B)$ is open in (X, u_1, u_2). Since f is open, $f(f^{-1}(B))$ is open in (Y, v_1, v_2) by Proposition 2.15. But f is a surjection, hence $f(f^{-1}(B)) = B$. Thus, B is open in (Y, v_1, v_2). Therefore, (Y, v_1, v_2) is a T_s-space. ∎

Proposition 5.6. *Let (X, u_1, u_2), (Y, v_1, v_2) and (Z, w_1, w_2) be biclosure spaces, and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ and $g : (Y, v_1, v_2) \to (Z, w_1, w_2)$ be maps. If f is semi-irresolute and g is semi-continuous, then $g \circ f$ is semi-continuous.*

Proposition 5.7. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a bijective map.*

(i) *If f is 1-continuous and f^{-1} is 2-continuous, then f is semi-irresolute.*

(ii) *If f is 2-continuous and f^{-1} is 1-continuous, then f^{-1} is semi-irresolute.*

Proof.

(i) Let B be semi-open in (Y, v_1, v_2). Then there exists an open subset H of (Y, v_1) such that $H \subseteq B \subseteq v_2 H$. Since f^{-1} is 2-continuous, $f^{-1} : (Y, v_2) \to (X, u_2)$ is continuous. Thus, $f^{-1}(v_2(H)) \subseteq u_2 f^{-1}(H)$, i.e. $f^{-1}(H) \subseteq f^{-1}(B) \subseteq u_2 f^{-1}(H)$. As f is 1-continuous, $f : (X, u_1) \to (Y, v_1)$ is continuous, hence $f^{-1}(H)$ is open in (X, u_1). Consequently, $f^{-1}(B)$ is semi-open in (X, u_1, u_2). Therefore, f is semi-irresolute.

(ii) Let A be semi-open in (X, u_1, u_2). Then there exists an open set G of (X, u_1) such that $G \subseteq A \subseteq u_2 G$. Since f is 2-continuous, $f : (X, u_2) \to (Y, v_2)$ is continuous. Thus, $f(u_2 G) \subseteq v_2 f(G)$, i.e. $f(G) \subseteq f(A) \subseteq v_2 f(G)$. But f^{-1} is 1-continuous, hence $f^{-1} : (Y, v_1) \to (X, u_1)$ is continuous. Since $f(G)$ is the inverse image of G under f^{-1}, $f(G)$ is open in (Y, v_1) . Consequently, $f(A)$ is semi-open in (Y, v_1, v_2). But $f(A)$ is the inverse image of A under f^{-1}, thus f^{-1} is semi-irresolute. ∎

6. Pre-semi-open maps in biclosure spaces

In this section, we introduce pre-semi-open maps obtained by using semi-open sets. We study some of their properties.

Definition 6.1. Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is called *pre-semi-open* (respectively, *pre-semi-closed*) if $f(A)$ is a semi-open (respectively, semi-closed) subset of (Y, v_1, v_2) for every semi-open (respectively, semi-closed) subset A of (X, u_1, u_2).

It is easy to show that the composition of two pre-semi-open maps in biclosure spaces is again a pre-semi-open map.

Clearly, if a map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is pre-semi-open, then f is semi-open. The converse need not be true as shown in the following example.

Example 6.2. In Example 2.16, the map f is semi-open but f is not pre-semi-open because $\{1\}$ is semi-open in (X, u_1, u_2) but $f(\{1\})$ is not semi-open in (Y, v_1, v_2).

Proposition 6.3. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. Then the following statements are equivalent:*

(i) *f is pre-semi-open*

(ii) *If $B \subseteq Y$ and C is a semi-closed subset of (X, u_1, u_2) such that $f^{-1}(B) \subseteq C$, then $B \subseteq E$ and $f^{-1}(E) \subseteq C$ for some semi-closed subset E of (Y, v_1, v_2).*

Proof. (i)$\to$ (ii) Let B be a subset of Y and let C be a semi-closed subset of (X, u_1, u_2) such that $f^{-1}(B) \subseteq C$. Then $f(X - C)$ is a semi-open subset of (Y, v_1, v_2). Put $E = Y - f(X - C)$. Then E is semi-closed in (Y, v_1, v_2) and $X - C \subseteq X - f^{-1}(B) = f^{-1}(Y - B)$. Hence, $f(X - C) \subseteq f(f^{-1}(Y - B)) \subseteq Y - B$. Thus, $Y - (Y - B) \subseteq Y - f(X - C)$, i.e. $B \subseteq E$ and $f^{-1}(E) = f^{-1}(Y - f(X - C)) = X - f^{-1}(f(X - C)) \subseteq X - (X - C) = C$. Therefore, E is a semi-closed subset of (Y, v_1, v_2) such that $B \subseteq E$ and $f^{-1}(E) \subseteq C$.

(ii)$\to$ (i) Let A be a semi-open subset of (X, u_1, u_2). Then $X - A$ is semi-closed in (X, u_1, u_2) and $f^{-1}(Y - f(A)) = X - f^{-1}(f(A)) \subseteq X - A$ where $Y - f(A)$ is a subset of Y. By the assumption, there is a semi-closed subset E of (Y, v_1, v_2) such that $Y - f(A) \subseteq E$ and $f^{-1}(E) \subseteq X - A$. Hence, $Y - E \subseteq f(A)$ and $A \subseteq X - f^{-1}(E)$. It follows that $Y - E \subseteq f(A) \subseteq f(X - f^{-1}(E)) = f(f^{-1}(Y - E)) \subseteq Y - E$, i.e. $f(A) = Y - E$. Thus, $f(A)$ is semi-open in (Y, v_1, v_2). Therefore, f is pre-semi-open. ■

Proposition 6.4. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. If f is pre-semi-open, then for every $y \in Y$ and every semi-closed subset C of (X, u_1, u_2) such that $f^{-1}(\{y\}) \subseteq C$, there exists a semi-closed subset E of (Y, v_1, v_2) such that $y \in E$ and $f^{-1}(E) \subseteq C$.*

Proof. Let $y \in Y$ and let C be a semi-closed subset of (X, u_1, u_2) such that $f^{-1}(\{y\}) \subseteq C$. Since $\{y\} \subseteq Y$, there exists a semi-closed subset E of (Y, v_1, v_2) such that $y \in E$ and $f^{-1}(E) \subseteq C$ by Proposition 6.3. ■

The converse of the previous statement is not true in general as can be seen from the following example.

Example 6.5. Let $X = \{1, 2, 3\} = Y$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{1\} = u_1\{2\} = u_1\{1, 2\} = \{1, 2\}$ and $u_1\{3\} = u_1\{1, 3\} = u_1\{2, 3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{3\} = \{3\}$ and $u_2\{1\} = u_2\{2\} = u_2\{1, 2\} = u_2\{1, 3\} = u_2\{2, 3\} = u_2X = X$. Define a closure operator v_1 on Y by $v_1\emptyset = \emptyset$, $v_1\{1\} = \{1\}$, $v_1\{2\} = \{2\}$, $v_1\{3\} = v_1\{1, 2\} = v_1\{1, 3\} = v_1\{2, 3\} = v_1Y = Y$ and define a closure operator v_2 on Y by $v_2\emptyset = \emptyset$ and $v_2\{1\} = v_2\{2\} = v_2\{3\} = v_2\{1, 2\} = v_2\{1, 3\} = v_2\{2, 3\} = v_2Y = Y$. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be an identity map. Then there are only three semi-closed subsets of (X, u_1, u_2), namely $\emptyset$, $\{1, 2\}$ and X. Moreover, we can see that there are only four semi-closed subsets of (Y, v_1, v_2), namely $\emptyset$, $\{1\}$, $\{2\}$ and Y. Then for every $y \in Y$ and every semi-closed subset C of (X, u_1, u_2) such that $f^{-1}(\{y\}) \subseteq C$, there exists a semi-closed subset E of (Y, v_1, v_2) such that $y \in E$ and $f^{-1}(E) \subseteq C$. But f is not pre-semi-open because $\{3\}$ is semi-open in (X, u_1, u_2) but $f(\{3\})$ is not semi-open in (Y, v_1, v_2).

Proposition 6.6. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a map. Then the following statements are equivalent:*

(i) *f is pre-semi-closed.*

(ii) *If* $D \subseteq Y$ *and* A *is a semi-open subset of* (X, u_1, u_2) *such that* $f^{-1}(D) \subseteq A$*, then* $D \subseteq M$ *and* $f^{-1}(M) \subseteq A$ *for some semi-open subset* M *of* (Y, v_1, v_2)*.*

(iii) *If* $y \in Y$ *and* A *is a semi-open subset of* (X, u_1, u_2) *such that* $f^{-1}(\{y\}) \subseteq A$*, then* $y \in M$ *and* $f^{-1}(M) \subseteq A$ *for some semi-open subset* M *of* (Y, v_1, v_2)*.*

Proof. (i) $\rightarrow$ (ii) The proof is a minor modification of the proof (i)$\rightarrow$(ii) in Proposition 6.3.

(ii)$\rightarrow$ (iii) Let $y \in Y$ and A be a semi-open subset of (X, u_1, u_2) such that $f^{-1}(\{y\}) \subseteq A$. By (ii), put $D = \{y\}$. Then there exists a semi-open subset M of (Y, v_1, v_2) such that $y \in M$ and $f^{-1}(M) \subseteq A$.

(iii)$\rightarrow$(i) Let C be a semi-closed subset of (X, u_1, u_2). Then $X - C$ is semi-open in (X, u_1, u_2) and $f^{-1}(Y - f(C)) = X - f^{-1}(f(C)) \subseteq X - C$. Let $y \in Y - f(C) \subseteq Y$ and put $A = X - C$. Then $f^{-1}(\{y\}) \subseteq X - C = A$. By (iii), there exists a semi-open subset M_y of (Y, v_1, v_2) such that $y \in M_y$ and $f^{-1}(M_y) \subseteq A = X - C$, i.e. $C \subseteq X - f^{-1}(M_y)$. Hence, $f(C) \subseteq f(X - f^{-1}(M_y)) = f(f^{-1}(Y - M_y)) \subseteq Y - M_y$. Thus, $y \in M_y \subseteq Y - f(C)$ for all $y \in Y - f(C)$. It follows that $Y - f(C) = \cup_{y \in Y - f(C)} M_y$. By Proposition 3.4, $\cup_{y \in Y - f(C)} M_y$ is semi-open in (Y, v_1, v_2). Consequently, $f(C)$ is semi-closed in (Y, v_1, v_2). Therefore, f is pre-semi-closed. ■

Proposition 6.7. *Let* (X, u_1, u_2)*,* (Y, v_1, v_2) *and* (Z, w_1, w_2) *be biclosure spaces and let* $f : (X, u_1, u_2) \rightarrow (Y, v_1, v_2)$ *and* $g : (Y, v_1, v_2) \rightarrow (Z, w_1, w_2)$ *be maps.*

(i) *If* f *is a semi-irresolute surjection and* $g \circ f$ *is pre-semi-open, then* g *is pre-semi-open.*

(ii) *If* g *is a semi-irresolute injection and* $g \circ f$ *is pre-semi-open, then* f *is pre-semi-open.*

(iii) *If* f *is a pre-semi-open surjection and* $g \circ f$ *is semi-irresolute, then* g *is semi-irresolute.*

(iv) *If* g *is a pre-semi-open injection and* $g \circ f$ *is semi-irresolute, then* f *is semi-irresolute.*

Proof. (i) Let B be semi-open in (Y, v_1, v_2). Since f is semi-irresolute, $f^{-1}(B)$ is semi-open in (X, u_1, u_2). But $g \circ f$ is pre-semi-open and f is surjective, hence $g \circ f(f^{-1}(B)) = g(B)$ is semi-open in (Z, w_1, w_2). Therefore, g is pre-semi-open.

The proofs of (ii)-(iv) are minor modifications of that of (i) ∎

Proposition 6.8. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a continuous, pre-semi-open and injective map. Then (X, u_1, u_2) is a T_s-space if (Y, v_1, v_2) is a T_s-space.*

Proof. Let (Y, v_1, v_2) be a T_s-space and let A be a semi-open subset of (X, u_1, u_2). Since f is pre-semi-open, $f(A)$ is semi-open in (Y, v_1, v_2). But (Y, v_1, v_2) is a T_s-space, hence $f(A)$ is open in (Y, v_1, v_2). As f is continuous, $f^{-1}(f(A))$ is open in (X, u_1, u_2) by Proposition 2.17. Since f is injective, $f^{-1}(f(A)) = A$. Thus, A is open in (X, u_1, u_2) Therefore, (X, u_1, u_2) is a T_s-space. ∎

Proposition 6.9. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ is a 1-open and 2-continuous map, then f is pre-semi-open.*

Proof. Let A be semi-open in (X, u_1, u_2). Then there exists an open subset G of (X, u_1) such that $G \subseteq A \subseteq u_2 G$. Consequently, $f(G) \subseteq f(A) \subseteq f(u_2 G)$. Since f is 2-continuous, $f : (X, u_2) \to (Y, v_2)$ is continuous. Hence, $f(u_2 G)) \subseteq v_2 f(G)$, i.e. $f(G) \subseteq f(A) \subseteq v_2 f(G)$. But f is 1-open, thus $f : (X, u_1) \to (Y, v_1)$ is open. It follows that $f(G)$ is open in (Y, v_1). Thus, $f(A)$ is a semi-open set in (Y, v_1, v_2). Therefore, f is pre-semi-open. ∎

Definition 6.10. A map $f : (X, u_1, u_2) \to (Y, v_1, v_2)$, where (X, u_1, u_2) and (Y, v_1, v_2) are biclosure spaces, is called a *semi-homeomorphism* if f is bijective, semi-irresolute and pre-semi-open.

It is easy to show that the composition of two semi-homeomorphisms of biclosure spaces is again a semi-homeomorphism.

Remark 6.11. The concepts of a homeomorphism and a semi-homeomorphism are independent as can be seen from two following examples.

Example 6.12. In Example 3.12, the map f is a semi-homeomorphism but f is not open. Consequently, f is not a homeomorphism.

Example 6.13. Let $X = \{1,2,3\} = Y$ and define a closure operator u_1 on X by $u_1\emptyset = \emptyset$, $u_1\{2\} = u_1\{3\} = u_1\{2,3\} = \{2,3\}$ and $u_1\{1\} = u_1\{1,2\} = u_1\{1,3\} = u_1X = X$. Define a closure operator u_2 on X by $u_2\emptyset = \emptyset$, $u_2\{1\} = \{1,3\}$ and $u_2\{2\} = u_2\{3\} = u_2\{1,2\} = u_2\{1,3\} = u_2\{2,3\} = u_2X = X$. Define a closure operator v_1 on Y by $v_1\emptyset = \emptyset$, $v_1\{2\} = v_1\{3\} = v_1\{2,3\} = \{2,3\}$ and $v_1\{1\} = v_1\{1,2\} = v_1\{1,3\} = v_1Y = Y$. Define a closure operator v_2 on Y by $v_2\emptyset = \emptyset$ and $v_2\{1\} = v_2\{2\} = v_2\{3\} = v_2\{1,2\} = v_2\{1,3\} = v_2\{2,3\} = v_2Y = Y$. Let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be the identity map. Then f is a homeomorphism but f is not semi-irresolute because $f^{-1}(\{1,2\})$ is not semi-open in (X, u_1, u_2) while $\{1,2\}$ is semi-open in (Y, v_1, v_2), i.e. f is not semi-homeomorphism.

Proposition 6.14. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a bijective map. Then f is pre-semi-open if and only if f is pre-semi-closed.*

As a direct consequence of Proposition 6.14, we have:

Proposition 6.15. *Let (X, u_1, u_2) and (Y, v_1, v_2) be biclosure spaces and let $f : (X, u_1, u_2) \to (Y, v_1, v_2)$ be a bijective semi-irresolute map, then the following statements are equivalent:*

(i) *f is a semi-homeomorphism,*

(ii) *f is a pre-semi-closed map,*

(iii) *f is a pre-semi-open map.*

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