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# HYPERIDENTITIES IN MANY-SORTED ALGEBRAS

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**Abstract**

The theory of hyperidentities generalizes the equational theory of universal algebras and is applicable in several fields of science, especially in computers sciences (see e.g., [2, 1]). The main tool to study hyperidentities is the concept of a hypersubstitution. Hypersubstitutions of many-sorted algebras were studied in [3]. On the basis of hypersubstitutions one defines a pair of closure operators which turns out to be a conjugate pair. The theory of conjugate pairs of additive closure operators can be applied to characterize solid varieties, i.e., varieties in which every identity is satisfied as a hyperidentity (see [4]). The aim of this paper is to apply the theory of conjugate pairs of additive closure operators to many-sorted algebras.

**Keywords:** hypersubstitution; hyperidentity; heterogeneous algebra.

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## 1. Preliminaries

Hyperidentities in one-based algebras were considered by many authors (for references see e.g., [4, 2]). An identity $s \approx t$ is satisfied as a hyperidentity in the one-based algebra $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ of type $\tau$ if after any replacements of the operation symbols occurring in $s$ and $t$ by terms of the same arity the arising equation is satisfied in $\mathcal{A}$. These replacements can be described by hypersubstitutions, i.e., mappings from the set of operation symbols into the set of all terms of type $\tau$. Hypersubstitutions cannot only be applied to terms or equations but also to algebras. This gives a pair of additive closure operators which are related to each other by the so-called conjugate property and which form a conjugate pair of additive closure operators (see [4]). A variety of one-based algebras is called solid if every identity is satisfied as a hyperidentity. Characterizations of solid varieties are based on the theory of conjugate pairs of additive closure operators. For more background see [4].

In this paper we want to apply the theory of conjugate pairs of additive closure operators to many-sorted algebras and identities and want to define hyperidentities and solid varieties of many-sorted algebras.

Many-sorted algebras occur in various branches of mathematics. They have found their way into computer science through abstract data type specifications. Many-sorted algebras, varieties and quasivarieties of many-sorted algebras are the mathematical fundament of approaches to abstract data types in programming and specification languages. For basic concepts on many-sorted algebras we refer the reader to [5].

The concept of terms in many-sorted algebras was discussed in [5]. First we want to give a slightly different version of the definitions and results from [3].

Let $I$ be a non-empty set, let $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ , $n \in \mathbb{N}^+$, let $I^* := \bigcup_{n\geq 1} I^n$ and $\Sigma \subseteq I^* \times I$. Then we define $\Sigma_n := \Sigma \cap I^{n+1}$. For $\gamma \in \Sigma$ let $\gamma(l)$ denote the $l$-th component of $\gamma$. Let $K_\gamma$ be a set of indices with respect to $\gamma$. If $|K_\gamma| = 1$, we will drop the index.

**Definition 1.1.** Let $n \in \mathbb{N}^+$ and $X^{(n)} := (X_i^{(n)})_{i\in I}$ be an $I$-sorted set of variables, also called an $n$-element $I$-sorted alphabet, with $X_i^{(n)} := \{x_{i1}, \ldots, x_{in}\}, i \in I$ and let $((f_\gamma)_k)_{k\in K_\gamma,\gamma\in\Sigma}$ be an indexed set of $\Sigma$-sorted operation symbols. Then for each $i \in I$ a set $W_n(i)$ which is called the set of all $n$-ary $\Sigma$-terms of sort $i$, is inductively defined as follows:

(i) $W_0^n(i) := X_i^{(n)}, i \in I$,

(ii) $W_{l+1}^n(i) := W_l^n(i) \cup \{f_\gamma(t_{k_1}, \ldots, t_{k_n}) \mid \gamma = (k_1, \ldots, k_n; i) \in \Sigma, t_{k_j} \in W_l^n(k_j), 1 \le j \le n\}, l \in \mathbb{N}$. (Here we inductively assume that the sets $W_l^n(i)$ are already defined for all sorts $i \in I$).

Then $W_n(i) := \bigcup_{l=0}^{\infty} W_l^n(i)$ and we set $W(i) := \bigcup_{n\in\mathbb{N}^+} W_n(i)$. Let $X_i := \bigcup_{n\in\mathbb{N}^+} X_i^{(n)}$ and $X := (X_i)_{i\in I}$. Let $W_\Sigma(X) := (W(i))_{i\in I}$. The set $W_\Sigma(X)$ is called $I$-sorted set of all $\Sigma$-terms and its elements are called $I$-sorted $\Sigma$-terms.

For any $n \in \mathbb{N}^+, i \in I$ we set $\Lambda_n(i) := \{(w; i) \in I^{n+1} \mid w \in I^n, \exists\, m \in \mathbb{N}^+, \exists\, \alpha \in \Sigma_m, \exists\, j\ (1 \le j \le m)(\alpha(j) = i)\}$. Let $\Lambda(i) := \bigcup_{n=1}^{\infty} \Lambda_n(i)$ and we set $\Lambda := \bigcup_{i\in I} \Lambda(i)$.

To define many-sorted hypersubstitutions we need the following superposition operation for $I$-sorted $\Sigma$-terms.

**Definition 1.2.** Let $t \in W(i), t_j \in W(k_j)$ where $1 \le j \le n, n \in \mathbb{N}$. Then the superposition operation

$$S_\beta : W(i) \times W(k_1) \times \cdots \times W(k_n) \to W(i)$$

for $\beta = (k_1, \ldots, k_n; i) \in \Lambda$, is defined inductively as follows:

1. If $t = x_{ij} \in X_i$, then

   1.1 $S_\beta(x_{ij}, t_1, \ldots, t_n) := x_{ij}$ for $i \ne k_j$ and

   1.2 $S_\beta(x_{ij}, t_1, \ldots, t_n) := t_j$ for $i = k_j$.

2. If $t = f_\gamma(s_1, \ldots, s_m) \in W(i)$ for $\gamma = (i_1, \ldots, i_m; i) \in \Sigma$ and $s_q \in W_n(i_q), 1 \le q \le m, m \in \mathbb{N}$, and if we assume that $S_{\beta_q}(s_q, t_1, \ldots, t_n)$ with $\beta_q = (k_1, \ldots, k_n; i_q) \in \Lambda$ are already defined, then $S_\beta(f_\gamma(s_1, \ldots, s_m), t_1, \ldots, t_n) := f_\gamma(S_{\beta_1}(s_1, t_1, \ldots, t_n), \ldots, S_{\beta_m}(s_m, t_1, \ldots, t_n))$.

**Definition 1.3.** Let $i \in I$ and $((f_\gamma)_k)_{k\in K_\gamma,\gamma\in\Sigma}$ be an indexed set of $\Sigma$-sorted operation symbols. Let $\Sigma_m(i) := \{\gamma \in \Sigma_m \mid \gamma(m+1) = i\}, m \in \mathbb{N}^+$ and let

$$\Sigma(i) := \bigcup_{m\geq 1} \Sigma_m(i).$$

Any mapping

$$\sigma_i : \{(f_\gamma)_k \mid k \in K_\gamma, \gamma \in \Sigma(i)\} \to W(i), i \in I,$$

which preserves arities, is said to be a $\Sigma$-hypersubstitution of sort $i$. Let $\Sigma(i)$-$Hyp$ be the set of all $\Sigma$-hypersubstitutions of sort $i$. The $I$-sorted mapping $\sigma := (\sigma_i)_{i\in I}$ is called an $I$-sorted $\Sigma$-hypersubstitution. Let $\Sigma$-$Hyp$ be the set of all $I$-sorted $\Sigma$-hypersubstitutions. Any $I$-sorted $\Sigma$-hypersubstitution $\sigma$ can inductively be extended to an $I$-sorted mapping $\hat{\sigma} := (\hat{\sigma}_i)_{i\in I}$. The $I$-sorted mapping

$$\hat{\sigma} : W_\Sigma(X) \to W_\Sigma(X)$$

is defined by the following steps: For each $i \in I$ we define

(i) $\hat{\sigma}_i[x_{ij}] := x_{ij}$ for any variable $x_{ij} \in X_i$.

(ii) $\hat{\sigma}_i[f_\gamma(t_1,\ldots,t_n)] := S_\gamma(\sigma_i(f_\gamma), \hat{\sigma}_{k_1}[t_1],\ldots,\hat{\sigma}_{k_n}[t_n])$, where $\gamma = (k_1,\ldots, k_n; i) \in \Sigma$ and $t_q \in W(k_q), 1 \leq q \leq n, n \in \mathbb{N}$, assumed that $\hat{\sigma}_{k_q}[t_q]$, are already defined.

Using the extension $\hat{\sigma}_i$, we define $(\sigma_1)_i \circ_i (\sigma_2)_i := (\hat{\sigma}_1)_i \circ (\sigma_2)_i$. Then we have $((\sigma_1)_i \circ_i (\sigma_2)_i)^\wedge = (\hat{\sigma}_1)_i \circ (\hat{\sigma}_2)_i$. Together with the identity mapping $(\sigma_{id})_i$ the set $\Sigma(i)$-$Hyp$ forms a monoid (see [3]).

Now we want to describe the connection between heterogeneous algebras and $\Sigma$-terms.

Let $A$ be an $I$-sorted set. Then $\mathcal{A}$ is said to be a $\Sigma$-algebra if it has the form

$$\mathcal{A} = \Big(A; (((f_\gamma)_k)^{\mathcal{A}})_{k\in K_\gamma,\gamma\in\Sigma}\Big)$$

where $((f_\gamma)_k)^{\mathcal{A}} : A_{k_1} \times \cdots \times A_{k_n} \to A_i$ if $\gamma = (k_1, \ldots, k_n; i) \in \Sigma$. Let $Alg(\Sigma)$ be the collection of all $\Sigma$-algebras. To connect $\Sigma$-terms with $\Sigma$-algebras we need to consider operations on $I$-sorted sets. Let $A$ be an $I$-sorted set, $n \in \mathbb{N}^+$, $(\omega; i) \in I^* \times I$. Then $\omega$ is called input sequence on $A$ and $i$ is called output sort.

**Definition 1.4.** Let $A$ be an $I$-sorted set, let $\omega = (k_1, \ldots, k_n) \in I^n, n \in \mathbb{N}^+$ be an input sequence on $A$. Then we define the $q$-th $n$-ary projection operation

$$e_q^{\omega,A} : A_{k_1} \times \cdots \times A_{k_n} \to A_{k_q}, 1 \leq q \leq n$$

of the input sequence $\omega$ on $A$ by

$$e_q^{\omega,A}(a_1, \ldots, a_n) := a_q.$$

We denote by

$$O^{(\omega,i)}(A) := \{f \mid f : A_{k_1} \times \cdots \times A_{k_n} \to A_i\}$$

the set of all $n$-ary operations on $A$ with input sequence $\omega$ and output sort $i$.

In particular we denote by

$$O^\omega(A) := (O^{(\omega,i)}(A))_{i \in I}$$

the $I$-sorted set of all $n$-ary operations on $A$ with the same input sequence $\omega$.

Finally we introduce

$$O(A) := \bigcup_{\omega \in I^*} O^\omega(A)$$

as the $I$-sorted set of all finitary operations on the $I$-sorted set $A$.

**Definition 1.5.** Let $A$ be an $I$-sorted set and let $\omega = (s_1, \ldots, s_n), \omega' = (s'_1, \ldots, s'_m)$ be input sequences on $A$. Then the superposition operation

$$S^{\omega,i}_{\omega'} : O^{(\omega,i)}(A) \times O^{(\omega',s_1)}(A) \times \cdots \times O^{(\omega',s_n)}(A) \to O^{(\omega',i)}(A)$$

is defined by

$$S^{\omega,i}_{\omega'}(f, g_1, \ldots, g_n) := f[g_1, \ldots, g_n], \text{ with}$$

$$f[g_1, \ldots, g_n](a_1, \ldots, a_m) := f(g_1(a_1, \ldots, a_m), \ldots, g_n(a_1, \ldots, a_m))$$

for all $(a_1, \ldots, a_m) \in A_{s'_1} \times \cdots \times A_{s'_m}$.

Using these composition operations we may consider a many-sorted algebra, which satisfies similar identities as clones in the one-sorted case.

**Theorem 1.6.** *Let $A$ be an $I$-sorted set. Then the many-sorted algebra*

$$\left((O^{\omega}(A))_{\omega \in I^*}; \left(S^{\omega,i}_{\omega'}\right)_{(\omega,i),(\omega',i) \in I^* \times I}, \left(e^{\omega,A}_j\right)_{\omega \in I^*, 1 \leq j \leq |\omega|}\right)$$

*(where $|\omega|$ is the length of the sequence $\omega$) satisfies the following identities:*

1) $S^{\omega,i}_{\omega''}\left(f, S^{\omega',s_1}_{\omega''}(g_1, h_1, \ldots, h_m), \ldots, S^{\omega',s_n}_{\omega''}(g_n, h_1, \ldots, h_m)\right)$

$$= S^{\omega',i}_{\omega''}\left(S^{\omega,i}_{\omega'}(f, g_1, \ldots g_n), h_1, \ldots, h_m\right) \text{ where}$$

$$\omega = (s_1, \ldots, s_n) \in I^n,\ \omega' = (s'_1, \ldots, s'_m) \in I^m,\ \omega'' = (s''_1, \ldots, s''_p) \in\ I^p,$$

*and*

$f \in O^{(\omega,i)}(A), \quad g_j \in O^{(\omega',s_j)}(A), \quad h_k \in O^{(\omega'',s'_k)}(A) \quad for \quad 1 \leq j \leq n,$

$1 \leq k \leq m,\ m,n \in \mathbb{N}.$

2) $S^{\omega,s_j}_{\omega'}\left(e^{\omega,A}_j, g_1, \ldots, g_n\right) = g_j$ *where* $\omega = (s_1, \ldots, s_n) \in I^n, \omega' \in I^m,$

*and*

$g_j \in O^{(\omega',s_j)}(A),\ 1 \leq j \leq n, m, n \in \mathbb{N}^+.$

3) $S^{\omega,i}_{\omega}\left(f, e^{\omega,A}_1, \ldots, e^{\omega,A}_n\right) = f$ *where* $f \in O^{(\omega,i)}(A),\ \omega \in I^n, n \in \mathbb{N}^+.$

The proofs are similar to the proofs of the corresponding propositions for $\Sigma$-terms (see [3]).

## 2. $I$-Sorted Identities and Model Classes

**Definition 2.1.** Let $n \in \mathbb{N}^+$ and $X^{(n)}$ be an $n$-element $I$-sorted alphabet and let $A$ be an $I$-sorted set. Let $\mathcal{A} \in Alg(\Sigma)$ be a $\Sigma$-algebra, and $t \in W_n(i), i \in I$. Let $f := (f_i)_{i\in I}$, where $f_i : X^{(n)}_i \to A_i$ is an $I$-sorted evaluation mapping of variables from $X^{(n)}$ by elements in $A$. Each mapping $f_i$ can be extended in a canonical way to a mapping $\bar{f}_i : W_n(i) \to A_i$. Then $t^{\mathcal{A}} : A^{X^{(n)}} \to A_i$ is defined by

$$t^{\mathcal{A}}(f) := \bar{f}_i(t) \text{ for all } f \in A^{X^{(n)}},$$

where $\bar{f}_i$ is the extension of the evaluation mapping $f_i : X^{(n)}_i \to A_i$. The operation $t^{\mathcal{A}}$ is called the $n$-ary $\Sigma$-term operation on $\mathcal{A}$ induced by the $n$-ary $\Sigma$-term $t$ of sort $i$. We have $x^{\mathcal{A}}_{k_q q} = e^{\omega,A}_q, 1 \leq q \leq n$, where $\omega = (k_1, \ldots, k_n)$, since for $f \in A^{X^{(n)}}$ we have

$$
\begin{aligned}
x^{\mathcal{A}}_{k_q q}(f) &= \bar{f}_{k_q}(x_{k_q q}) \\
&= f_{k_q}(x_{k_q q}) \\
&= e^{\omega,A}_q(a_1,\ldots,a_{q-1},f_{k_q}(x_{k_q q}),a_{q+1},\ldots,a_n)
\end{aligned}
$$

for all $a_j \in A_{k_j}$ such that $j \in \{1,\ldots,q-1,q+1,\ldots,n\}$.

Let $W^A(i)$ be the set of all $\Sigma$-term operations on $\mathcal{A}$ induced by the $\Sigma$-terms of sort $i$. We set $W^{\mathcal{A}}_{\Sigma}(X) := (W^A(i))_{i\in I}$ and call it the $I$-sorted set of $\Sigma$-term operations on $\mathcal{A}$ induced by the $\Sigma$-terms.

**Definition 2.2.** Let $t \in W(i)$, $t_j \in W(k_j)$ where $1 \leq j \leq n, n \in \mathbb{N}$. Then the superposition operation

$$S^A_\alpha : W^A(i) \times W^A(k_1) \times \cdots \times W^A(k_n) \to W^A(i)$$

where $\alpha = (k_1,\ldots,k_n;i) \in \Lambda$, is inductively defined in the following way:

1) If $t = x_{ij} \in X_i$, then

 1.1) $S^A_\alpha\Big(x^{\mathcal{A}}_{ij}, t^{\mathcal{A}}_1,\ldots,t^{\mathcal{A}}_n\Big) := x^{\mathcal{A}}_{ij}$ for $i \neq k_j$ and

 1.2) $S^A_\alpha\Big(x^{\mathcal{A}}_{ij}, t^{\mathcal{A}}_1,\ldots,t^{\mathcal{A}}_n\Big) := t^{\mathcal{A}}_j$ for $i = k_j$.

2) If $t = f_\gamma(s_1,\ldots,s_m) \in W(i)$ where $\gamma = (i_1,\ldots,i_m;i) \in \Sigma$, $s_q \in W(i_q), 1 \leq q \leq m, m \in \mathbb{N}$ and assume that $S^A_{\alpha_q}(s^{\mathcal{A}}_q, t^{\mathcal{A}}_1,\ldots,t^{\mathcal{A}}_n)$, where $\alpha_q = (k_1,\ldots,k_n;i_q) \in \Lambda$, are already defined, then

$$S_\alpha^{\mathcal{A}}\Big((f_\gamma(s_1,\ldots,s_m))^{\mathcal{A}}, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\Big)$$

$$:= f_\gamma^{\mathcal{A}}\Big(S_{\alpha_1}^{\mathcal{A}}(s_1^{\mathcal{A}}, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}),\ldots,S_{\alpha_m}^{\mathcal{A}}(s_m^{\mathcal{A}}, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}})\Big).$$

**Example 2.3.** Let $I=\{1,2\}$, $X^{(2)}=(X_i^{(2)})_{i\in I}$, $\Sigma=\{(1,2;1),(2,1;2)\}$. Let $\mathcal{A}$ be a $\Sigma$-algebra and let $t=f_{(1,2;1)}(f_{(1,2;1)}(x_{11},x_{21}),f_{(2,1;2)}(x_{22},x_{11}))\in W(1)$, $t_1\in W(2)$, and $t_2\in W(1)$. Then

$$S_{(2,1;1)}^{\mathcal{A}}\Big(t^{\mathcal{A}}t_1^{\mathcal{A}}t_2^{\mathcal{A}}\Big)=S_{(2,1;1)}^{\mathcal{A}}\Big((f_{(1,2;1)}(f_{(1,2;1)}(x_{11},x_{21}),f_{(2,1;2)}(x_{22},x_{11})))^{\mathcal{A}}t_1^{\mathcal{A}}t_2^{\mathcal{A}}\Big)$$

$$=f_{(1,2;1)}^{\mathcal{A}}\Big(S_{(2,1;1)}^{\mathcal{A}}((f_{(1,2;1)}(x_{11},x_{21}))^{\mathcal{A}}t_1^{\mathcal{A}}t_2^{\mathcal{A}}\Big),$$

$$S_{(1,2;2)}^{\mathcal{A}}\Big((f_{(2,1;2)}(x_{22},x_{11}))^{\mathcal{A}},t_1^{\mathcal{A}},t_2^{\mathcal{A}}\Big)\Big)$$

$$=f_{(1,2;1)}^{\mathcal{A}}\Big(f_{(1,2;1)}^{\mathcal{A}}\Big(S_{(2,1;1)}^{\mathcal{A}}\Big(x_{11}^{\mathcal{A}},t_1^{\mathcal{A}},t_2^{\mathcal{A}}\Big),S_{(2,1;2)}^{\mathcal{A}}\Big(x_{21}^{\mathcal{A}},t_1^{\mathcal{A}},t_2^{\mathcal{A}}\Big)\Big),$$

$$f_{(2,1;2)}^{\mathcal{A}}\Big(S_{(2,1;2)}^{\mathcal{A}}\Big(x_{22}^{\mathcal{A}},t_1^{\mathcal{A}},t_2^{\mathcal{A}}\Big),S_{(2,1;1)}^{\mathcal{A}}\Big(x_{11}^{\mathcal{A}},t_1^{\mathcal{A}},t_2^{\mathcal{A}}\Big)\Big)\Big)$$

$$=f_{(1,2;1)}^{\mathcal{A}}\Big(f_{(1,2;1)}^{\mathcal{A}}\Big(x_{11}^{\mathcal{A}},t_1^{\mathcal{A}}\Big),f_{(2,1;2)}^{\mathcal{A}}\Big(x_{22}^{\mathcal{A}},x_{11}^{\mathcal{A}}\Big)\Big).$$

**Proposition 2.4.** *Let $\mathcal{A}$ be a $\Sigma$-algebra and $f_\gamma(t_1,\ldots,t_n)\in W_n(i)$ where $\gamma=(i_1,\ldots,i_n,i)\in\Sigma$, $t_q\in W_n(i_q), 1\le q\le n, n\in\mathbb{N}$. Then*

$$\Big(f_\gamma\Big(t_1,\ldots,t_n\Big)\Big)^{\mathcal{A}}=f_\gamma^{\mathcal{A}}\Big(t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\Big).$$

***Proof.*** Let $f\in A^{X^{(n)}}$, then

$$\Big(f_\gamma\big(t_1,\ldots,t_n\big)\Big)^{\mathcal{A}}(f) = \bar{f}_i\Big(f_\gamma\big(t_1,\ldots,t_n\big)\Big)$$

$$= f_\gamma^{\mathcal{A}}\Big(\bar{f}_{i_1}(t_1),\ldots,\bar{f}_{i_n}(t_n)\Big)$$

$$= f_\gamma^{\mathcal{A}}\Big(t_1^{\mathcal{A}}(f),\ldots,t_n^{\mathcal{A}}(f)\Big)$$

$$= f_\gamma^{\mathcal{A}}\Big(t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\Big)(f). \qquad \blacksquare$$

**Lemma 2.5.** *Let $\mathcal{A}$ be a $\Sigma$-algebra. For $t \in W(i), t_j \in W(k_j)$, $1 \leq j \leq n, n \in \mathbb{N}$ we have:*

$$S_\alpha^{\mathcal{A}}\Big(t^{\mathcal{A}}, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\Big) = \Big(S_\alpha(t,t_1,\ldots,t_n)\Big)^{\mathcal{A}}$$

*where $\alpha = (k_1,\ldots,k_n;i) \in \Lambda$.*

***Proof.*** We will give a proof by induction on the complexity of the $\Sigma$-term $t$.

1) If $t = x_{ij} \in X_i$, then

1.1) for $i \neq k_j$,

$$S_\alpha^{\mathcal{A}}\Big(t^{\mathcal{A}}, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\Big) = S_\alpha^{\mathcal{A}}\Big(x_{ij}^{\mathcal{A}}, t_1^{\mathcal{A}},\ldots,t_n^{\mathcal{A}}\Big)$$

$$= x_{ij}^{\mathcal{A}}$$

$$= \Big(S_\alpha(x_{ij},t_1,\ldots,t_n)\Big)^{\mathcal{A}}$$

$$= \Big(S_\alpha(t,t_1,\ldots,t_n)\Big)^{\mathcal{A}},$$

1.2) and for $i = k_j$,

$$\begin{aligned} S_\alpha^A\Big(t^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big) &= S_\alpha^A\Big(x_{ij}^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big) \\ &= t_j^{\mathcal{A}} \\ &= \Big(S_\alpha(x_{ij}, t_1, \ldots, t_n)\Big)^{\mathcal{A}} \\ &= \Big(S_\alpha(t, t_1, \ldots, t_n)\Big)^{\mathcal{A}}. \end{aligned}$$

2) If $t = f_\gamma(s_1, \ldots, s_m) \in W(i)$, where $\gamma = (i_1, \ldots, i_m; i) \in \Sigma$ and $s_q \in W(i_q), 1 \le q \le m, m \in \mathbb{N}$, and if we assume that the equations

$$S_{\alpha_q}^A\Big(s_q^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big) = \Big(S_{\alpha_q}(s_q, t_1, \ldots, t_n)\Big)^{\mathcal{A}},$$

where $\alpha_q = (k_1, \ldots, k_n; i_q) \in \Lambda$, are satisfied, then for $f \in A^{X^{(n)}}$ we have

$$\begin{aligned} &S_\alpha^A\Big(t^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big)(f) \\ &= S_\alpha^A\Big(\Big(f_\gamma(s_1, \ldots, s_m)\Big)^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big)(f) \\ &= f_\gamma^{\mathcal{A}}\Big(S_{\alpha_1}^A\Big((s_1^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big)(f), \ldots, S_{\alpha_m}^A\Big(s_m^{\mathcal{A}}, t_1^{\mathcal{A}}, \ldots, t_n^{\mathcal{A}}\Big)(f)\Big) \\ &= f_\gamma^{\mathcal{A}}\Big(\Big(S_{\alpha_1}(s_1, t_1, \ldots, t_n)\Big)^{\mathcal{A}}(f), \ldots, \Big(S_{\alpha_m}(s_m, t_1, \ldots, t_n)\Big)^{\mathcal{A}}(f)\Big) \\ &= f_\gamma^{\mathcal{A}}\Big(\bar{f}_{i_i}\Big(S_{\alpha_1}(s_1, t_1, \ldots, t_n)\Big), \ldots, \bar{f}_{i_m}\Big(S_{\alpha_m}(s_m, t_1, \ldots, t_n)\Big)\Big) \end{aligned}$$

$$= \bar{f}_i\Big(f_\gamma\Big(S_{\alpha_1}(s_1,t_1,\ldots,t_n),\ldots,S_{\alpha_m}(s_m,t_1,\ldots,t_n)\Big)\Big)$$

$$= \Big(f_\gamma\Big(S_{\alpha_1}(s_1,t_1,\ldots,t_n),\ldots,S_{\alpha_m}(s_m,t_1,\ldots,t_n)\Big)\Big)^{\mathcal{A}}(f)$$

$$= \Big(S_\alpha\Big(f_\gamma(s_1,\ldots,s_m),t_1,\ldots,t_n\Big)\Big)^{\mathcal{A}}(f)$$

$$= \Big(S_\alpha\Big(t,t_1,\ldots,t_n\Big)\Big)^{\mathcal{A}}(f). \qquad \blacksquare$$

Now we can define equations and identities.

**Definition 2.6.** A $\Sigma$-equation of sort $i$ in $X$ is a pair $(s_i,t_i)$ of elements from $W(i), i \in I$. Such pairs are more commonly written as $s_i \approx_i t_i$. The $\Sigma$-equation $s_i \approx_i t_i$ is said to be a $\Sigma$-identity of sort $i$ in the $\Sigma$-algebra $\mathcal{A}$ if $s_i^{\mathcal{A}} = t_i^{\mathcal{A}}$, that is, if the $\Sigma$-term operations induced by $s_i$ and $t_i$, respectively, on the $\Sigma$-algebra $\mathcal{A}$ are equal.

In this case we also say that the $\Sigma$-equation $s_i \approx_i t_i$ is satisfied or modelled by the $\Sigma$-algebra $\mathcal{A}$, and write $\mathcal{A} \models_i s_i \approx_i t_i$. If the $\Sigma$-equation $s_i \approx_i t_i$ is satisfied by every $\Sigma$-algebra $\mathcal{A}$ of a class $K_0$ of $\Sigma$-algebras, we write $K_0 \models_i s_i \approx_i t_i$. For a set $F(i)$ of equations of sort $i$ we write $\mathcal{A} \models_i F(i)$ if $\mathcal{A} \models_i s_i \approx_i t_i$ for all $(s_i,t_i) \in F(i)$.

**Example 2.7.** Let $I = \{1,2\}$, $X^{(2)} := (X_i^{(2)})_{i\in I}$ be a 2-element $I$-sorted alphabet, and $\Sigma = \{(1,1;1),(2,1;1)\}$. Let $\mathcal{V} = (A; f^{\mathcal{V}}_{(2,1;1)}, f^{\mathcal{V}}_{(1,1;1)})$ where $f^{\mathcal{V}}_{(2,1;1)}, f^{\mathcal{V}}_{(1,1;1)}$ correspond to $\circ$, $+$, respectively, and $A := (V,\mathbb{R})$ is the universe of a real vector space. Then the $\Sigma$-equation

$$f_{(2,1;1)}\Big(x_{21}, f_{(1,1;1)}(x_{11},x_{12})\Big)$$

$$\approx_1 f_{(1,1;1)}\Big(f_{(2,1;1)}(x_{21},x_{11}), f_{(2,1;1)}(x_{21},x_{12})\Big) \in W(1)^2$$

is a $\Sigma$-identity of sort 1 in $\mathcal{V}$, that is,

$$\mathcal{V} \models_1 f_{(2,1;1)}\Big(x_{21}, f_{(1,1;1)}(x_{11}, x_{12})\Big)$$

$$\approx_1 f_{(1,1;1)}\Big(f_{(2,1;1)}(x_{21}, x_{11}), f_{(2,1;1)}(x_{21}, x_{12})\Big)$$

since for $f \in A^{X^{(2)}}$ we have

$$\begin{aligned} f_{(2,1;1)}\Big(x_{21}, f_{(1,1;1)}(x_{11}, x_{12})\Big)^{\mathcal{V}}(f) &= \bar{f}_1\Big(f_{(2,1;1)}(x_{21}, f_{(1,1;1)}(x_{11}, x_{12}))\Big) \\ &= f^{\mathcal{V}}_{(2,1;1)}\Big(\bar{f}_2(x_{21}), \bar{f}_1\Big(f_{(1,1;1)}(x_{11}, x_{12})\Big)\Big) \\ &= f^{\mathcal{V}}_{(2,1;1)}\Big(\bar{f}_2(x_{21}), f^{\mathcal{V}}_{(1,1;1)}\Big(\bar{f}_1(x_{11}), \bar{f}_1(x_{12})\Big)\Big) \\ &= f^{\mathcal{V}}_{(2,1;1)}\Big(f_2(x_{21}), f^{\mathcal{V}}_{(1,1;1)}\Big(f_1(x_{11}), f_1(x_{12})\Big)\Big) \end{aligned}$$

and

$$\begin{aligned} & f_{(1,1;1)}\Big(f_{2,1;1}(x_{21}, x_{11}), f_{(2,1;1)}(x_{21}, x_{12})\Big)^{\mathcal{V}}(f) \\ &= \bar{f}_1\Big(f_{(1,1;1)}\Big(f_{(2,1;1)}(x_{21}, x_{11}),\ f_{(2,1;1)}(x_{21}, x_{12})\Big)\Big) \\ &= f^{\mathcal{V}}_{(1,1;1)}\Big(\bar{f}_1\Big(f_{(2,1;1)}(x_{21}, x_{11})\Big),\ \bar{f}_1\Big(f_{(2,1;1)}(x_{21}, x_{12})\Big)\Big) \\ &= f^{\mathcal{V}}_{(1,1;1)}\Big(f^{\mathcal{V}}_{(2,1;1)}\Big(\bar{f}_2(x_{21}), \bar{f}_1(x_{11})\Big),\ f^{\mathcal{V}}_{(2,1;1)}\Big(\bar{f}_2(x_{21}), \bar{f}_1(x_{12})\Big)\Big) \\ &= f^{\mathcal{V}}_{(1,1;1)}\Big(f^{\mathcal{V}}_{(2,1;1)}\Big(f_2(x_{21}), f_1(x_{11})\Big),\ f^{\mathcal{V}}_{(2,1;1)}\Big(f_2(x_{21}), f_1(x_{12}))\Big)\Big). \end{aligned}$$

Therefore,

$$\Big(f_{(2,1;1)}\Big(x_{21}, f_{(1,1;1)}(x_{11}, x_{12})\Big)\Big)^{\mathcal{V}}$$

$$= \Big(f_{(1,1;1)}\Big(f_{2,1;1}(x_{21}, x_{11}), f_{(2,1;1)}(x_{21}, x_{12})\Big)\Big)^{\mathcal{V}}.$$

Now we extend the usual Galois-connection between identities and algebras to the many-sorted case.

Let $K_0 \subseteq Alg(\Sigma)$ and $L(i) \subseteq W(i)^2$. Then a mapping

$$\Sigma(i)\text{-}Id : P(Alg(\Sigma)) \to P\Big(W(i)^2\Big)$$

is defined by

$$\Sigma(i)\text{-}IdK_0 := \Big\{(s_i, t_i) \in W(i)^2 \mid (\forall \mathcal{A} \in K_0)(\mathcal{A} \models_i s_i \approx_i t_i)\Big\}$$

and a mapping $\Sigma(i)\text{-}Mod : P(W(i)^2) \to P(Alg(\Sigma))$ is defined by

$$\Sigma(i)\text{-}ModL(i) := \{\mathcal{A} \in Alg(\Sigma) \mid (\forall (s_i, t_i) \in L(i))(\mathcal{A} \models_i s_i \approx_i t_i)\}.$$

In the next propositions, we will show that these two mappings satisfy the Galois-connection properties.

**Proposition 2.8.** *Let $i \in I$ and let $K_0, K_1, K_2 \subseteq Alg(\Sigma)$. Then*

(1) $K_1 \subseteq K_2 \Rightarrow \Sigma(i)\text{-}IdK_2 \subseteq \Sigma(i)\text{-}IdK_1$,

(2) $K_0 \subseteq \Sigma(i)\text{-}Mod\Sigma(i)\text{-}IdK_0$.

***Proof.***

(1) Assume that $K_1 \subseteq K_2$ and let $s_i \approx_i t_i \in \Sigma(i)\text{-}IdK_2$. Then for all $\mathcal{A} \in K_2$, we have $\mathcal{A} \models_i s_i \approx_i t_i$. Because of $K_1 \subseteq K_2$, we obtain $\mathcal{A} \models_i s_i \approx_i t_i$, for all $\mathcal{A} \in K_1$. This means that $s_i \approx_i t_i \in \Sigma(i)\text{-}IdK_1$, and then $\Sigma(i)\text{-}IdK_2 \subseteq \Sigma(i)\text{-}IdK_1$.

(2) Let $\mathcal{A} \in K_0$. Then $\mathcal{A} \models_i \Sigma(i)\text{-}IdK_0$, means that $\mathcal{A} \in \Sigma(i)\text{-}Mod\Sigma(i)\text{-}IdK_0$ and then $K_0 \subseteq \Sigma(i)\text{-}Mod\Sigma(i)\text{-}IdK_0$.

■

**Proposition 2.9.** *Let $L(i), L_1(i), L_2(i) \subseteq W(i)^2$ be subsets of the set of all $\Sigma$-equations of sort $i \in I$. Then*

(1) $L_1(i) \subseteq L_2(i) \Rightarrow \Sigma(i)\text{-}ModL_2(i) \subseteq \Sigma(i)\text{-}ModL_1(i)$,

(2) $L(i) \subseteq \Sigma(i)\text{-}Id\Sigma(i)\text{-}ModL(i)$.

***Proof.***

(1) Assume that $L_1(i) \subseteq L_2(i)$ and let $\mathcal{A} \in \Sigma(i)\text{-}ModL_2(i)$. Then $\mathcal{A} \models_i s_i \approx_i t_i$ for all $s_i \approx_i t_i \in L_2(i)$, but we have $L_1(i) \subseteq L_2(i)$, so that $\mathcal{A} \models_i s_i \approx_i t_i$ for all $s_i \approx_i t_i \in L_1(i)$. It follows that $\mathcal{A} \in \Sigma(i)\text{-}ModL_1(i)$ and then $\Sigma(i)\text{-}ModL_2(i) \subseteq \Sigma(i)\text{-}ModL_1(i)$.

(2) Let $s_i \approx_i t_i \in L(i)$. Then we have $\Sigma(i)\text{-}ModL(i) \models_i s_i \approx_i t_i$, that is $s_i \approx_i t_i \in \Sigma(i)\text{-}Id\Sigma(i)\text{-}ModL(i)$ and then $L(i) \subseteq \Sigma(i)\text{-}Id\Sigma(i)\text{-}ModL(i)$.

■

From both propositions, we have that $(\Sigma(i)\text{-}Mod, \Sigma(i)\text{-}Id)$ is a Galois connection between $Alg(\Sigma)$ and $W(i)^2$ with respect to the relation

$$\models_i := \Big\{(\mathcal{A}, (s_i, t_i)) \in Alg(\Sigma) \times W(i)^2 \mid \mathcal{A} \models_i s_i \approx_i t_i\Big\}.$$

The fixed points with respect to the closure operator $\Sigma(i)\text{-}Mod\Sigma(i)\text{-}Id$ are called $\Sigma$-varieties of sort $i$ and the fixed points with respect to the closure operator $\Sigma(i)\text{-}Id\Sigma(i)\text{-}Mod$ are called $\Sigma$-equational theories of sort $i$.

## 3. Application of Σ-Hypersubstitutions

Now we apply Σ-hypersubstitutions to many-sorted algebras and to many-sorted equations.

**Definition 3.1.** Let $A$ be an $I$-sorted set, let $\mathcal{A} := (A; (((f_\gamma)_k)^{\mathcal{A}})_{k\in K_\gamma,\gamma\in\Sigma})$ be a Σ-algebra and let $\sigma \in \Sigma\text{-}Hyp$. Then we define the Σ-algebra

$$\sigma(\mathcal{A}) := \left(A; \left((\sigma_i((f_\gamma)_k))^{\mathcal{A}}\right)_{k\in K_\gamma,\gamma\in\Sigma(i),i\in I}\right).$$

This Σ-algebra is called the Σ-algebra derived from $\mathcal{A}$ and $\sigma$, for short derived Σ-algebra.

For illustration we consider the following example.

**Example 3.2.** Let $I = \{1,2\}$, $\Sigma = \{(1,2,1),(2,1,2)\}$, $K_{(1,2,1)} = \{1,2\}$, $A = (A_1, A_2)$, $\mathcal{A} = ((A_1,A_2); ((f_{(1,2,1)})_1)^{\mathcal{A}}, ((f_{(1,2,1)})_2)^{\mathcal{A}}, f^{\mathcal{A}}_{(2,1,2)})$. Let $\sigma = (\sigma_1, \sigma_2) \in \Sigma\text{-}Hyp$. Then we have

$$\begin{aligned}&\sigma(\mathcal{A})\\ &= \Big((A_1,A_2);\ \Big(\sigma_1((f_{(1,2,1)}))_1\Big)^{\mathcal{A}},\ \Big(\sigma_1((f_{(1,2,1)}))_2\Big)^{\mathcal{A}},\ \Big(\sigma_2(f_{(2,1,2)})\Big)^{\mathcal{A}}\Big).\end{aligned}$$

**Theorem 3.3.** *Let $A$ be an $I$-sorted set and $\mathcal{A} := (A; (((f_\gamma)_k)^{\mathcal{A}})_{k\in K_\gamma,\gamma\in\Sigma})$ be a Σ-algebra. Let $\sigma \in \Sigma\text{-}Hyp$ and $t \in W(i), i \in I$. Then $t^{\sigma(\mathcal{A})} = (\hat{\sigma}_i[t])^{\mathcal{A}}$.*

***Proof.*** We will give a proof by induction on the complexity of the Σ-term $t$.

1) If $t = x_{ij} \in X_i$ where $1 \le j \le n, n \in \mathbb{N}$, then for $f \in A^{X^{(n)}}$ we have

$$\begin{aligned} t^{\sigma(\mathcal{A})}(f) &= x_{ij}^{\sigma(\mathcal{A})}(f) \\ &= \bar{f}_i(x_{ij}) \\ &= x_{ij}^{\mathcal{A}}(f) \\ &= (\hat{\sigma}_i[x_{ij}])^{\mathcal{A}}(f) \\ &= (\hat{\sigma}_i[t])^{\mathcal{A}}(f). \end{aligned}$$

2) If $t = f_\gamma(s_1, \ldots, s_m) \in W(i)$ where $\gamma = (i_i, \ldots, i_m; i) \in \Sigma, s_q \in W(i_q)$, $1 \leq q \leq m, m \in \mathbb{N}$ and assume that $s_q^{\sigma(\mathcal{A})} = \hat{\sigma}_{i_q}[s_q]^{\mathcal{A}}$ are satisfied, then for $f \in A^{X^{(n)}}$ we have

$$\begin{aligned} t^{\sigma(\mathcal{A})}(f) &= (f_\gamma(s_1, \ldots, s_m))^{\sigma(\mathcal{A})}(f) \\ &= \bar{f}_i(f_\gamma(s_1, \ldots, s_m)) \\ &= f_\gamma^{\sigma(\mathcal{A})}(\bar{f}_{i_1}(s_1), \ldots, \bar{f}_{i_m}(s_m)) \\ &= f_\gamma^{\sigma(\mathcal{A})}(s_1^{\sigma(\mathcal{A})}(f), \ldots, s_m^{\sigma(\mathcal{A})}(f)) \\ &= \sigma_i(f_\gamma)^{\mathcal{A}}(\hat{\sigma}_{i_1}[s_1]^{\mathcal{A}}(f), \ldots, \hat{\sigma}_{i_m}[s_m]^{\mathcal{A}}(f)) \\ &= \sigma_i(f_\gamma)^{\mathcal{A}}(\hat{\sigma}_{i_1}[s_1]^{\mathcal{A}}, \ldots, \hat{\sigma}_{i_m}[s_m]^{\mathcal{A}})(f) \\ &= S_\gamma^{\mathcal{A}}(\sigma_i(f_\gamma)^{\mathcal{A}}, \hat{\sigma}_{i_1}[s_1]^{\mathcal{A}}, \ldots, \hat{\sigma}_{i_m}[s_m]^{\mathcal{A}})(f) \\ &= (S_\gamma(\sigma_i(f_\gamma), \hat{\sigma}_{i_1}[s_1], \cdots, \hat{\sigma}_{i_m}[s_m]))^{\mathcal{A}}(f) \text{ by Lemma 2.5} \\ &= (\hat{\sigma}_i[f_\gamma(s_1, \ldots, s_m)])^{\mathcal{A}}(f) \\ &= (\hat{\sigma}_i[t])^{\mathcal{A}}(f). \end{aligned}$$

■

**Lemma 3.4.** *Let $\mathcal{A} \in Alg(\Sigma)$, $\sigma_1, \sigma_2 \in \Sigma$-$Hyp$. Then we have*

$$\Big((\sigma_1)_i(f_\gamma)\Big)^{\sigma_2(\mathcal{A})} = \Big(((\sigma_2)_i \circ_i (\sigma_1)_i)(f_\gamma)\Big)^{\mathcal{A}},$$

*for $\gamma \in \Sigma(i), i \in I$.*

***Proof.*** By Theorem 3.3, we have

$$\begin{aligned}\Big((\sigma_1)_i(f_\gamma)\Big)^{\sigma_2(\mathcal{A})} &= \Big((\hat{\sigma}_2)_i[(\sigma_1)_i(f_\gamma)]\Big)^{\mathcal{A}}\\ &= \Big(((\hat{\sigma}_2)_i \circ (\sigma_1)_i)(f_\gamma)\Big)^{\mathcal{A}}\\ &= \Big(((\sigma_2)_i \circ_i (\sigma_1)_i)(f_\gamma)\Big)^{\mathcal{A}}.\end{aligned}$$

■

Let $\sigma_1, \sigma_2$ be elements in $\Sigma$-$Hyp$. Then we set $\sigma_1 \diamond \sigma_2 := ((\sigma_1)_i \circ_i (\sigma_2)_i)_{i \in I}$.

**Lemma 3.5.** *Let $A$ be an $I$-sorted set, let $\mathcal{A} = (A; (((f_\gamma)_k)^{\mathcal{A}})_{k \in K_\gamma, \gamma \in \Sigma})$ be a $\Sigma$-algebra, and $\sigma_1, \sigma_2 \in \Sigma$-$Hyp$. Then we have*

$$\sigma_1(\sigma_2(\mathcal{A})) = (\sigma_2 \diamond \sigma_1)(\mathcal{A}).$$

***Proof.*** By Lemma 3.4, we have

$$\begin{aligned}\sigma_1(\sigma_2(\mathcal{A})) &= \Big(A; \Big(((\sigma_1)_i((f_\gamma)_k)^{\sigma_2(\mathcal{A})}\Big)_{k \in K_\gamma, \gamma \in \Sigma(i), i \in I}\Big)\\ &= \Big(A; \Big((((\sigma_2)_i \circ_i (\sigma_1)_i)((f_\gamma)_k)^{\mathcal{A}}\Big)_{k \in K_\gamma, \gamma \in \Sigma(i), i \in I}\Big)\\ &= (\sigma_2 \diamond \sigma_1)(\mathcal{A}).\end{aligned}$$

■

**Theorem 3.6.** *Let $A$ be an $I$-sorted set, $\mathcal{A} := (A; (((f_\alpha)_k)^{\mathcal{A}})_{k\in K_\alpha,\alpha\in\Sigma})$, and $\sigma_{id} \in \Sigma$-$Hyp$. Then we have*

$$\sigma_{id}(\mathcal{A}) = \mathcal{A}.$$

***Proof.*** We will show that $((\sigma_{id})_i(f_\alpha)_k)^{\mathcal{A}} = f_\alpha^{\mathcal{A}}$ for all $k \in K_\alpha, \alpha \in \Sigma$. Assume that $\alpha = (k_1, \ldots, k_n; i) \in \Sigma$ and $\omega = (k_1, \ldots, k_n) \in I^n$. Then

$$\begin{aligned}\Big((\sigma_{id})_i(f_\alpha)\Big)^{\mathcal{A}} &= \Big(f_\alpha(x_{k_11}, \ldots, x_{k_nn})\Big)^{\mathcal{A}}\\ &= f_\alpha^{\mathcal{A}}\Big(x_{k_11}^{\mathcal{A}}, \ldots, x_{k_nn}^{\mathcal{A}}\Big)\\ &= f_\alpha^{\mathcal{A}}\Big(e_1^{\omega,A}, \ldots, e_n^{\omega,A}\Big)\\ &= f_\alpha^{\mathcal{A}}.\end{aligned}$$

■

**Definition 3.7.** A $\Sigma$-algebra $\mathcal{A}$ is said to hypersatisfy the $\Sigma$-identity $s_i \approx_i t_i$ of sort $i \in I$, if for every $\Sigma$-hypersubstitution of sort $i$, i.e., $\sigma_i \in \Sigma(i)$-$Hyp$, the $\Sigma$-identity $\hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i]$ holds in $\mathcal{A}$.

In this case we say that the $\Sigma$-identity $s_i \approx_i t_i$ of sort $i$ is satisfied as a $\Sigma$-hyperidentity of sort $i$ in $\mathcal{A}$ and write $\mathcal{A} \underset{\Sigma-hyp}{\models_i} s_i \approx_i t_i$, that is

$$\mathcal{A} \underset{\Sigma-hyp}{\models_i} s_i \approx_i t_i :\Leftrightarrow \forall \sigma_i \in \Sigma(i)\text{-}Hyp \ \ (\mathcal{A} \models_i \hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i]).$$

Let us consider the following example.

**Example 3.8.** Let $I = \{1, 2\}$, $X^{(2)} := (X_i^{(2)})_{i\in I}$ and let $\Sigma = \{(1, 1; 1), (2, 2; 2)\}$. Let $\mathcal{B}_i := (B_i; \circ_i)$ be bands. Then $f_{(i,i,i)}(x_{ij}, x_{ij}) \approx_i x_{ij}$ are hyperidentities in $\mathcal{B}_i, i \in I$. Let $\mathcal{B} := (B; \circ)$ be a double band, where $B := (B_i)_{i\in I}$, $\circ := (\circ_i)_{i\in I}$. Then $f_{(i,i,i)}(x_{ij}, x_{ij}) \approx_i x_{ij}$ are $\Sigma$-hyperidentities of sort $i$ in $\mathcal{B}$.

Let $K_0 \subseteq Alg(\Sigma)$ be a set of $\Sigma$-algebras, and let $L(i) \subseteq W(i)^2$ be a set of $\Sigma$-equations of sort $i$. Then we define a mapping

$$H\Sigma(i)\text{-}Id : P(Alg(\Sigma)) \to P\Big(W(i)^2\Big)$$

by

$$H\Sigma(i)\text{-}IdK_0 := \Big\{(s_i, t_i) \in W(i)^2 \Big| (\forall \mathcal{A} \in K_0)\Big(\mathcal{A} \underset{\Sigma\text{-}hyp}{\models_i} s_i \approx_i t_i\Big)\Big\}$$

and a mapping $H\Sigma(i)\text{-}Mod : P(W(i)^2) \to P(Alg(\Sigma))$ by

$$H\Sigma(i)\text{-}ModL(i) := \Big\{\mathcal{A} \in Alg(\Sigma) \Big| (\forall (s_i, t_i) \in L(i))\Big(\mathcal{A} \underset{\Sigma\text{-}hyp}{\models_i} s_i \approx_i t_i\Big)\Big\}.$$

We get that $(H\Sigma(i)\text{-}Mod, H\Sigma(i)\text{-}Id)$ is also a Galois connection between $Alg(\Sigma)$ and $W(i)^2$ with respect to the relation

$$\underset{\Sigma-hyp}{\models_i} := \Big\{(\mathcal{A}, (s_i, t_i)) \in Alg(\Sigma) \times W(i)^2 \Big| \mathcal{A} \underset{\Sigma-hyp}{\models_i} s_i \approx_i t_i\Big\}.$$

**Definition 3.9.** Let $K_0 \subseteq Alg(\Sigma)$ be a subclass of $\Sigma$-algebras and let $L(i) \subseteq W(i)^2$ be a set of $\Sigma$-equations of sort $i$. Then we set

$$\chi^{\Sigma\text{-}E(i)}[s_i \approx_i t_i] := \{\hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i] \mid \sigma_i \in \Sigma(i)\text{-}Hyp\}$$

and

$$\chi^{\Sigma\text{-}A}[\mathcal{A}] := \{\sigma(\mathcal{A}) \mid \sigma \in \Sigma\text{-}Hyp\}.$$

We define two operators

$$\chi^{\Sigma\text{-}E(i)} : P\Big(W(i)^2\Big) \to P\Big(W(i)^2\Big)$$

by

$$\chi^{\Sigma\text{-}E(i)}[L(i)] := \bigcup_{s_i \approx_i t_i \in L(i)} \chi^{\Sigma\text{-}E(i)}[s_i \approx_i t_i]$$

and

$$\chi^{\Sigma\text{-}A} : P(Alg(\Sigma)) \to P(Alg(\Sigma))$$

by

$$\chi^{\Sigma\text{-}A}[K_0] := \bigcup_{\mathcal{A} \in K_0} \chi^{\Sigma\text{-}A}[\mathcal{A}].$$

**Proposition 3.10.** *Let $L(i), L_k(i) \subseteq W(i)^2$ be sets of $\Sigma$-equations of sort $i \in I$ with $k = 1, 2$. Then*

(i) $L(i) \subseteq \chi^{\Sigma-E(i)}[L(i)]$,

(ii) $L_1(i) \subseteq L_2(i) \Rightarrow \chi^{\Sigma-E(i)}[L_1(i)] \subseteq \chi^{\Sigma-E(i)}[L_2(i)]$,

(iii) $\chi^{\Sigma-E(i)}[L(i)] = \chi^{\Sigma-E(i)}[\chi^{\Sigma-E(i)}[L(i)]]$.

***Proof.***

(i) Let $s_i \approx_i t_i \in L(i)$. Then since $s_i = (\hat{\sigma}_{id})_i[s_i]$ and $t_i = (\hat{\sigma}_{id})_i[t_i]$, we have $(\hat{\sigma}_{id})_i[s_i] = s_i \approx_i t_i = (\hat{\sigma}_{id})_i[t_i] \in \chi^{\Sigma-E(i)}[L(i)]$ and then $L(i) \subseteq \chi^{\Sigma-E(i)}[L(i)]$.

(ii) Assume that $L_1(i) \subseteq L_2(i)$ and let $\hat{\sigma}[s_i] \approx_i \hat{\sigma}[t_i] \in \chi^{\Sigma-E(i)}[L_1(i)]$. Then $s_i \approx_i t_i \in L_1(i)$ but $L_1(i) \subseteq L_2(i)$, so that $s_i \approx_i t_i \in L_2(i)$ and $\hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i] \in \chi^{\Sigma-E(i)}[L_2(i)]$. We have $\chi^{\Sigma-E(i)}[L_1(i)] \subseteq \chi^{\Sigma-E(i)}$ $[L_2(i)]$.

(iii) By (i) we have $\chi^{\Sigma-E(i)}[L(i)] \subseteq \chi^{\Sigma-E(i)}[\chi^{\Sigma-E(i)}[L(i)]]$. Let $\hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i] \in \chi^{\Sigma-E(i)}[\chi^{\Sigma-E(i)}[L(i)]]$. Then $s_i \approx_i t_i \in \chi^{\Sigma-E(i)}[L(i)]$, and there exists $\rho_i \in \Sigma(i)$-$Hyp$ and $u_i \approx_i v_i \in L(i)$ such that $s_i = \hat{\rho}_i[u_i]$ and $t_i = \hat{\rho}_i[v_i]$, and we have

$$\begin{aligned}\hat{\sigma}_i[s_i] &= \hat{\sigma}_i[\hat{\rho}_i[u_i]] \\ &= (\hat{\sigma}_i \circ \hat{\rho}_i)\ [u_i] \\ &= (\sigma_i \circ_i \rho_i)\hat{}[u_i] \\ &= \hat{\lambda}_i[u_i],\ \text{where } \lambda_i = \sigma_i \circ_i \rho_i \in \Sigma(i)\text{-}Hyp,\end{aligned}$$

and

$$\begin{aligned}\hat{\sigma}_i[t_i] &= \hat{\sigma}_i[\hat{\rho}_i[v_i]] \\ &= (\hat{\sigma}_i \circ \hat{\rho}_i)\ [v_i] \\ &= (\sigma_i \circ_i \rho_i)\hat{}[v_i] \\ &= \hat{\lambda}_i[v_i].\end{aligned}$$

Then we set

$$\hat{\lambda}_i[u_i] = \hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i] = \hat{\lambda}_i[v_i] \in \chi^{\Sigma-E(i)}[L(i)],$$

and then

$$\chi^{\Sigma-E(i)}[\chi^{\Sigma-E(i)}[L(i)]] \subseteq \chi^{\Sigma-E(i)}[L(i)].$$

■

**Proposition 3.11.** *Let* $K_0, K_1, K_2 \subseteq Alg(\Sigma)$ *be classes of* $\Sigma$*-algebras. Then*

(i) $K_0 \subseteq \chi^{\Sigma-A}[K_0]$,

(ii) $K_1 \subseteq K_2 \Rightarrow \chi^{\Sigma-A}[K_1] \subseteq \chi^{\Sigma-A}[K_2]$,

(iii) $\chi^{\Sigma-A}[K_0] = \chi^{\Sigma-A}[\chi^{\Sigma-A}[K_0]]$.

***Proof.***

(i) Let $\mathcal{A} \in K_0$. Then since $\mathcal{A} = \sigma_{id}(\mathcal{A}) \in \chi^{\Sigma-A}[K_0]$, we have $K_0 \subseteq \chi^{\Sigma-A}[K_0]$.

(ii) Assume that $K_1 \subseteq K_2$ and let $\sigma(\mathcal{A}) \in \chi^{\Sigma-A}[K_1]$. Then $\mathcal{A} \in K_1$ by our assumption that $\mathcal{A} \in K_2$, with $\sigma(\mathcal{A}) \in \chi^{\Sigma-A}[K_2]$, and then $\chi^{\Sigma-A}[K_1] \subseteq \chi^{\Sigma-A}[K_2]$.

(iii) By (i), we have $\chi^{\Sigma-A}[K_0] \subseteq \chi^{\Sigma-A}[\chi^{\Sigma-A}[K_0]]$. We will show that $\chi^{\Sigma-A}[\chi^{\Sigma-A}[K_0]] \subseteq \chi^{\Sigma-A}[K_0]$. Let $\sigma(\mathcal{A}) \in \chi^{\Sigma-A}[\chi^{\Sigma-A}[K_0]]$. Then $\mathcal{A} \in \chi^{\Sigma-A}[K_0]$, and there exists $\rho \in \Sigma\text{-}Hyp$ and $\mathcal{B} \in K_0$ such that $\mathcal{A} = \rho(\mathcal{B})$. We have

$$\begin{aligned}\sigma(\mathcal{A}) &= \sigma(\rho(\mathcal{B}))\\ &= (\rho \diamond \sigma)(\mathcal{B})\\ &= \lambda(\mathcal{B}), \text{ where } \lambda = \rho \diamond \sigma \in \Sigma\text{-}Hyp.\end{aligned}$$

Thus we have $\sigma(\mathcal{A}) = \lambda(\mathcal{B}) \in \chi^{\Sigma-A}[K_0]$ and then $\chi^{\Sigma-A}[\chi^{\Sigma-A}[K_0]] \subseteq \chi^{\Sigma-A}[K_0]$.

∎

**Lemma 3.12.** *Let $\mathcal{A} \in Alg(\Sigma)$ be a $\Sigma$-algebra, let $s_i \approx_i t_i \in W(i)^2$ be a $\Sigma$-equation of sort $i \in I$, and $\sigma \in \Sigma$-$Hyp$. Then*

$$\sigma(\mathcal{A}) \models_i s_i \approx_i t_i \Longleftrightarrow \mathcal{A} \models_i \hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i].$$

***Proof.*** We obtain

$$\begin{aligned} \sigma(\mathcal{A}) \models_i s_i \approx_i t_i &\Longleftrightarrow s_i^{\sigma(\mathcal{A})} = t_i^{\sigma(\mathcal{A})} \\ &\Longleftrightarrow \hat{\sigma}_i[s_i]^{\mathcal{A}} = \hat{\sigma}_i[t_i]^{\mathcal{A}} \\ &\Longleftrightarrow \mathcal{A} \models_i \hat{\sigma}_i[s_i] \approx_i \hat{\sigma}_i[t_i]. \end{aligned}$$

■

The next theorem needs the concept of a conjugate pair of additive closure operators (see [4]).

**Theorem 3.13.** *The pair $(\chi^{\Sigma-A}, \chi^{\Sigma-E(i)})$ is a conjugate pair of completely additive closure operators of sort $i$ with respect to the relation $\models_i$.*

***Proof.*** By Definition 3.9, Propositions 3.10–3.11, and Lemma 3.12. ■

Now we may apply the theory of conjugate pairs of additive closure operators (see e.g., [4]) and obtain the following propositions:

**Lemma 3.14** ([4]). *For all $K_0 \subseteq Alg(\Sigma)$ and for all $L(i) \subseteq W(i)^2$ the following properties hold:*

(i) $H\Sigma(i)$-$ModL(i) = \Sigma(i)$-$Mod\chi^{\Sigma-E(i)}[L(i)]$,

(ii) $H\Sigma(i)$-$ModL(i) \subseteq \Sigma(i)$-$ModL(i)$,

(iii) $\chi^{\Sigma-A}[H\Sigma(i)$-$ModL(i)] = H\Sigma(i)$-$ModL(i)$,

(iv) $\chi^{\Sigma-E(i)}[\Sigma(i)\text{-}IdH\Sigma(i)\text{-}ModL(i)] = \Sigma(i)\text{-}IdH\Sigma(i)\text{-}ModL(i)$,

(v) $H\Sigma(i)\text{-}ModH\Sigma(i)\text{-}IdK_0 = \Sigma(i)\text{-}Mod\Sigma(i)\text{-}Id\chi^{\Sigma-A}[K_0]$, *and*

(i)′ $H\Sigma(i)\text{-}IdK_0 = \Sigma(i)\text{-}Id\chi^{\Sigma-A}[K_0]$,

(ii)′ $H\Sigma(i)\text{-}IdK_0 \subseteq \Sigma(i)\text{-}IdK_0$,

(iii)′ $\chi^{\Sigma-E(i)}[H\Sigma(i)\text{-}IdK_0] = H\Sigma(i)\text{-}IdK_0$,

(iv)′ $\chi^{\Sigma-A}[\Sigma(i)\text{-}ModH\Sigma(i)\text{-}IdK_0] = \Sigma(i)\text{-}ModH\Sigma(i)\text{-}IdK_0$,

(v)′ $H\Sigma(i)\text{-}IdH\Sigma(i)\text{-}ModL(i) = \Sigma(i)\text{-}Id\Sigma(i)\text{-}Mod\chi^{\Sigma-E(i)}[L(i)]$.

## 4. $I$-Sorted Solid $\Sigma$-Varieties

**Definition 4.1.** Let $K_0 \subseteq Alg(\Sigma)$ be a subclass of $\Sigma$-algebras. Then $K_0$ is called a solid model class of sort $i$ or a solid $\Sigma$-variety of sort $i$ if every $\Sigma$-identity of sort $i$ is satisfied as a $\Sigma$-hyperidentity of sort $i$:

$$K_0 \underset{\Sigma-hyp}{\models_i} \Sigma(i)\text{-}IdK_0.$$

$K_0$ is called an $I$-sorted solid model class if every $\Sigma$-identity of sort $i$ is satisfied as a $\Sigma$-hyperidentity of sort $i$ for all $i \in I$, that is,

$$K_0 \underset{\Sigma-hyp}{\models_i} \Sigma(i)\text{-}IdK_0 \text{ for all } i \in I.$$

$L(i)$ is said to be a $\Sigma$-equational theory of sort $i$ if there exists a class of $\Sigma$-algebras $K_0$ such that $L(i) = \Sigma(i)\text{-}IdK_0$. Then we set $L := (L(i))_{i\in I}$. This $I$-sorted set is called $I$-sorted $\Sigma$-equational theory.

Using the propositions of Lemma 3.14 one obtains the following characterization of solid $\Sigma$-varieties of sort $i$ and solid $\Sigma$-equational theories of sort $i$ (see e.g., [4]).

**Theorem 4.2** ([4]). *Let $K_0$ be a $\Sigma$-variety of sort $i$. Then the following properties are equivalent:*

(i) $K_0 = H\Sigma(i)\text{-}ModH\Sigma(i)\text{-}IdK_0$,

(ii) $\chi^{\Sigma-A}[K_0] = K_0$,

(iii) $\Sigma(i)\text{-}IdK_0 = H\Sigma(i)\text{-}IdK_0$,

(iv) $\chi^{\Sigma-E(i)}[\Sigma(i)\text{-}IdK_0] = \Sigma(i)\text{-}IdK_0$.

**Theorem 4.3** ([4]). *Let $L(i)$ be a $\Sigma$-equational theory of sort $i$. Then the following properties are equivalent:*

(i) $L(i) = H\Sigma(i)\text{-}IdH\Sigma(i)\text{-}ModL(i)$,

(ii) $\chi^{\Sigma-E(i)}[L(i)] = L(i)$,

(iii) $\Sigma(i)\text{-}ModL(i) = H\Sigma(i)\text{-}ModL(i)$,

(iv) $\chi^{\Sigma-A}[\Sigma(i)\text{-}ModL(i)] = \Sigma(i)\text{-}ModL(i)$.

## 5. $I$-SORTED COMPLETE LATTICES

Let $\mathcal{H}(i)$ be the class of all fixed points with respect to the closure operator $\Sigma(i)\text{-}Mod\Sigma(i)\text{-}Id$:

$$\mathcal{H}(i) := \{K_0 \subseteq Alg(\Sigma) \mid K_0 = \Sigma(i)\text{-}Mod\Sigma(i)\text{-}IdK_0\},$$

that is, $\mathcal{H}(i)$ is the class of all $\Sigma$-varieties of sort $i$. Then $\mathcal{H}(i)$ forms a complete lattice of $\Sigma$-varieties of sort $i$. Let $\mathcal{H}y(i)$ be the class of all fixed points with respect to the closure operator $H\Sigma(i)$-$ModH\Sigma(i)$-$Id$:

$$\mathcal{H}y(i) := \{K_0 \subseteq Alg(\Sigma) \mid K_0 = H\Sigma(i)\text{-}ModH\Sigma(i)\text{-}IdK_0\},$$

that is, $\mathcal{H}y(i)$ is the class of all solid $\Sigma$-varieties of sort $i$. Then $\mathcal{H}y(i)$ forms a complete lattice of solid $\Sigma$-varieties of sort $i$ and $\mathcal{H}y(i)$ is a complete sublattice of $\mathcal{H}(i)$. We set $\mathcal{H} := (\mathcal{H}(i))_{i\in I}$ and $\mathcal{H}y := (\mathcal{H}y(i))_{i\in I}$. $\mathcal{H}$ is called an $I$-sorted complete lattice. $\mathcal{H}y$ is called an $I$-sorted complete sublattice of $\mathcal{H}$, since for every $i \in I, \mathcal{H}y(i)$ is a complete sublattice of $\mathcal{H}(i)$. Dually

Let $\mathcal{L}(i)$ be the class of all fixed points with respect to the closure operator $\Sigma(i)$-$Id\Sigma(i)$-$Mod$:

$$\mathcal{L}(i) := \{L(i) \subseteq W(i)^2 \mid L(i) = \Sigma(i)\text{-}Id\Sigma(i)\text{-}ModL(i)\},$$

that is, $\mathcal{L}(i)$ is the class of all $\Sigma$-equational theories of sort $i$. Then $\mathcal{L}(i)$ forms a complete lattice of $\Sigma$-equational theories of sort $i$. Let $\mathcal{L}y(i)$ be the class of all fixed points with respect to the closure operator $H\Sigma(i)$-$IdH\Sigma(i)$-$Mod$:

$$\mathcal{L}y(i) := \{L(i) \subseteq W(i)^2 \mid L(i) = H\Sigma(i)\text{-}IdH\Sigma(i)\text{-}ModL(i)\},$$

that is, $\mathcal{L}y(i)$ is the class of all solid $\Sigma$-equational theories of sort $i$. Then $\mathcal{L}y(i)$ forms a complete lattice of solid $\Sigma$-equational theories of sort $i$ and $\mathcal{L}y(i)$ is a complete sublattice of $\mathcal{L}(i)$. We set $\mathcal{L} := (\mathcal{L}(i))_{i\in I}$ and $\mathcal{L}y := (\mathcal{L}y(i))_{i\in I}$. $\mathcal{L}$ is called an $I$-sorted complete lattice. $\mathcal{L}y$ is called an $I$-sorted complete sublattice of $\mathcal{L}$, since for every $i \in I, \mathcal{L}y(i)$ is a complete sublattice of $\mathcal{L}(i)$.

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