Discussiones Mathematicae
General Algebra and Applications 28 (2008) 209–225

NOETHERIAN AND ARTINIAN PSEUDO MV-ALGEBRAS

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Abstract

The notions of Noetherian pseudo MV-algebras and Artinian pseudo MV-algebras are introduced and their characterizations are established. Characterizations of them via fuzzy ideals are also given.

Keywords: pseudo MV-algebra, (fuzzy) ideal, Noetherian (Artinian) pseudo MV-algebra.

2000 Mathematics Subject Classification: 06D35.

1. Introduction

Pseudo MV-algebras were introduced by Georgescu and Iorgulescu in [7] and [8], and independently by Rachůnek in [10] (he uses the name generalized MV-algebra) as a non-commutative generalization of MV-algebras which were introduced by Chang in [1]. As it is well known, MV-algebras are an algebraic counterpart of the Łukasiewicz many valued propositional logic. Therefore pseudo MV-algebras are an algebraic model of a non-commutative generalization of the Łukasiewicz logic, which allows two different negations (see [11] for details).

The theory of fuzzy sets was first developed by Zadeh in [13]. Since then this idea has been applied to other algebraic structures such as semigroups, groups, rings, ideals, modules, vector spaces and topologies.

Recently, Jun and Walendziak in [9] applied the concept of fuzzy sets to pseudo MV-algebras. They introduced the notions of fuzzy ideal and fuzzy implicative ideal in a pseudo MV-algebra, gave their characterizations and provided conditions for a fuzzy set to be a fuzzy (implicative) ideal. Further, the author in [4], [5] and [6] introduced the fuzzy maximal and fuzzy prime ideals of pseudo MV-algebras and obtained some related properties.

It is well known that every (proper) ideal of pseudo MV-algebra is an intersection of prime ideals. But there are pseudo MV-algebras in which every ideal has such decomposition as finite. These algebras are called Noetherian and this paper is devoted to them and also to dual case Artinian.

The paper is organized as follows. In Section 2 we recall some basic definitions and results of pseudo MV-algebras. In Section 3 we introduce the notions of Noetherian pseudo MV-algebras and Artinian pseudo MV-algebras and investigate some of their related properties. Further, we characterize them in terms of fuzzy ideals in Section 4.

2. Preliminaries

Let $A = (A, \oplus, ^-, ^\sim, 0, 1)$ be an algebra of type $(2, 1, 1, 0, 0)$. Set $x \cdot y = (y^- \oplus x^-)^\sim$ for any $x, y \in A$. We consider that the operation $\cdot$ has priority to the operation $\oplus$, i.e., we will write $x \oplus y \cdot z$ instead of $x \oplus (y \cdot z)$. The algebra A is called a *pseudo MV-algebra* if for any $x, y, z \in A$ the following conditions are satisfied:

(A1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(A2) $x \oplus 0 = 0 \oplus x = x$,

(A3) $x \oplus 1 = 1 \oplus x = 1$,

(A4) $1^\sim = 0, 1^- = 0$,

(A5) $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$,

(A6) $x \oplus x^\sim \cdot y = y \oplus y^\sim \cdot x = x \cdot y^- \oplus y = y \cdot x^- \oplus x$,

(A7) $x \cdot (x^- \oplus y) = (x \oplus y^\sim) \cdot y$,

(A8) $(x^-)^\sim = x$.

Throughout this paper A will denote a pseudo MV-algebra.

As it is shown in [8], if we define

$$x \leqslant y \Longleftrightarrow x^{-} \oplus y = 1,$$

then $(A, \leqslant)$ is a bounded distributive lattice in which the join $x \vee y$ and the meet $x \wedge y$ of any two elements x and y are given by:

$$x \vee y = x \oplus x^{\sim} \cdot y = x \cdot y^{-} \oplus y;$$

$$x \wedge y = x \cdot (x^{-} \oplus y) = (x \oplus y^{\sim}) \cdot y.$$

Definition 2.1. A subset I of A is called an ideal of A if it satisfies:

(I1) $0 \in I$,

(I2) if $x, y \in I$, then $x \oplus y \in I$,

(I3) if $x \in I$, $y \in A$ and $y \leqslant x$, then $y \in I$.

Under this definition, $\{0\}$ and A are the simplest examples of ideals.

Proposition 2.2 (Walendziak [12]). *Let I be a nonempty subset of A. Then I is an ideal of A if and only if I satisfies conditions* (I2) *and*

(I3$'$) *if $x \in I$, $y \in A$, then $x \wedge y \in I$.*

Denote by $\mathcal{I}(A)$ the set of ideals of A and note that $\mathcal{I}(A)$ ordered by set inclusion is a complete lattice.

Remark 2.3. Let $I \in \mathcal{I}(A)$. If $x, y \in I$, then $x \cdot y, x \wedge y, x \vee y \in I$.

For every subset $W \subseteq A$, the smallest ideal of A which contains W, i.e., the intersection of all ideals $I \supseteq W$, is said to be the ideal *generated* by W, and will be denoted by $(W]$. If W is a finite set, then an ideal $(W]$ is said to be *finitely generated*. We will write $(a_1, a_2, \ldots, a_n]$ instead of $(\{a_1, a_2, \ldots, a_n\}]$.

Definition 2.4. Let I be a proper ideal of A (i.e., $I \neq A$). Then I is called *prime* if, for all $I_1, I_2 \in \mathcal{I}(A)$, $I = I_1 \cap I_2$ implies $I = I_1$ or $I = I_2$.

Definition 2.5. An ideal I of A is called *normal* if it satisfies the condition:

(N) for all $x, y \in I$, $x \cdot y^{-} \in I \Longleftrightarrow y^{\sim} \cdot x \in I$.

Following [8], for any normal ideal I of A, we define the congruence on A:

$$x \sim_I y \Longleftrightarrow x \cdot y^- \vee y \cdot x^- \in I.$$

We denote by x/I the congruence class of an element $x \in A$ and on the set $A/I = \{x/I : x \in A\}$ we define the operations:

$$x/I \oplus y/I = (x \oplus y)/I,$$

$$(x/I)^- = (x^-)/I,$$

$$(x/I)^\sim = (x^\sim)/I.$$

The resulting quotient algebra $A/I = (A/I, \oplus, ^-, ^\sim, 0/I, 1/I)$ becomes a pseudo MV-algebra, called *the quotient algebra of A by the normal ideal I.* Observe that for all $x, y \in A$,

$$x/I \cdot y/I = (x \cdot y)/I,$$

$$x/I \vee y/I = (x \vee y)/I,$$

$$x/I \wedge y/I = (x \wedge y)/I.$$

It is clear that:

$$x/I = 0/I \Longleftrightarrow x \in I. \tag{1}$$

For $P \subseteq A$, we set $P/I = \{x/I : x \in P\}$.

Proposition 2.6. *Let I be a normal ideal of A and let $M \subseteq A/I$. Then M is an ideal of A/I if and only if there is an ideal $P \supseteq I$ of A such that $M = P/I$.*

Proof. Assume that M is an ideal of A/I. Let $P = \{x \in A : x/I \in M\}$. Clearly, $P \supseteq I$. Observe that P is an ideal of A. Indeed, let $x, y \in P$. Then $x/I, y/I \in M$. Hence $(x \oplus y)/I = x/I \oplus y/I \in M$. Thus $x \oplus y \in P$ and the condition (I2) is satisfied. Now assume $x \in P$ and $y \in A$. Then $x/I \in M$ and we have $(x \wedge y)/I = x/I \wedge y/I \in M$. So $x \wedge y \in P$ and the condition (I3') is also satisfied. Therefore, by Proposition 2.2, P is an ideal of A. Obviously, $M = P/I$.

Conversely, assume that there is an ideal $P \supseteq I$ of A such that $M = P/I$. Let $a, b \in M$. Then $a = x/I, b = y/I$, where $x, y \in P$. Hence $x \oplus y \in P$.

Thus $a \oplus b = x/I \oplus y/I = (x \oplus y)/I \in P/I = M$ and (I2) is satisfied. Now assume $a \in M$ and $b \in A/I$. Then $a = x/I$, where $x \in P$ and $b = y/I$, where $y \in A$. We have $x \wedge y \in P$ and hence $a \wedge b = x/I \wedge y/I = (x \wedge y)/I \in P/I = M$. So (I3') is also satisfied. Therefore, by Proposition 2.2, M is an ideal of A/I. ∎

Definition 2.7. Let A and B be pseudo MV-algebras. A function $f : A \to B$ is a *homomorphism* if and only if it satisfies, for each $x, y \in A$, the following conditions:

(H1) $f(0) = 0$,

(H2) $f(x \oplus y) = f(x) \oplus f(y)$,

(H3) $f(x^-) = (f(x))^-$,

(H4) $f(x^\sim) = (f(x))^\sim$.

Remark 2.8. We also have for any $x, y \in A$:

(a) $f(1) = 1$,

(b) $f(x \cdot y) = f(x) \cdot f(y)$,

(c) $f(x \vee y) = f(x) \vee f(y)$,

(d) $f(x \wedge y) = f(x) \wedge f(y)$.

The *kernel* of a homomorphism $f : A \to B$ is the set

$$\mathrm{Ker}(f) = \{x \in A : f(x) = 0\}.$$

Note that Ker(f) is an ideal of a pseudo MV-algebra A.

Proposition 2.9 (Georgescu and Iorgulescu [7]). *A homomorphism $f : A \to B$ is injective if and only if* $\mathrm{Ker}(f) = \{0\}$.

Now we review some fuzzy logic concepts. First, for $\Gamma \subseteq [0,1]$ we define $\bigwedge \Gamma = \inf \Gamma$ and $\bigvee \Gamma = \sup \Gamma$. Obviously, if $\Gamma = \{\alpha, \beta\}$, then $\alpha \wedge \beta = \min\{\alpha, \beta\}$ and $\alpha \vee \beta = \max\{\alpha, \beta\}$. Recall that a fuzzy set in A is a function $\mu : A \to [0,1]$.

Definition 2.10. A fuzzy set μ in a pseudo MV-algebra A is called a *fuzzy ideal* of A if it satisfies:

(d1) $\mu(x \oplus y) \geqslant \mu(x) \wedge \mu(y)$ for all $x, y \in A$,

(d2) for all $x, y \in A$, if $y \leqslant x$, then $\mu(y) \geqslant \mu(x)$.

It is easily seen that (d2) implies

(d3) $\mu(0) \geqslant \mu(x)$ for all $x \in A$.

Proposition 2.11 (Jun and Walendziak [9]). *Let μ be a fuzzy set in A. Then μ is a fuzzy ideal of A if and only if it satisfies* (d1) *and*

(d4) $\mu(x \wedge y) \geqslant \mu(x)$ for all $x, y \in A$.

It is shown in [9] that if I is an ideal of A, then a fuzzy set

$$\mu_I(x) = \begin{cases} \alpha & \text{if } x \in I, \\ \beta & \text{otherwise,} \end{cases}$$

where $\alpha, \beta \in [0,1]$ with $\alpha > \beta$, is a fuzzy ideal of A. In particular, we have that the characteristic function

$$\chi_I(x) = \begin{cases} 1 & \text{if } x \in I, \\ 0 & \text{otherwise} \end{cases}$$

of an ideal I of A is a fuzzy ideal of A.

Proposition 2.12 (Jun and Walendziak [9]). *Let μ be a fuzzy set in A. Then μ is a fuzzy ideal of A if and only if its level subset*

$$U(\mu; \alpha) = \{x \in A : \mu(x) \geqslant \alpha\}$$

is empty or is an ideal of A for all $\alpha \in [0,1]$.

Proposition 2.13. *Let μ be a fuzzy ideal of A. Then the set $P(\mu) = \{x \in A : \mu(x) > 0\}$ is an ideal of A when it is nonempty.*

Proof. Assume that μ is a fuzzy ideal of A such that $P(\mu) \neq \varnothing$. Obviously, $0 \in P(\mu)$. Let $x, y \in A$ be such that $x, y \in P(\mu)$. Then $\mu(x) > 0$ and $\mu(y) > 0$. It follows from (d1) that $\mu(x \oplus y) \geqslant \mu(x) \wedge \mu(y) > 0$ so that $x \oplus y \in P(\mu)$. Now, let $x, y \in A$ be such that $x \in P(\mu)$ and $y \leqslant x$. Then, by (d2), we have $\mu(y) \geqslant \mu(x)$, and since $\mu(x) > 0$, we obtain $\mu(y) > 0$. So $y \in P(\mu)$. Thus $P(\mu)$ is an ideal of A. ∎

3. Noetherian and Artinian pseudo MV-algebras

In this section we study Noetherian pseudo MV-algebras and Artinian pseudo MV-algebras. Although some of theorems presented below are well-known (in the theory of rings, for example), we give their proofs. Let's start from some definitions.

Definition 3.1. A pseudo MV-algebra A satisfies the *maximal condition* if each nonempty set of ideals of A has a maximal element.

Definition 3.2. A pseudo MV-algebra A is said to satisfy the *ascending chain condition* if for every ascending sequence $I_1 \subseteq I_2 \subseteq \cdots$ of ideals of A there exists $k \in \mathbb{N}$ such that $I_n = I_k$ for all $n \geqslant k$.

Definition 3.3. A pseudo MV-algebra A is called *Noetherian* if it satisfies the ascending chain condition.

Now we have two simple theorems. The first of them characterizes Noetherian pseudo MV-algebras and the second describes ideals of such pseudo MV-algebras.

Theorem 3.4. *Let A be a pseudo MV-algebra. The following conditions are equivalent:*

(a) *A is Noetherian,*

(b) *A satisfies the maximal condition,*

(c) *each ideal of A is finitely generated.*

Proof. (a) $\Rightarrow$ (b): Assume that A is Noetherian. Then A satisfies the ascending chain condition. Let $\mathfrak{I}$ be any nonempty set of ideals of A and suppose that $\mathfrak{I}$ has no maximal element. Take $I_1 \in \mathfrak{I}$. Since I_1 is not a maximal element of $\mathfrak{I}$, there exists an ideal $I_2 \in \mathfrak{I}$ such that $I_1 \subset I_2$. Repeating the argument we obtain a strictly ascending sequence $I_1 \subset I_2 \subset \cdots$ of ideals of A, which is a contradiction. Therefore A satisfies the maximal condition.

(b) $\Rightarrow$ (c): Assume that A satisfies the maximal condition and I is any ideal of A. Let $\mathfrak{I}$ be the set of all finitely generated ideals of A contained in I. The set $\mathfrak{I}$ is nonempty, because $\{0\} \in \mathfrak{I}$. By the maximal condition, $\mathfrak{I}$ has a maximal element. Denote it by I_1. Suppose that $I_1 \neq I$ and I_1 is generated by the elements $a_1, a_2, \ldots, a_n$. Then there exists an element $b \in I$ such that $b \notin I_1$. Let I_2 be an ideal generated by $a_1, a_2, \ldots, a_n, b$. Then $I_1 \subset I_2$ and $I_2 \in \mathfrak{I}$. This contradicts the maximality of I_1. Hence $I_1 = I$ and therefore I is finitely generated.

(c) $\Rightarrow$ (a): Assume that each ideal of A is finitely generated. Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending sequence of ideals of A. Then it is clear that $I = \bigcup_{k=1}^{\infty} I_k$ is a finitely generated ideal of A. Let $a_1, a_2, \ldots, a_n \in A$ be the generators of I. This means that $a_k \in I_{m_k}$ for some $m_k \in \mathbb{N}$ and $k = 1, 2, \ldots, n$. Let $m = \max\{m_1, m_2, \ldots, m_n\}$. Then $a_k \in I_m$ for $k = 1, 2, \ldots, n$. Since I is the minimal ideal containing a_k for $k = 1, 2, \ldots, n$, it follows that $I \subseteq I_m$. Thus $I_k = I_m$ for all $k \geqslant m$. Therefore A satisfies the ascending chain condition and so it is Noetherian. ■

Theorem 3.5. *Let A be Noetherian. Then every ideal of A can be written as the intersection of a finite number of prime ideals.*

Proof. Assume that A is Noetherian. Let $\mathfrak{I}$ be the set of all ideals of A, which cannot be written as the intersection of a finite number of prime ideals. Assume that $\mathfrak{I}$ is nonempty. Since A is Noetherian, we have, by Theorem 3.4, that the set $\mathfrak{I}$ has a maximal element. Denote it by I. Then, since I cannot be written as the intersection of a finite number of prime ideals, it is not prime. Thus we have $I = I_1 \cap I_2$, where I_1 and I_2 are ideals of A such that $I \subsetneq I_1$ and $I \subsetneq I_2$. Then $I_1, I_2 \notin \mathfrak{I}$. Hence I_1, I_2 can both be written as the intersection of a finite number of prime ideals. Thus the same is true for I, which is a contradiction. Therefore $\mathfrak{I}$ is empty and theorem is proved. ■

Theorem 3.6. *If A is Noetherian, then A/I is Noetherian for every normal ideal I of A.*

Proof. Let A be Noetherian and I be a normal ideal of A. Let $J_1 \subseteq J_2 \subseteq \cdots$ be an ascending sequence of ideals of A/I. If $p : A \to A/I$ is the canonical epimorphism, then $p^{-1}(J_1) \subseteq p^{-1}(J_2) \subseteq \cdots$ is an ascending sequence of ideals of A. Since A is Noetherian, there is $k \in \mathbb{N}$ such that $p^{-1}(J_n) = p^{-1}(J_k)$ for all $n \geqslant k$. Then $J_n = p(p^{-1}(J_n)) = p(p^{-1}(J_k)) = J_k$ for all $n \geqslant k$, because p is surjective. Thus A/I satisfies the ascending chain condition and so it is Noetherian. ∎

Theorem 3.7. *If A is Noetherian and $f : A \to A$ is a surjective homomorphism, then f is injective.*

Proof. Assume that A is Noetherian and $f : A \to A$ is a surjective homomorphism. Suppose that f is not injective. Then, by Proposition 2.9, $\mathrm{Ker}(f) \neq \{0\}$. It is obvious that

$$\mathrm{Ker}(f) \subseteq \mathrm{Ker}\left(f^2\right) \subseteq \cdots, \text{ where } f^k = f \circ f \circ \cdots \circ f \ (k \text{ times}).$$

We claim that $\mathrm{Ker}(f^n) \neq \mathrm{Ker}\big(f^{2n}\big)$ for all $n \in \mathbb{N}$. Indeed, suppose that $\mathrm{Ker}(f^n) = \mathrm{Ker}\big(f^{2n}\big)$. Let $y \in \mathrm{Ker}(f^n)$. Since f^n is surjective, there exists $x \in A$ such that $y = f^n(x)$. So $0 = f^n(y) = (f^n \circ f^n)(x) = f^{2n}(x)$, which implies that $x \in \mathrm{Ker}\big(f^{2n}\big) = \mathrm{Ker}(f^n)$. Thus $y = f^n(x) = 0$, i.e., $\mathrm{Ker}(f^n) = \{0\}$. This means that also $\mathrm{Ker}(f) = \{0\}$, which is a contradiction. Therefore we have a strictly ascending sequence

$$\mathrm{Ker}(f) \subset \mathrm{Ker}\left(f^2\right) \subset \cdots$$

of ideals of A. This cannot happen, because A is Noetherian. Thus f is injective. ∎

Now we define and investigate Artinian pseudo MV-algebras.

Definition 3.8. A pseudo MV-algebra A satisfies the *minimal condition* if each nonempty set of ideals of A has a minimal element.

Definition 3.9. A pseudo MV-algebra A is said to satisfy the *descending chain condition* if for every descending sequence $I_1 \supseteq I_2 \supseteq \cdots$ of ideals of A there exists $k \in \mathbb{N}$ such that $I_n = I_k$ for all $n \geqslant k$.

Definition 3.10. A pseudo MV-algebra A is called *Artinian* if it satisfies the descending chain condition.

The following simple theorem characterizes Artinian pseudo MV-algebras.

Theorem 3.11. *Let A be a pseudo MV-algebra. Then the following conditions are equivalent:*

(a) *A is Artinian,*

(b) *A satisfies the minimal condition.*

Proof. (a) $\Rightarrow$ (b): Assume that A is Artinian. Then A satisfies the descending chain condition. Let $\mathcal{I}$ be any nonempty set of ideals of A and suppose that $\mathcal{I}$ has no minimal element. Take $I_1 \in \mathcal{I}$. Since I_1 is not a minimal element of $\mathcal{I}$, there exists an ideal $I_2 \in \mathcal{I}$ such that $I_2 \subset I_1$. Repeating the argument we obtain a strictly descending sequence $I_1 \supset I_2 \supset \cdots$ of ideals of A, which is a contradiction. Therefore A satisfies the minimal condition.

(b) $\Rightarrow$ (a): Assume that A satisfies the minimal condition. Let $I_1 \supseteq I_2 \supseteq \cdots$ be a descending sequence of ideals of A. Then the set $\{I_n : n = 1, 2, \ldots\}$ of ideals has a minimal element. Denote it by I_k. Hence we have that $I_n = I_k$ for all $n \geqslant k$. Thus A satisfies the descending chain condition, i.e., it is Artinian. ■

Theorem 3.12. *If A is Artinian, then A/I is Artinian for every normal ideal I of A.*

Proof. Let A be Artinian and I be a normal ideal of A. Let $J_1 \supseteq J_2 \supseteq \cdots$ be a descending sequence of ideals of A/I. If $p : A \to A/I$ is the canonical epimorphism, then $p^{-1}(J_1) \supseteq p^{-1}(J_2) \supseteq \cdots$ is a descending sequence of ideals of A. Since A is Artinian, there is $k \in \mathbb{N}$ such that $p^{-1}(J_n) = p^{-1}(J_k)$ for all $n \geqslant k$. Since p is the canonical epimorphism, $J_n = J_k$ for all $n \geqslant k$. Thus A/I satisfies the descending chain condition and so it is Artinian. ■

Definition 3.13. A pseudo MV-algebra A is *finitely cogenerated* if for every family $\{I_j : j \in J\}$ of ideals of A such that $\bigcap_{j\in J} I_j = \{0\}$ there exists a finite subset K of J such that $\bigcap_{j\in K} I_j = \{0\}$.

Theorem 3.14. *If A satisfies the minimal condition, then A/I is finitely cogenerated for every normal ideal I of A.*

Proof. Assume that A satisfies the minimal condition and I is a normal ideal of A. We have to prove that for every family $\{M_j : j \in J\}$ of ideals of A/I such that $\bigcap_{j\in J} M_j = \{0/I\}$ there exists a finite subset K of J such that $\bigcap_{j\in J} M_j = \{0/I\}$. By Proposotion 2.6, $M_j = I_j/I$ for some ideal I_j of A containing I. Since $\bigcap_{j\in J} M_j = \{0/I\}$, we have $\bigcap_{j\in J} I_j = I$ by (1). Let $\mathcal{I} = \{\bigcap_{l\in L} I_l : L \subseteq J \text{ is finite}\}$. Then $\mathcal{I}$ is a nonempty family of ideals of A. Since A satisfies the minimal condition, $\mathcal{I}$ has a minimal element. Let $\bigcap_{k\in K} I_k$, where $K \subseteq J$ and K is finite, be this minimal element. We have $I \subseteq \bigcap_{k\in K} I_k$. Suppose that $\bigcap_{k\in K} I_k \neq I$. Then we can find x such that $x \in I_k$ for all $k \in K$ and $x \notin I_m$ for some $m \in J - K$. But $K \cup \{m\}$ is a finite subset of J and $\left(\bigcap_{k\in K} I_k\right) \cap I_m \subseteq \bigcap_{k\in K} I_k$. Hence, by the minimality of $\bigcap_{k\in K} I_k$, we obtain $\left(\bigcap_{k\in K} I_k\right) \cap I_m = \bigcap_{k\in K} I_k$, that is $\bigcap_{k\in K} I_k \subseteq I_m$. So $x \in I_m$, which is a contradiction. Thus $\bigcap_{k\in K} I_k = I$ and hence $\bigcap_{j\in K} M_j = \{0/I\}$. Therefore A/I is finitely cogenerated. ■

By Theorems 3.11 and 3.14, we have the following corollary.

Corollary 3.15. *If A is Artinian, then A/I is finitely cogenerated for every normal ideal I of A.*

Since $\{0\}$ is the trivial normal ideal of a pseudo MV-algebra A and we can associate $A/\{0\}$ with A, we obtain the following corollary.

Corollary 3.16. *If A is Artinian, then it is finitely cogenerated.*

We shall end this section with two examples.

Example 3.17. Let $A = \{(1, y) \in \mathbb{R}^2 : y \geqslant 0\} \cup \{(2, y) \in \mathbb{R}^2 : y \geqslant 0\}$, $\mathbf{0} = (1, 0)$, $\mathbf{1} = (2, 0)$. For any $(a, b), (c, d) \in A$, we define operations $\oplus, ^-, ^\sim$ as follows:

$$(a,b) \oplus (c,d) = \begin{cases} (1, b+d) & \text{if } a = c = 1, \\ (2, ad+b) & \text{if } ac = 2 \text{ and } ad + b \leqslant 0, \\ (2, 0) & \text{in other cases,} \end{cases}$$

$$(a,b)^{-} = \left(\frac{2}{a}, -\frac{2b}{a}\right),$$

$$(a,b)^{\sim} = \left(\frac{2}{a}, -\frac{b}{a}\right).$$

Then $A = (A, \oplus, ^{-}, ^{\sim}, \mathbf{0}, \mathbf{1})$ is a pseudo MV-algebra. Let $I = \{(1,y) \in \mathbb{R}^2 : y \geqslant 0\}$. Then I is the unique proper ideal of A. Therefore A is Noetherian as well as Artinian pseudo MV-algebra.

Example 3.18. Let B be the set of all increasing bijective functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x \leqslant f(x) \leqslant x+1 \text{ for all } x \in \mathbb{R}.$$

Define the operations $\oplus, ^{-}, ^{\sim}$ and constants 0 and 1 as follows:

$$\begin{aligned}
(f \oplus g)(x) &= \min\{f(g(x)), x+1\}, \\
f^{-}(x) &= f^{-1}(x) + 1, \\
f^{\sim}(x) &= f^{-1}(x+1), \\
0(x) &= x, \\
1(x) &= x+1.
\end{aligned}$$

Then $B = (B, \oplus, ^{-}, ^{\sim}, 0, 1)$ is a pseudo MV-algebra. Note that for an arbitrary $r \in \mathbb{R}$ a set

$$I_r = \{f \in B : f(r) = r\}$$

is an ideal of B (see [2], [3]). Now take a set

$$J_s = \{f \in B : f(x) = x \text{ for all } x \geqslant s\}$$

for any $s \in \mathbb{R}$. It is easy to see that

$$J_s = \bigcap_{r \geqslant s} I_r,$$

i.e., J_s is the ideal of B for any $s \in \mathbb{R}$. Moreover, we have that $J_{s_1} \subseteq J_{s_2}$ for $s_1 \leqslant s_2$. Now we can take an ascending sequence

$$J_1 \subseteq J_2 \subseteq \cdots$$

of ideals of B which does not stop in any time. Thus B is not Noetherian. Observe also that if we take a descending sequence

$$J_{-1} \supseteq J_{-2} \supseteq \cdots$$

of ideals of B, then it never stops, and therefore B is not Artinian as well.

4. Fuzzy characterizations of Noetherian and Artinian pseudo MV-algebras

In this section we characterize Noetherian pseudo MV-algebras and Artinian pseudo MV-algebras using some fuzzy concepts, in particular, fuzzy ideals.

Theorem 4.1. *Let A be a pseudo MV-algebra. The following statements are equivalent:*

(a) *A is Noetherian,*

(b) *for each fuzzy ideal μ of A, $\mathrm{Im}(\mu) = \{\mu(x) : x \in A\}$ is a well-ordered set.*

Proof. (a) $\Rightarrow$ (b): Assume that A is Noetherian and μ is a fuzzy ideal of A such that $\mathrm{Im}(\mu)$ is not a well-ordered subset of $[0,1]$. Then there exists a strictly decreasing sequence $\{\mu(x_n)\}$, where $x_n \in A$. Let $t_n = \mu(x_n)$ and $U_n = U(\mu; t_n) = \{x \in A : \mu(x) \geqslant t_n\}$. Then, by Proposition 2.12, U_n is an ideal of A for every $n \in \mathbb{N}$. So $U_1 \subset U_2 \subset \cdots$ is a strictly ascending sequence of ideals of A. This contradicts the assumption that A is Noetherian. Therefore $\mathrm{Im}(\mu)$ is a well-ordered set for each fuzzy ideal μ of A.

(b) $\Rightarrow$ (a): Assume that the condition (b) is satisfied and A is not Noetherian. Then there exists a strictly ascending sequence

$$I_1 \subset I_2 \subset \cdots$$

of ideals of A. Let μ be a fuzzy set in A such that

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin I_n \text{ for every } n \in \mathbb{N}, \\ \frac{1}{k} & \text{if } x \in I_k - I_{k-1} \text{ for } k = 1, 2, \ldots, \end{cases}$$

where $I_0 = \varnothing$. We show that μ is a fuzzy ideal of A. We begin by proving that μ satisfies (d1). Let $x, y \in A$. We have three cases.

Case 1. $x \notin I_n$ for all $n \in \mathbb{N}$ or $y \notin I_n$ for all $n \in \mathbb{N}$.
Then $\mu(x) = 0$ or $\mu(y) = 0$. Thus $\mu(x \oplus y) \geqslant \mu(x) \wedge \mu(y) = 0$.

Case 2. $x \in I_k - I_{k-1}$ and $y \in I_l - I_{l-1}$ for $k \geqslant l$.
Then $\mu(x) = \frac{1}{k} \leqslant \mu(y) = \frac{1}{l}$. Since $x, y \in I_k$, $x \oplus y \in I_k$. Hence $\mu(x \oplus y) \geqslant \frac{1}{k} = \mu(x) = \mu(x) \wedge \mu(y)\}$.

Case 3. $x \in I_k - I_{k-1}$ and $y \in I_l - I_{l-1}$ for $k \leqslant l$.
Analogous.

Therefore (d1) is satisfied. Now, we prove that μ satisfies (d4). Let $x, y \in A$. We have two cases.

Case 1. $x \notin I_n$ for all $n \in \mathbb{N}$.
Then $\mu(x \wedge y) \geqslant \mu(x) = 0$.

Case 2. $x \in I_k - I_{k-1}$ for some $k = 1, 2, \ldots$.
Then $\mu(x) = \frac{1}{k}$. Since $x \wedge y \leqslant x$, we have $x \wedge y \in I_k$ and so $\mu(x \wedge y) \geqslant \frac{1}{k} = \mu(x)$.

Therefore (d4) is also satisfied. Thus, by Proposition 2.11, μ is a fuzzy ideal of A, but $\mathrm{Im}(\mu)$ is not a well-ordered set, which is a contradiction. Hence A is Noetherian. ∎

Corollary 4.2. *Let A be a pseudo MV-algebra. If for every fuzzy ideal μ of A, $\mathrm{Im}(\mu)$ is a finite set, then A is Noetherian.*

Theorem 4.3. *Let A be a pseudo MV-algebra and let $T = \{t_1, t_2, \ldots\} \cup \{0\}$, where $\{t_n\}$ is a strictly decreasing sequence in $(0, 1)$. Then the following conditions are equivalent:*

(a) *A is Noetherian,*

(b) *for each fuzzy ideal μ of A, if* $\mathrm{Im}(\mu) \subseteq T$*, then there exists $k \in \mathbb{N}$ such that* $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0\}$.

Proof. (a) $\Rightarrow$ (b): Assume that A is Noetherian. Let μ be a fuzzy ideal of A such that $\mathrm{Im}(\mu) \subseteq T$. From Theorem 4.1 we know that $\mathrm{Im}(\mu)$ is a well-ordered subset of $[0, 1]$. Thus there exists $k \in \mathbb{N}$ such that $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{0\}$.

(b) $\Rightarrow$ (a): Assume that (b) is true. Suppose that A is not Noetherian. Then there exists a strictly ascending sequence

$$I_1 \subset I_2 \subset \cdots .$$

of ideals of A. Let $I = \bigcup_{n \in \mathbb{N}} I_n$. Then I is an ideal of A. Define a fuzzy set μ in A by

$$\mu(x) = \begin{cases} 0 & \text{if } x \notin I, \\ t_m & \text{where } m = \min\{n \in \mathbb{N} : x \in I_n\}. \end{cases}$$

It is easy to see that μ is a fuzzy ideal of A. This contradicts our assumption. Thus A is Noetherian. ∎

The following theorem characterizing Artinian pseudo MV-algebras is dual to Theorem 4.3. Therefore its proof is left to the reader.

Theorem 4.4. *Let A be a pseudo MV-algebra and let* $T = \{t_1, t_2, \ldots\} \cup \{1\}$*, where $\{t_n\}$ is a strictly increasing sequence in $(0, 1)$. Then the following conditions are equivalent:*

(a) *A is Artinian,*

(b) *for each fuzzy ideal μ of A, if* $\mathrm{Im}(\mu) \subseteq T$*, then there exists $k \in \mathbb{N}$ such that* $\mathrm{Im}(\mu) \subseteq \{t_1, t_2, \ldots, t_k\} \cup \{1\}$.

Corollary 4.5. *Let A be a pseudo MV-algebra. If for every fuzzy ideal μ of A,* $\mathrm{Im}(\mu)$ *is a finite set, then A is Artinian.*

Acknowledgements

The author wishes to express his thanks to Professor A. Walendziak for his several helpful comments.

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Received 5 Februay 2008
Revised 9 April 2008
Revised 22 April 2008