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WREATH PRODUCT OF A SEMIGROUP AND A Γ-SEMIGROUP

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Abstract

Let $S = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two nonempty sets. S is called a Γ-semigroup if $a\alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$. In this paper we study the semidirect product of a semigroup and a Γ-semigroup. We also introduce the notion of wreath product of a semigroup and a Γ-semigroup and investigate some interesting properties of this product.

Keywords: semigroup, Γ-semigroup, orthodox semigroup, right(left) orthodox Γ-semigroup, right(left) inverse semigroup, right(left) inverse Γ-semigroup, right(left)α- unity, Γ-group, semidirect product, wreath product.

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1. Introduction

The notion of a Γ-semigroup has been introduced by Sen and Saha [7] in the year 1986. Many classical notions of semigroup have been extended to Γ-semigroup. In [1] and [2] we have introduced the notions of right inverse Γ-semigroup and right orthodox Γ-semigroup. In [6] we have studied the semidirect product of a monoid and a Γ-semigroup as a generalization of [4] and [5]. We have obtained necessary and sufficient conditions for a semidirect product of the monoid and a Γ-semigroup to be right (left) orthodox Γ-semigroup and right (left) inverse Γ-semigroup. In [9] Zhang has studied the semidirect product of semigroups and also studied wreath product of semigroups. In this paper we generalize the results of Zhang to the semidirect product of a semigroup and a Γ-semigroup. We also study the wreath product of a semigroup and a Γ-semigroup.

2. Preliminaries

We now recall some definitions and results relating our discussion.

Definition 2.1. Let $S = \{a, b, c, \dots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \dots\}$ be two nonempty sets. S is called a Γ-semigroup if

(i) $a\alpha b \in S$, for all $\alpha \in \Gamma$ and $a, b \in S$ and

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Let S be an arbitrary semigroup. Let 1 be a symbol not representing any element of S. We extend the binary operation defined on S to $S \cup \{1\}$ by defining $11 = 1$ and $1a = a1 = a$ for all $a \in S$. It can be shown that $S \cup \{1\}$ is a semigroup with identity element 1. Let $\Gamma = \{1\}$. If we take $ab = a1b$, it can be shown that the semigroup S is a $\Gamma-$semigroup where $\Gamma = \{1\}$. Thus a semigroup can be considered to be a Γ-semigroup.

Let S be a Γ-semigroup and x be a fixed element of Γ. We define $a.b = axb$ for all $a, b \in S$. We can show that $(S, .)$ is a semigroup and we denote this semigroup by S_x.

Definition 2.2. Let S be a Γ-semigroup. An element $a \in S$ is said to be regular if $a \in a\Gamma S\Gamma a$ where $a\Gamma S\Gamma a = \{a\alpha b\beta a : b \in S, \alpha, \beta \in \Gamma\}$. S is said to be regular if every element of S is regular.

We now describe some examples of regular Γ-semigroup.

In [7] we find the following interesting example of a regular Γ-semigroup.

Example 2.3. Let S be the set of all 2×3 matrices and Γ be the set of all 3×2 matrices over a field. Then for all $A, B, C \in S$ and $P, Q \in \Gamma$ we have $APB \in S$ and since the matrix multiplication is associative, we have $(APB)QC = AP(BQC)$. Hence S is a Γ-semigroup. Moreover it is regular shown in [7].

Here we give another example of a regular Γ-semigroup.

Example 2.4. Let S be a set of all negative rational numbers. Obviously S is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} : p \text{ is prime }\}$. Let $a, b, c \in S$ and $\alpha, \beta \in \Gamma$. Now if $a\alpha b$ is equal to the usual product of rational numbers a, α, b, then $a\alpha b \in S$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$. Hence S is a Γ-semigroup. Let $a = \frac{m}{n} \in \Gamma$ where $m > 0$ and $n < 0$. $m = p_1 p_2 ... p_k$ where p_i's are prime. $\frac{p_1 p_2 \cdots p_k}{n}(-\frac{1}{p_1})\frac{n}{p_2 \cdots p_{k-1}}(-\frac{1}{p_k})\frac{m}{n} = \frac{p_1 p_2 \cdots p_k}{n}$. Thus taking $b = \frac{n}{p_2 \cdots p_{k-1}}$, $\alpha = (-\frac{1}{p_1})$ and $\beta = (-\frac{1}{p_k})$ we can say that a is regular. Hence S is a regular Γ-semigroup.

Definition 2.5 [7]**.** Let S be a Γ-semigroup and $\alpha \in \Gamma$. Then $e \in S$ is said to be an α-idempotent if $e\alpha e = e$. The set of all α-idempotents is denoted by E_α. We denote $\bigcup_{\alpha \in \Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent elements of S.

Definition 2.6 [7]**.** Let $a \in M$ and $\alpha, \beta \in \Gamma$. An element $b \in M$ is called an (α, β)-inverse of a if $a = a\alpha b\beta a$ and $b = b\beta a\alpha b$. In this case we write $b \in V_\alpha^\beta(a)$.

Definition 2.7 [2]**.** A regular Γ-semigroup M is called a right (left) orthodox Γ-semigroup if for any α-idempotent e and β-idempotent f, $e\alpha f$ (resp. $f\alpha e$) is a β-idempotent.

Example 2.8 [2]**.** Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. S denotes the set of all mappings from A to B. Here members of S are described by the images of the elements 1, 2, 3. For example the map $1 \to 4, 2 \to 5, 3 \to 4$ is written as $(4, 5, 4)$ and $(5, 5, 4)$ denotes the map $1 \to 5, 2 \to 5, 3 \to 4$. A map from B to A

is described in the same fashion. For example $(1,2)$ denotes $4 \to 1, 5 \to 2$. Now $S = \Big\{(4,4,4),(4,4,5),(4,5,4),(4,5,5),(5,5,5),(5,4,5),\ (5,4,4),\ (5,5,4)\Big\}$ and let $\Gamma = \{(1,1),(1,2),(2,3),(3,1)\}$. Let $f, g \in S$ and $\alpha \in \Gamma$. We define $f\alpha g$ by $(f\alpha g)(a) = f\alpha\Big(g(a)\Big)$ for all $a \in A$. So $f\alpha g$ is a mapping from A to B and hence $f\alpha g \in S$ and we can show that $(f\alpha g)\beta h = f\alpha(g\beta h)$ for all $f, g, h \in S$ and $\alpha, \beta \in \Gamma$. We can show that each element x of S is an α-idempotent for some $\alpha \in \Gamma$ and hence each element is regular. Thus S is a regular Γ-semigroup. It is an idempotent Γ-semigroup. Moreover we can show that it is a right orthodox Γ-semigroup.

Theorem 2.9 [2]**.** *A regular Γ-semigroup M is a right orthodox Γ-semigroup if and only if for $a, b \in M, V_{\alpha_1}^{\beta}(a) \cap V_{\alpha}^{\beta}(b) \neq \phi$ for some $\alpha, \alpha_1, \beta \in \Gamma$ implies that $V_{\alpha_1}^{\delta}(a) = V_{\alpha}^{\delta}(b)$ for all $\delta \in \Gamma$.*

Definition 2.10 [1]**.** A regular Γ-semigroup is called a right (left) inverse Γ-semigroup if for any α-idempotent e and for any β-idempotent f, $e\alpha f\beta e = f\beta e$ ($e\beta f\alpha e = e\beta f$).

Theorem 2.11 [7]**.** *Let S be a Γ-semigroup. If S_α is a group for some $\alpha \in S$ then S_α is a group for all $\alpha \in \Gamma$.*

Definition 2.12 [7]**.** A Γ-semigroup S is called a Γ-group if S_α is a group for some $\alpha \in \Gamma$.

Definition 2.13 [8]**.** A regular semigroup S is said to be a right (left) inverse semigroup if for any $e, f \in E(S)$, $efe = fe(efe = ef)$.

Definition 2.14 [3]**.** A semigroup S is called orthodox semigroup if it is regular and the set of all idempotents forms a subsemigroup.

Definition 2.15 [7]**.** A nonempty subset I of a Γ-semigroup S is called a right (resp. left) ideal if $I\Gamma S \subseteq I$(resp. $S\Gamma I \subseteq I$). If I is both a right ideal and a left ideal then we say that I is an ideal of S.

Definition 2.16 [7]**.** A Γ-semigroup S is called right (resp. left) simple if it contains no proper right (resp. left) ideal i.e, for every $a \in S, a\Gamma S = S$(resp. $S\Gamma a = S$). A Γ- semigroup is said to be simple if it has no proper ideals.

Theorem 2.17 [7]**.** *Let S be a Γ- semigroup. S is a Γ- group if and only if it is both left simple and right simple.*

3. Semidirect product of a semigroup and a Γ-semigroup

Let S be a semigroup and T be a Γ-semigroup. Let $End(T)$ denote the set of all endomorphisms on T i.e., the set of all mappings $f : T \to T$ satisfying $f(a\alpha b) = f(a)\alpha f(b)$ for all $a, b \in T$, $\alpha \in \Gamma$. Clearly $End(T)$ is a semigroup. Let $\phi : S \not\to End(T)$ be a given antimorphism i.e, $\phi(sr) = \phi(r)\phi(s)$ for all $r, s \in S$. If $s \in S$ and $t \in T$, we write t^s for $(\phi(s))(t)$ and $T^s = \{t^s : t \in T\}$. Let $S \times_\phi T = \{(s, t) : s \in S, t \in T\}$. We define $(s_1, t_1)\, \alpha(s_2, t_2) = (s_1 s_2, t_1^{s_2}\, \alpha t_2)$ for all $(s_i, t_i) \in S \times_\phi T$ and $\alpha \in \Gamma$. Then $S \times_\phi T$ is a Γ-semigroup. This Γ-semigroup $S \times_\phi T$ is called the semidirect product of the semigroup S and the Γ-semigroup T. In [6] we have studied the semidirect product $S \times_\phi T$ assuming that S is a monoid. In this paper we investigate the properties of the semidirect product $S \times_\phi T$ without taking 1 in S.

Lemma 3.1. *Let $S \times_\phi T$ be a semidirect product of a semigroup S and a Γ-semigroup T. Then*

(i) $(t\alpha u)^s = t^s \alpha u^s$ *for all $s \in S$, $t, u \in T$ and $\alpha \in \Gamma$.*

(ii) $(t^s)^r = (t)^{sr}$ *for all $s, r \in S$ and $t \in T$.*

Proof. Let $s, r \in S$, $\alpha \in \Gamma$ and $t, u \in T$. Now $(t\alpha u)^s = (\phi(s))(t\alpha u) = (\phi(s))(t)\alpha(\phi(s))(u) = t^s \alpha u^s$ Hence (i) follows. Again $(t^s)^r = (\phi(r))(t^s) = (\phi(r))((\phi(s))(t)) = (\phi(r)\phi(s))(t) = (\phi(sr))(t) = (t)^{sr}$. Thus (ii) follows.

Theorem 3.2. *Let $S \times_\phi T$ be a semidirect product of a semigroup S and a Γ-semigroup T. Then T^x is a Γ-semigroup for all $x \in S$ where $T^x = \{t^x : t \in T\}$. If moreover $S \times_\phi T$ is a regular Γ - semigroup then S is a regular semigroup and T^e is a regular Γ-semigroup for all $e \in E(S)$.*

Proof. The first part is clear from the above lemma. Let $S \times_\phi T$ be regular. For $(s, t) \in S \times_\phi T$, there exist $(s', t') \in S \times_\phi T$ and $\alpha, \beta \in \Gamma$ such that $(s, t) = (s, t)\alpha(s', t')\beta(s, t) = (ss's, t^{s's}\alpha(t')^s\beta t)$ and $(s', t') = (s', t')\beta(s, t)\alpha(s', t') = (s'ss', (t')^{ss'}\beta t^{s'}\alpha t')$. This implies $s' \in V(s)$. Let $e \in E(S)$, then for (e, t^e), there exist $(s', t') \in S \times_\phi T$ and $\alpha, \beta \in \Gamma$ such that $(e, t^e) = (e, t^e)\alpha(s', t')\beta(e, t^e) = (es'e, t^{es'e}\alpha t'^e\beta t^e)$ and $(s', t') = (s', t')\beta(e, t^e)\alpha(s', t') = (s'es', (t')^{es'}\beta t^{es'}\alpha t')$. Hence $s' \in V(e)$ and we have $t^e = t^e\alpha t'^e\beta t^e$ and $t'^e = t'^e\beta t^e\alpha t'^e$. i.e, $t'^e \in V_\alpha^\beta(t^e)$. Hence T^e is a regular Γ-semigroup.

Theorem 3.3. *Let S be a semigroup and T be a Γ-semigroup, $\phi : S \not\rightarrow End(T)$ be a given antimorphism. If the semidirect product $S \times_\phi T$ is*

(i) *a right (left) orthodox Γ-semigroup then S is an orthodox semigroup and T^e is a right (left) orthodox Γ-semigroup for every idempotent $e \in S$,*

(ii) *a right (left) inverse Γ-semigroup then S is a right (left) inverse semigroup and T^e is a right (left) inverse Γ-semigroup.*

Proof.

(i) Let $S \times_\phi T$ be a right orthodox Γ-semigroup. Let $e, g \in E(S)$ and t^e be an α-idempotent and u^e be a β-idempotent in T^e. Then $(e, t^e)\alpha(e, t^e) = (e, t^e\alpha t^e) = (e, t^e)$, i.e., (e, t^e) is an α-idempotent. Similarly (e, u^e) is a β-idempotent. Again $(g, u^{eg})\beta(g, u^{eg}) = (g, u^{eg}\beta u^{eg}) = (g, (u^e\beta u^e)^g) = (g, u^{eg})$. Thus (g, u^{eg}) is a β-idempotent of $S \times_\phi T$. Now $(e, (t^e\alpha u^e)\beta(t^e\alpha u^e)) = (e, (t^e\alpha u^e))\beta(e, (t^e\alpha u^e)) = ((e, t^e)\alpha(e, u^e))\beta((e, t^e)\alpha(e, u^e)) = (e, t^e)\alpha(e, u^e) = (e, t^e\alpha u^e)$ which shows that $t^e\alpha u^e$ is a β-idempotent and hence T^e is a right orthodox Γ-semigroup. Again since $S \times_\phi T$ is a right orthodox Γ-semigroup we have $((eg)^2, (t^{eg}\alpha u^{eg})^{eg}\beta t^{eg}\alpha u^{eg}) = (eg, t^{eg}\alpha u^{eg})\beta(eg, t^{eg}\alpha u^{eg}) = ((e, t^e)\alpha(g, u^{eg}))\beta((e, t^e)\alpha(g, u^{eg})) = (e, t^e)\alpha(g, u^{eg}) = (eg, t^{eg}\alpha u^{eg})$. Thus $(eg)^2 = eg$ which shows that S is orthodox.

(ii) Suppose that $S \times_\phi T$ is a right inverse Γ-semigroup. Let $e, g \in E(S)$ and t^e be an α-idempotent and u^e be a β-idempotent in T^e. Then (e, t^e) is an α-idempotent, $(e, u^e), (g, u^{eg})$ are β-idempotents of $S \times_\phi T$. Now $(e, t^e\alpha u^e\beta t^e) = (e, t^e)\alpha(e, u^e)\beta(e, t^e) = (e, u^e)\beta(e, t^e) = (e, u^e\beta t^e)$ and $(ege, t^{ege}\alpha u^{ege}\beta t^e) = (e, t^e)\alpha(g, u^{eg})\beta(e, t^e) = (g, u^{eg})\beta(e, t^e) = (ge, u^{ege}\beta t^e)$. So we have $t^e\alpha u^e\beta t^e = u^e\beta t^e$ and $ege = ge$. Consequently we have S is a right inverse semigroup and T^e is a right inverse Γ-semigroup.

The proofs of the following two theorems are almost similar to our Lemma 3.3 and Lemma 3.4 proved in [6]. For completeness we give the proof here.

Theorem 3.4. *Let $S \times_\phi T$ be the semidirect product of a semigroup S and a Γ-semigroup T corresponding to a given antimorphism $\phi : S \not\rightarrow End(T)$ and let $(s, t) \in S \times_\phi T$, then*

(i) *if* $(s',t') \in V_\alpha^\beta((s,t))$ *then* $(s',t') \in V_\alpha^\beta((s,t^{s's}))$. *In particular if* $s \in E(S)$, *then* $(s,(t')^s\beta t^{s's}\alpha t') \in V_\alpha^\beta((s,t^{s's}))$ *and*

(ii) *if* t^s *is an* α*-idempotent and* $s' \in V(s)$, *then* $(s',t^{ss'}) \in V_\alpha^\alpha((s,t^s))$.

Proof.

(i) Since $(s',t') \in V_\alpha^\beta((s,t))$ we have,

$$(s',t') = (s',t')\beta(s,t)\alpha(s',t') = (s'ss',(t')^{ss'}\beta t^{s'}\alpha t')$$

and

$$(s,t) = (s,t)\alpha(s',t')\beta(s,t) = (ss's,t^{s's}\alpha(t')^s\beta t).$$

This shows that

$$s' \in V(s) \text{ and } t^{s's}\alpha(t')^s\beta t = t \tag{1}$$

$$(t')^{ss'}\beta t^{s'}\alpha t' = t'. \tag{2}$$

From (1) we have, $(t^{s's}\alpha(t')^s\beta t)^{s's} = (t)^{s's}$ i.e., $t^{s's}\alpha(t')^s\beta t^{s's} = t^{s's}$ and from (2), $((t')^{ss'}\beta t^{s'}\alpha t')^s = (t')^s$ i.e., $(t')^s\beta t^{s's}\alpha(t')^s = (t')^s$. Now $(s',t')\beta(s,t^{s's})\alpha(s',t') = (s'ss',(t')^{ss'}\beta t^{s'ss'}\alpha t') = (s',t')$ by (2) and $(s,t^{s's})\alpha(s',t')\beta(s,t^{s's}) = (ss's,t^{s'ss's}\alpha(t')^s\beta t^{s's}) = (s,t^{s's}\alpha(t')^s\beta t^{s's}) = (s,t^{s's})$. Thus we have $(s',t') \in V_\alpha^\beta((s,t^{s's}))$. Again if $s \in E(S)$, $((t')^s\beta t^{s's}\alpha t')^s = (t')^s\beta t^{s's}\alpha(t')^s = (t')^s$ and $(s,t^{s's})\alpha(s,(t')^s\beta t^{s's}\alpha t')$ $\beta(s,t^{s's}) = (sss,t^{s's}\alpha((t')^s\beta t^{s's}\alpha t')^s\beta t^{s's}) = (s,t^{s's}\alpha(t')^s\beta t^{s's}) = (s,t^{s's})$ and $(s,(t')^s\beta t^{s's}\alpha t')\beta\ (s,t^{s's})\alpha(s,(t')^s\beta t^{s's}\alpha t') = (s,((t')^s\beta t^{s's}\alpha t')^s$ $\beta t^{s'ss}\alpha(t')^s\beta t^{s's}\alpha t') = (s,(t')^s\beta t^{s's}\alpha(t')^s\ \beta t^{s's}\alpha t') = (s,(t')^s\ \beta t^{s's}\alpha t')$. Hence $(s,(t')^s\beta t^{s's}\alpha t') \in V_\alpha^\beta(s,t^{s's})$.

(ii) $(s,t^s)\alpha(s',t^{ss'})\alpha(s,t^s) = (ss's,t^{ss's}\alpha t^{ss's}\alpha t^s) = (s,t^s)$ since t^s is an α-idempotent and $(s',t^{ss'})\alpha(s,t^s)\alpha(s',t^{ss'}) = (s'ss',t^{ss'ss'}\alpha t^{ss'}\alpha t^{ss'})$ $= (s',t^{ss'}\alpha t^{ss'}\alpha t^{ss'}) = (s',(t^s\alpha t^s\alpha t^s)^{s'}) = (s',t^{ss'})$ i.e., $(s',t^{ss'}) \in V_\alpha^\alpha(s,t^s)$.

Theorem 3.5. *Let* S *be a semigroup and* T *be a* Γ*-semigroup and* $S \times_\phi T$ *be the semidirect product corresponding to a given antimorphism* $\phi : S \not\to End(T)$. *Moreover, if* $t \in t^e\Gamma T$ *for every* $e \in E(S)$ *and every* $t \in T$, *then*

(i) (e,t) *is an* α*-idempotent if and only if* $e \in E(S)$ *and* t^e *is an* α*-idempotent and*

(ii) *if* (e,t) *is an* α*-idempotent, then* $(e,t^e) \in V_\alpha^\alpha((e,t))$.

Proof.
(i) If (e,t) is an α-idempotent then

$$(e,t)=(e,t)\alpha(e,t)=(e^2,t^e\alpha t) \text{ i.e., } e=e^2 \text{ and } t^e\alpha t=t. \tag{3}$$

So, $t^e=(t^e\alpha t)^e=t^e\alpha t^e$ which implies that t^e is an α-idempotent. Conversely, let $e\in E(S)$ and t^e be an α-idempotent. Since $t\in t^e\Gamma T$, $t=t^e\beta t_1$ for some $\beta\in\Gamma$, $t_1\in T$ and hence $t^e\alpha t=t^e\alpha t^e\beta t_1=t$. Thus $(e,t)\alpha(e,t)=(e,t^e\alpha t)=(e,t)$ i.e., (e,t) is an α-idempotent.

(ii) If (e,t) is an α-idempotent, from (i) $e\in E(S)$ and t^e is an α-idempotent. Now $(e,t)\alpha(e,t^e)\alpha(e,t)=(e,t^e\alpha t^e\alpha t)=(e,t^e\alpha t)=(e,t)$ from (3) and $(e,t^e)\alpha(e,t)\alpha\ (e,t^e)=(e,t^e\alpha t^e\alpha t^e)=(e,t^e)$. Thus $(e,t^e)\in V_\alpha^\alpha((e,t))$.

Theorem 3.6. *Let S be a semigroup and T be a Γ-semigroup. Let $\phi:S\not\to End(T)$ be a given antimorphism. Then the semidirect product $S\times_\phi T$ is a right (left) orthodox Γ-semigroup if and only if*

(i) *S is an orthodox semigroup and T^e is a right (left) orthodox Γ-semigroup for every $e\in E(S)$,*

(ii) *for every $e\in E(S)$ and every $t\in T$, $t\in t^e\Gamma T$ and*

(iii) *for every α-idempotent t^e, t^{ge} is an α-idempotent, where $e,g\in E(S)$, $t\in T$.*

Proof. Suppose $S\times_\phi T$ is a right orthodox Γ-semigroup. Then by Theorem 3.3 S is an orthodox semigroup and T^e is a right orthodox Γ-semigroup for every $e\in E(S)$. For (ii), let $(e,t)\in S\times_\phi T$ with $e\in E(S)$ and let $(e',t')\in V_\alpha^\beta((e,t))$ for some $\alpha,\beta\in\Gamma$. Then by Theorem 3.4 $(e',t'),(e',(t')^e\beta t^{e'e}\alpha t')\in V_\alpha^\beta((e,t^{e'e}))$. Thus $V_\alpha^\beta((e,t))\cap V_\alpha^\beta((e,t^{e'e}))\neq\phi$ and hence by Theorem 2.9, $V_\alpha^\beta((e,t))=V_\alpha^\beta((e,t^{e'e}))$. So $(e,(t')^e\beta t^{e'e}\alpha t')\in V_\alpha^\beta((e,t))$. Thus $(e,t)=(e,t)\alpha(e,(t')^e\beta t^{e'e}\alpha t')\beta(e,t)=(e,t^e\alpha(t')^e\beta t^{e'e}\alpha(t')^e\beta t)$ and hence $t=t^e\alpha(t')^e\beta t^{e'e}\alpha(t')^e\beta t\in t^e\Gamma T$.

For (iii) we shall first show that for an α-idempotent t^e of T if $e\in E(S), t^{e'}$ is an α-idempotent for any $e'\in V(e)$. If $e\in E(S)$ and t^e is an α-idempotent, then by Theorem 3.5, (e,t) is an α-idempotent in $S\times_\phi T$ and $(e,t^e)\in V_\alpha^\alpha((e,t))$. Again since t^e is an α-idempotent (e,t^e) is also an α-idempotent and thus $(e,t^e)\in V_\alpha^\alpha((e,t^e))$ i.e., $V_\alpha^\alpha((e,t^e))\cap V_\alpha^\alpha((e,t))\neq\phi$ and so $V_\alpha^\alpha((e,t^e))=V_\alpha^\alpha((e,t))$ and by Theorem 3.5 $(e',t^{ee'})\in V_\alpha^\alpha((e,t^e))$ i.e.,

$(e', t^{ee'}) \in V_\alpha^\alpha((e,t))$. Thus $(e,t) = (e,t)\alpha(e',t^{ee'})\alpha(e,t) = (ee'e, t^{e'e}\alpha t^{ee'e}\alpha t)$ $= (e, t^{e'e}\alpha t^e \alpha t) = (e, t^{e'e}\alpha t)$ [since $t = t^e\beta u$ for some $\beta \in \Gamma$, $u \in T, t^e\alpha t = t$]. So $t = t^{e'e}\alpha t$ and hence $t^{e'} = (t^{e'e}\alpha t)^{e'} = t^{e'}\alpha t^{e'}$. Thus $t^{e'}$ is an α-idempotent. Let $e, g \in E(S)$ and suppose that t^e is an α-idempotent for $t \in T$, then $t^{eg}\alpha t^{eg} = (t^e\alpha t^e)^g = t^{eg}$ i.e, t^{eg} is an α-idempotent and we have $eg \in E(S)$ and $ge \in V(eg)$ since S is orthodox. Then by the above fact t^{ge} is an α-idempotent.

We now prove the converse part. Suppose S and T satisfy (i), (ii) and (iii). Let $(s,t) \in S \times_\phi T$. Since S is regular, there exists $s' \in S$ such that $s = ss's$ and $s' = s'ss'$. We take $e = s's$, then $e \in E(S)$. By (ii) $t \in t^e\Gamma T$ which implies $t = t^e\beta u$ for some $\beta \in \Gamma$, $u \in T$. Let $t' = v^{s'}$ where $v^e \in V_\gamma^\delta(t^e)$ where $\gamma, \delta \in \Gamma$. Now $t^{s's}\gamma(t')^s\delta t = t^{s's}\gamma v^{s's}\delta t^e\beta u = (t\gamma v\delta t)^e\beta u = (t^e\gamma v^e\delta t^e)\beta u = t^e\beta u = t$ i.e, $(s,t) = (ss's, t^{s's}\gamma(t')^s\delta t) = (s,t)\gamma(s',t')\delta(s,t)$. Again $(t')^{ss'}\delta t^{s'}\gamma t' = (v^{s'})^{ss'}\delta t^{s'}\gamma v^{s'} = v^{s'}\delta t^{s'}\gamma v^{s'} = v^{s'ss'}\delta t^{s'ss'}\gamma v^{s'ss'} = (v^e\delta t^e\gamma v^e)^{s'} = v^{es'} = v^{s'ss'} = v^{s'} = t'$ i.e., $(s',t') = (s'ss', (t')^{ss'}\delta t^{s'}\gamma t') = (s',t')\delta(s,t)\gamma(s',t')$. Thus we have $(s',t') \in V_\gamma^\delta(s,t)$ which yields $S \times_\phi T$ is a regular Γ-semigroup.

Now let (e,t) be an α-idempotent and (g,u) be a β-idempotent. Then by Theorem 3.5 $e, g \in E(S)$, t^e is an α-idempotent and u^g is a β-idempotent. By (iii) t^{ge} is an α-idempotent, u^{eg} is a β-idempotent and $t^{geg}\alpha t^{geg} = (t^{ge}\alpha t^{ge})^g = t^{geg}$ i.e., t^{geg} is an α-idempotent. By our assumption $e, g \in E(S)$ and $(t^g\alpha u)^{eg} = t^{geg}\alpha u^{eg}$ is a β-idempotent. Thus by Theorem 3.5 $(e,t)\alpha(g,u) = (eg, t^g\alpha u)$ is a β-idempotent which shows that $S \times_\phi T$ is a right orthodox Γ-semigroup.

Theorem 3.7. *Let S be a semigroup, T be a Γ-semigroup and $\phi : S \not\rightarrow End(T)$ be a given antimorphism. Then the semidirect product $S \times_\phi T$ is a right inverse Γ-semigroup if and only if*

(i) *$S$is a right inverse semigroup and T^e is a right inverse Γ-semigroup for every $e \in E(S)$ and*

(ii) *for every $e \in E(S)$ and every $t \in T$, $t \in t^e\Gamma T$.*

Proof. Let $S \times_\phi T$ be a right inverse Γ-semigroup. Then by Theorem 3.3 S is a right inverse semigroup and T^e is a right inverse Γ-semigroup for every $e \in E(S)$. Again since every right inverse Γ-semigroup is a right orthodox Γ-semigroup from the above theorem, condition (ii) holds.

Conversely, suppose that S and T satisfy (i) and (ii). Then by Theorem 3.2 $S \times_\phi T$ is a regular Γ-semigroup . Let (e,t) be an α-idempotent and (g,u)

be a β-idempotent in $S \times_\phi T$. Then by Theorem 3.5 $e, g \in E(S)$, t^e is an α-idempotent, u^g is a β-idempotent. From (ii) $t = t^e\gamma v$ for some $\gamma \in \Gamma$, $v \in T$ and thus $t^e\alpha t = t$ and similarly $u^g\beta u = u$. So $u^{ge} = (u^g\beta u)^{ge} = u^{ge}\beta u^{ge}$ and $t^{ge} = (t^e\alpha t)^{ge} = t^{ege}\alpha t^{ge} = t^{ge}\alpha t^{ge}$ since S is a right inverse semigroup. Now by (ii) we have $u^e\beta t = (u^e\beta t)^{ge}\delta v_1$ for some $\delta \in \Gamma$, $v_1 \in T$ and hence $u^e\beta t = u^{ege}\beta t^{ge}\delta v_1 = u^{ge}\beta t^{ge}\delta v_1$. Thus we have $(e, t)\alpha(g, u)\beta(e, t) = (ege, t^{ge}\alpha u^e\beta t)$ $= (ge, t^{ge}\alpha u^{ge}\beta t^{ge}\delta v_1) = (ge, u^{ge}\beta t^{ge}\delta v_1) = (ge, u^e\beta t) = (g, u)\beta(e, t)$ which implies $S \times_\phi T$ is a right inverse Γ-semigroup.

Theorem 3.8. *Let S be a semigroup, T be a Γ-semigroup and $\phi : S \not\to End(T)$ be a given antimorphism. Then the semidirect product $S \times_\phi T$ is a left inverse Γ-semigroup if and only if*

(i) *S is a left inverse semigroup and T^e is a left inverse Γ-semigroup for every $e \in E(S)$ and*

(ii) *for every $e \in E(S)$ and every $t \in T$, $t = t^e$.*

Proof. Let $S \times_\phi T$ be a left inverse Γ-semigroup. Then by Theorem 3.3 S is a left inverse semigroup and T^e is a left inverse Γ-semigroup. For (ii) let (e, u) be an α-idempotent in $S \times_\phi T$. Then $(e, u) = (e, u)\alpha(e, u) = (e, u^e\alpha u)$ i.e., $u^e\alpha u = u$. Again $(e, u^e)\alpha(e, u^e) = (e, u^{ee}\alpha u^e) = (e, u^e)$ which yields (e, u^e) is an α-idempotent and we have $(e, u^e)\alpha(e, u) = (e, u^e\alpha u) = (e, u)$. Since $S \times_\phi T$ is a left inverse Γ-semigroup, $(e, u) = (e, u^e)\alpha(e, u) = (e, u^e)\alpha(e, u)\alpha(e, u^e)$ $= (e, u^{eee}\alpha u^{ee}\alpha u^e) = (e, (u^e\alpha u)^{ee}\alpha u^e) = (e, u^{ee}\alpha u^e) = (e, u^e)$ i.e., $u = u^e$. Thus if (e, u) is an α-idempotent then $u = u^e$. Now $(e, t) \in S \times_\phi T$ with $e \in E(S)$ and let $(e', t') \in V_\gamma^\delta((e, t))$ for some $\gamma, \delta \in \Gamma$. Then we get $e' \in V(e)$, $t^{e'e}\gamma(t')^e\delta t = t$ i.e., $t^{e'e}\gamma(t')^{ee'e}\delta t^{e'e} = t^{e'e}$ which implies $t^{e'e}\gamma(t')^e\delta t^{e'e} = t^{e'e}$. Since $(e'e, (t')^e\delta t) = (e', t')\delta(e, t)$ and $S \times_\phi T$ is left orthodox (since it is left inverse), $(e'e, (t')^e\delta t)$ is a γ-idempotent and hence $(t')^e\delta t = ((t')^e\delta t)^{e'e}$ $= (t')^e\delta t^{e'e}$. Thus $t^{e'e} = t^{e'e}\gamma(t')^e\delta t^{e'e} = t^{e'e}\gamma(t')^e\delta t = t$ and hence $t^e = (t^{e'e})^e = t^{e'e} = t$.

Conversely suppose that S and T satisfy (i) and (ii). Let $(s, t) \in S \times_\phi T$. Let $e \in E(S)$. Since S is regular there exists $s' \in S$ such that $s' \in V(s)$. From (ii) we have $t = t^e$. Since T^e is regular there exists $v \in T$ such that $v^e \in V_\gamma^\delta(t^e)$. We now take $t' = v^{s'}$. Now $t^{ss'}\gamma(t')^s\delta t = t^{s's}\gamma v^{s's}\delta t^e = t^e\gamma v^e\delta t^e = t^e = t$ and $(t')^{s's}\delta t^{s'}\gamma t' = (v^{s'})^{ss'}\delta t^{s'}\gamma v^{s'} = v^{s'}\delta t^{s'}\gamma v^{s'} = v^{s'ss'}\delta t^{s'ss'}\gamma v^{s'ss'} = (v^e\delta t^e\gamma v^e)^{s'} = v^{es'} = v^{s'ss'} = v^{s'} = t'$. Thus we have $(s', t') \in V_\gamma^\delta(s, t)$. Hence $S \times_\phi T$ is regular. Now let (e, t) be an α-idempotent and (g, u) be a

β-idempotent. Then $e^2 = e$ and $t = t^e\alpha t = t\alpha t$ [by (ii)] and similarly $g^2 = g$ and $u\beta u = u$ i.e., $e, g \in E(S)$ and t is an α-idempotent, u is a β-idempotent. Thus we have $(e,t)\beta(g,u)\alpha(e,t) = (ege, t^{ge}\beta u^e\alpha t) = (ege, t\beta u\alpha t)$ [by (ii)] $=(eg, t\beta u) = (eg, t^g\beta u) = (e,t)\beta(g,u)$. Thus $S \times_\phi T$ is a left inverse Γ-semigroup.

4. WREATH PRODUCT OF A SEMIGROUP AND A Γ-SEMIGROUP

In this section we introduce the notion of wreath product of a semigroup S and a Γ semigroup T. Let X be a nonempty set. Consider the set T^X of all mappings from X to T. For $f, g \in T^X$ and $\alpha \in \Gamma$, define $f\alpha g$ such that $T^X \times \Gamma \times T^X \to T^X$ by $(f\alpha g)(x) = f(x)\alpha g(x)$.

Before going to establish the relation between T and T^X we assume $\Gamma = \{\alpha\}$, a set consisting of single element. Then $(T, \cdot)$ becomes a semigroup where $a \cdot b = a\alpha b$ and T^X also becomes a semigroup where $f \cdot g = f\alpha g$. Suppose T is a regular Γ-semigroup. Then $(T, \cdot)$ is a semigroup. Let $f \in T^X$ and let $x \in X$. Now $f(x) \in T$ and $V(f(x)) \neq \phi$. We define $g : X \to T$ so that $g(x) \in V(f(x))$. Hence for each $x \in X$ we can choose a $g(x)$ such that $f(x)g(x)f(x) = f(x)$. Hence $fgf = f$ which implies that $(T^X, \cdot)$ is a regular semigroup and consequently T^X is a regular Γ-semigroup. In general we cannot extend the process when Γ contains more than one element. To explain this we consider the following example.

Example 4.1. Let $T = \{(a,0) : a \in Q\} \cup \{(0,b) : b \in Q\}$, Q denote the set of all rational numbers. Let $\Gamma = \{(0,5),(0,1),(3,0),(1,0)\}$. Defining $T \times \Gamma \times T \to T$ by $(a,b)(\alpha,\beta)(c,d) = (a\alpha c, b\beta d)$ for all $(a,b),(c,d) \in T$ and $(\alpha,\beta) \in \Gamma$, we can show that T is a Γ-semigroup. Now let $(a,0) \in T$. If $a = 0$ then $(a,0)$ is regular. Suppose $a \neq 0$, then $(a,0)(3,0)(\frac{1}{3a},0)(1,0)(a,0) = (a,0)$. Similarly we can show that $(0,b)$ is also regular. Hence T is a regular Γ-semigroup. Let us now take a set $X = \{x,y\}$, the set consisting of two elements and let us define a mapping $f : X \to T$ by $f(x) = (2,0)$ and $f(y) = (0,3)$. We now show that f is not regular in T^X. If possible let f be regular. Then there exists a mapping $g : X \to T$ and two elements $\alpha, \beta \in \Gamma$ such that $f\alpha g\beta f = f$. i.e., $f(p)\alpha g(p)\beta f(p) = f(p)$ for all $p \in X$. Now if $p = x$, then $\alpha, \beta \notin \{(0,5),(0,1)\}$, since the first component of $f(x)$ is nonzero but if $p = y$, then $\alpha, \beta \in \{(0,5),(0,1)\}$, since the second component of $f(y)$ is nonzero. Thus a contradiction arises. Hence T^X is not a regular Γ-semigroup.

Before further discussion about the relation between T and T^X we now give the following definition.

Definition 4.2. Let S be a Γ-semigroup. An element $e \in S$ is said to be a left(resp. right) γ-unity for some $\gamma \in \Gamma$ if $e\gamma a = a$(resp. $a\gamma e = a$) for all $a \in S$.

We now consider the following examples.

Example 4.3. Consider the Γ-semigroup S of Example 2.3. In this Γ-semigroup $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ is a left α-unity but not a right α-unity of S for $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Example 4.4. Let S be the set of all integers of the form 4n+1 and Γ be the set of all integers of the form 4n+3 where n is an integer. If $a\alpha b$ is $a + \alpha + b$ for all $a, b \in S$ and $\alpha \in \Gamma$ then S is a Γ-semigroup. Here 1 is a left (-1)- unity and also right (-1)- unity.

Example 4.5. Let us consider N, the set of all natural numbers. Let S be the set of all mappings from N to $N \times N$ and Γ be the set of all mappings from $N \times N$ to N. Then the usual mapping product of two elements of S cannot be defined. But if we take f, g from S and α from Γ the usual mapping product $f\alpha g$ can be defined. Also, we find that $f\alpha g \in S$ and $(f\alpha g)\beta h = f\alpha(g\beta h)$. Hence S is a Γ-semigroup. Now we know that the set $N \times N$ is countable. Hence there exists a bijective mapping $f \in S$. Since f is bijective, there exists $\alpha : N \times N \longrightarrow N$ such that $f\alpha$ is the identity mapping on $N \times N$ and αf is the identity mapping on N. Then $f\alpha g = g\alpha f = g$ for all $g \in S$. Hence f is both left α-unity and right α-unity of S.

Let S be a Γ-semigroup and e be a left α-unity. Then $S\Gamma e$ is a left ideal such that $e = e\alpha e \in S\Gamma e$. Also we note that the element e is both left and right α-unity of $S\Gamma e$ in $S\Gamma e$.

Suppose S is a regular Γ-semigroup with a left α-unity e. Then we show that $S\Gamma e$ is a regular Γ-semigroup with a unity . We only show that $S\Gamma e$ is regular. Let $a\gamma e \in S\Gamma e$. Since S is regular there exist $\beta, \delta \in \Gamma$ and $b \in S$ such that $a\gamma e = a\gamma e\beta b\delta a\gamma e$ i.e., $a\gamma e = a\gamma e\beta b\delta e\alpha a\gamma e = (a\gamma e)\beta(b\delta e)\alpha(a\gamma e)$. Since $b\delta e \in S\Gamma e$, $a\gamma e$ is regular. Hence $S\Gamma e$ is a regular Γ-semigroup.

Let us now consider T with a left γ-unity e and a right δ-unity g. Then the constant mapping $C_e : X \to T$ which is defined by $C_e(x) = e$ for all $x \in X$ is a left γ-unity of T^X. Similarly the constant mapping C_g is a right δ-unity of T^X.

Theorem 4.6. *Let T be a Γ-semigroup with a left γ-unity and a right δ-unity for some $\gamma, \delta \in \Gamma$. Then*

(i) *T^X is a regular Γ-semigroup if and only if T is a regular Γ-semigroup,*

(ii) *T^X is a right (resp. left) orthodox Γ-semigroup if and only if T is so and*

(iii) *T^X is a right (resp. left) inverse Γ-semigroup if and only if T is a right (resp. left) inverse Γ-semigroup.*

Proof. By $C_t, t \in T$ denotes the mapping in T^X such that $C_t(x) = t$ for all $x \in X$. Then it is clear that $(C_t)\alpha(C_u) = C_{(t\alpha u)}$ which shows that C_t is an α-idempotent if and only if t is an α-idempotent. Again we have that if f is an α-idempotent in T^X then $f(x)$ is an α-idempotent in T for all $x \in X$.

(i) Assume that T^X is a regular Γ-semigroup. Then for each $t \in T$ there exist $f \in T^X$ and $\alpha, \beta \in \Gamma$ such that $C_t\alpha f\beta C_t = C_t$ so that $t\alpha f(x)\beta t = t$ for all $x \in X$ which shows that t is regular in T. Consequently T is a regular Γ-semigroup. Conversely let T be regular and let e be a left γ-unity and g be a right δ-unity of T. Then for each $f \in T^X$ and for each $x \in X$, $f(x) \in T$ is a regular element and hence there exists a triplet $(\alpha_x, t_x, \beta_x) \in \Gamma \times T \times \Gamma$ such that $f(x)\alpha_x t_x\beta_x f(x) = f(x)$. i.e., $f(x) = (f(x)\delta g)\alpha_x t_x\beta_x(e\gamma f(x)) = f(x)\delta(g\alpha_x t_x\beta_x e)\gamma f(x)$. Define $h : X \to T$ by $h(x) = g\alpha_x t_x\beta_x e$. Then for all $y \in X$, we have

$$\begin{aligned}(f\delta h\gamma f)(y) &= f(y)\delta h(y)\gamma f(y)\\ &= f(y)\delta g\alpha_y t_y\beta_y e\gamma f(y)\\ &= f(y)\alpha_y t_y\beta_y f(y)\\ &= f(y).\end{aligned}$$

Hence f is regular in T^X. Consequently T^X is a regular Γ-semigroup.

(ii) Let $t, u \in T$ such that t be an α-idempotent and u be a β-idempotent. Then C_t is an α-idempotent and C_u is a β-idempotent in T^X. Now if T^X is a right orthodox Γ-semigroup then $(C_t\alpha C_u)\beta(C_t\alpha C_u) = C_t\alpha C_u$ i.e., $t\alpha u$ is a β-idempotent in T which implies T is also a right orthodox Γ-semigroup. Similarly we can show that if T^X is a left orthodox Γ-semigroup then T is so. Let f be an α-idempotent and h be a β-idempotent in T^X. Let us now suppose that T is a right(resp. left) orthodox Γ-semigroup. Then $f(x)\alpha h(x)$(resp. $f(x)\beta h(x)$) is a β-idempotent (resp. α-idempotent). Hence T^X is a right (resp. left) orthodox Γ-semigroup.

(iii) Let T^X be a right (resp. left) inverse Γ-semigroup and let $t, u \in T$ such that t is an α-idempotent and u be a β-idempotent. Then C_t is an α-idempotent and C_u is a β-idempotent in T^X and $C_t\alpha C_u\beta C_t = C_u\beta C_t$(resp. $C_t\beta C_u\alpha C_t = C_t\beta C_u$). Thus we have $t\alpha u\beta t = u\beta t$(resp. $t\beta u\alpha t = t\beta u$) which implies that T is a right(resp. left) inverse Γ-semigroup. Again let T be a right (resp. left) inverse Γ-semigroup. Let f be an α-idempotent and h be a β-idempotent in T^X. $f(x)\alpha h(x)\beta f(x) = h(x)\beta f(x)$ (resp. $f(x)\beta h(x)\alpha f(x) = f(x)\beta h(x)$) for all $x \in X$ i.e, $f\alpha h\beta f = h\beta f$ (resp. $f\beta h\alpha f = f\beta h$). Thus T^X is a right (resp. left)inverse Γ-semigroup.

Let us now suppose that the semigroup S acts on X from the left i.e., $sx \in X, s(rx) = (sr)x$ and $1x = x$ if S is a monoid, for every $r, s \in S$ and every $x \in X$. If S acts on X from left we call it left S set X.

For every Γ-semigroup T, it is known that $End(T)$ is a semigroup. Hence $End(T^X)$ is also a semigroup.

Let S be a semigroup, T a Γ-semigroup and X a nonempty set. Suppose S acts on X from left. Define $\phi : S \to End(T^X)$ by $((\phi(s))(f))(x) = f(sx)$ for all $s \in S, f \in T^X$ and $x \in X$. We now verify that $\phi(s) \in End(T^X)$. For this, let $f, g \in T^X, \alpha \in \Gamma$ and $x \in X$. Then $((\phi(s))(f\alpha g))(x) = (f\alpha g)(sx) = f(sx)\alpha g(sx) = ((\phi(s))(f))(x)\alpha((\phi(s))(g))(x) = ((\phi(s))(f))\alpha((\phi(s))(g))(x)$. Hence $(\phi(s))(f\alpha g) = ((\phi(s))(f))\alpha((\phi(s))(g))$, which implies that $\phi(s) \in End(T^X)$.

Let us now verify that $\phi : S \to End(T^X)$ is a semigroup antimorphism. For this let $s_1, s_2 \in S, f \in T^X$ and $x \in X$. Then $((\phi(s_1)\phi(s_2))(f))(x) = (\phi(s_1)(\phi(s_2)(f)))(x) = (\phi(s_2)(f))(s_1x) = f((s_2(s_1(x))) = f((s_2s_1)x) = (\phi(s_2s_1)(f))(x)$. Hence $\phi(s_2s_1) = \phi(s_1)\phi(s_2)$.

For this antimorphism $\phi : S \not\to End(T^X)$ we can define the semidirect product $S \times_\phi T^X$ of the semigroup S and the Γ-semigroup T^X. We call this semidirect product the wreath product of the semigroup S and the Γ-semigroup T relative to the left S-set X. We denote it by SW_XT. We also denote $\phi(s)(f)(x)$ by $f^s(x)$. Hence $f^s(x) = f(sx)$.

If $|T| = 1$, then $|T^X| = 1$ and hence throughout the paper we assume that $|T| \geq 2$. We now give the relation between T and $(T^X)^e$ for all $e \in E(S)$.

Similar to the Theorems 3.6 and 3.7 we have following Theorems.

Theorem 4.7. *Let S be a semigroup acting on the set X from the left and T be a Γ-semigroup with a left γ-unity and a right δ-unity for some $\gamma, \delta \in \Gamma$. Then*

(i) *T is a regular Γ-semigroup if and only if $(T^X)^e$ is a regular Γ-semigroup,*

(ii) *T is a right (resp. left) orthodox Γ-semigroup if and only if $(T^X)^e$ is so and*

(iii) *T is a right (resp. left) inverse Γ-semigroup if and only if $(T^X)^e$ is a right (resp. left) inverse Γ-semigroup.*

Theorem 4.8. *Let S be a semigroup acting on the set X from the left and T be a Γ-semigroup with a left γ-unity and a right δ-unity for some $\gamma, \delta \in \Gamma$. Then the wreath product SW_XT is a right(left) orthodox Γ-semigroup if and only if*

(i) *S is an orthodox semigroup and $(T^X)^e$ is a right(left) orthodox Γ-semigroup for every $e \in E(S)$*

(ii) *for every $x \in X, f \in T^X$ and $e \in E(S), f(x) \in f(ex)\Gamma T$ and*

(iii) *$f(ex)$ is an α-idempotent for every $x \in X$, implies that $f(gex)$ is an α-idempotent for every $g \in E(S)$ where $e \in E(S), f \in T^X$.*

We now prove the following Theorem.

Theorem 4.9. *Let S be an orthodox semigroup acting on the set X from the left and T be a right orthodox Γ-semigroup with a left γ-unity and a right δ-unity for some $\gamma, \delta \in \Gamma$. Then the following statements are equivalent.*

(a) *S and T^X satisfy* (ii) *and* (iii) *of Theorem* 4.8.

(b) *S permutes X or T is a Γ- group and $geX \subseteq eX$ for every $e, g \in E(S)$.*

Proof. (a) $\Longrightarrow$ (b): Let us suppose that T is not a Γ-group. Then there exists $z \in T$ such that $z\Gamma T \neq T$. Let e_δ be a left δ- unity in T. For $x \in X$, define $f_x : X \to T$ by $f_x(y) = e_\delta$ if $y = x$ and $f_x(y) = z$ if $y \neq x$. Then by (ii), $e_\delta = f_x(x) \in f_x(gx)\Gamma T$ for every $g \in E(S)$. If $f_x(gx) = z$ then $e_\delta \in z\Gamma T$. Thus $e_\delta = z\alpha v$ for some $v \in T$ and $\alpha \in \Gamma$. This implies that $u = e_\delta \delta u = z\alpha v\delta u$ for all $u \in T$. Hence $T = z\Gamma T$ which is a contradiction. Hence $f_x(gx) = e_\delta$ Thus we can conclude that $gx = x$ for all $g \in E(S)$. Let $a \in S$ and $x, y \in X$ such that $ax = ay$. For $a' \in V(a), a'a \in E(S)$ and $x = (a'a)x = (a'a)y = y$. Again $(aa')x = x$ implies that $a(a'x) = x$. Hence for each $a \in S$, the mapping $f_a : X \to X$ defined by $f_a(x) = ax$ is a permutation on X. This means that S permutes X.

Now T is a Γ- group. Note that e_δ is a δ- idempotent and since T is a Γ- group, $E_\delta(T) = \{e_\delta\}$. Let $t \neq e_\delta \in T$ and $e \in E(S)$. Define $h : X \longrightarrow T$ by $h(x) = e_\delta$ if $x \in eX$, otherwise $h(x) = t$. Now $h(ex) = e_\delta$ for every $x \in X$ and hence by (iii), $h(gex) = e_\delta$. This implies that $gex \in eX$ and hence $geX \subseteq eX$ for all $e, g \in E(S)$.

(b)$\Longrightarrow$ (a): The proof is almost similar to the proof of (2) $\Rightarrow$ (1) of Lemma 3.2 [5].

From Theorem 4.7 and 4.9 we conclude that

Theorem 4.10. *Let S be a semigroup acting on the set X from the left and T be a Γ-semigroup with a left γ-unity and a right δ-unity for some $\gamma, \delta \in \Gamma$. Then the wreath product SW_XT is a right orthodox Γ-semigroup if and only if*

(1) *S is an orthodox semigroup and T is a right orthodox Γ-semigroup and*

(2) *S permutes X or T is a Γ- group and $geX \subseteq eX$ for every $e, g \in E(S)$.*

Theorem 4.11. *Let S, T and X be as in Theorem* 4.10. *Then the wreath product SW_XT is a right inverse Γ-semigroup if and only if*

(i) *S is a right inverse semigroup and T is a right inverse Γ-semigroup and*

(ii) *S permutes X or T is a Γ-group.*

Proof. Suppose that SW_XT is a right inverse Γ-semigroup. Then by Theorem 3.7 and Theorem 4.7 we have S is a right inverse semigroup and T is a right inverse Γ-semigroup and by Theorem 4.10 we have S permutes X or T is a Γ-group.

Conversely suppose that S, T and X satisfy (i) and (ii). Then by Theorem 4.6 T^X is a right inverse Γ-semigroup. If T is a Γ-group, then $f(x) \in f(ex)\Gamma T$ for every $f \in T^X$, $e \in E(S), x \in X$. If S permutes X, then $f(x) \in f(x)\Gamma T = f(ex)\Gamma T$ since $ex = x$ for every $e \in E(S)$. Then by Theorem 3.7 $S \times_\alpha T^X = SW_X T$ is a right inverse Γ-semigroup.

Theorem 4.12. *Let S, T and X be as in Theorem 4.10. Then the wreath product $SW_X T$ is a left inverse Γ-semigroup if and only if S is a left inverse semigroup and T is a left inverse Γ-semigroup and S permutes X.*

Proof. By Theorem 3.8 and Theorem 4.7, we have $SW_X T$ is a left inverse Γ-semigroup if and only if S is a left inverse semigroup and T is a left inverse Γ-semigroup and $f(ex) = f(x)$ for every $f \in T^X$, $e \in E(S), x \in X$. The remaining part of the proof is almost similar to the proof of Corollary 3.7 [5].

Open problem:

(i) Find relation between T and T^X without assuming the existence of left α-unity and right β-unity in the Γ-semigroup T for some $\alpha, \beta \in \Gamma$.

(ii) Study the Wreath product of a semigroup S and a Γ-semigroup T without assuming the existence of left α-unity and right β-unity in T for some $\alpha, \beta \in \Gamma$.

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