

ON SOME PROPERTIES OF CHEBYSHEV POLYNOMIALS

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Abstract

Letting T_n (resp. U_n) be the n -th Chebyshev polynomials of the first (resp. second) kind, we prove that the sequences $(X^k T_{n-k})_k$ and $(X^k U_{n-k})_k$ for $n - 2 \lfloor n/2 \rfloor \leq k \leq n - \lfloor n/2 \rfloor$ are two basis of the \mathbb{Q} -vectorial space $\mathbb{E}_n[X]$ formed by the polynomials of $\mathbb{Q}[X]$ having the same parity as n and of degree $\leq n$. Also T_n and U_n admit remarkableness integer coordinates on each of the two basis.

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1. INTRODUCTION AND MAIN RESULTS

For each integer $n \geq 0$, $T_n = T_n(X)$ and $U_n = U_n(X)$ denote the unique polynomials, with integer coefficients, satisfying

$$\cos nx = T_n(\cos x) \text{ and } \sin((n+1)x) = \sin x U_n(\cos x), \quad (x \in \mathbb{R}).$$

The well known Simpson's formulae [2], for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$:

$$\cos nx = 2 \cos x \cos (n-1)x - \cos (n-2)x$$

$$\sin (n+1)x = 2 \cos x \sin nx - \sin (n-1)x$$

give the recurrence relations

$$(1) \quad T_n = 2XT_{n-1} - T_{n-2}, \quad \text{with } T_0 = 1, T_1 = X,$$

$$(2) \quad U_n = 2XU_{n-1} - U_{n-2}, \quad \text{with } U_0 = 1, U_1 = 2X.$$

We deduce (see [1]), for $n \geq 1$, the following relations

$$(3) \quad T_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-1-2k} \frac{n}{n-k} \binom{n-k}{k} X^{n-2k},$$

$$(4) \quad U_n = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-2k} \binom{n-k}{k} X^{n-2k}.$$

These relations allows to state that, for any $n \geq 0$, T_n and U_n belong to $\mathbb{E}_n[X]$, (3) and (4) are their decompositions in the canonical basis $\mathfrak{B}_n := (X^{n-2k})_{0 \leq k \leq \lfloor n/2 \rfloor}$

For instance, the first fifth values of thus polynomials are

$$\begin{array}{ll} T_0 = 1 & U_0 = 1 \\ T_1 = X & U_1 = 2X \\ T_2 = 2X^2 - 1 & U_2 = 4X^2 - 1 \\ T_3 = 4X^3 - 3X & U_3 = 8X^3 - 4X \\ T_4 = 8X^4 - 8X^2 + 1 & U_4 = 16X^4 - 12X^2 + 1 \end{array}$$

The main goal of this paper is to prove that the families $\mathfrak{T}_n := (X^k T_{n-k})_k$ and $\mathfrak{U}_n := (X^k U_{n-k})_k$ for $n - 2 \lfloor n/2 \rfloor \leq k \leq n - \lfloor n/2 \rfloor$ constitute two other basis of $\mathbb{E}_n[X]$ (Theorem 1) for which T_n and U_n admit remarkableness integer coordinates.

Our first result is the following

Theorem 1. *For any $n \geq 0$, \mathfrak{T}_n and \mathfrak{U}_n are two basis of $\mathbb{E}_n[X]$.*

So we can decompose T_{2n+1} and U_{2n+1} (resp. T_{2n} and U_{2n}) over each of the basis \mathfrak{T}_{2n+1} and \mathfrak{U}_{2n+1} (resp. \mathfrak{T}_{2n} and \mathfrak{U}_{2n}). Decompositions of T_{2n} over \mathfrak{T}_{2n} and U_{2n} over \mathfrak{U}_{2n} are trivial, it remains to examine the six decompositions:

1. The decomposition of T_{2n+1} over \mathfrak{T}_{2n+1} and U_{2n+1} over \mathfrak{U}_{2n+1} in Theorem 2.
2. The decomposition of T_{2n} over \mathfrak{U}_{2n} and U_{2n} over \mathfrak{T}_{2n} in Theorem 3.
3. The decomposition of T_{2n+1} over \mathfrak{U}_{2n+1} and U_{2n+1} over \mathfrak{T}_{2n+1} in Theorem 4.

Let us define the families of integers $(\alpha_{n,k})$, $(\beta_{n,k})$ and $(\gamma_{n,k})$ by the following equalities

$$(5) \quad -(1 - 2X)^n = \sum_{k \geq 0} \alpha_{n,k} X^k,$$

$$(6) \quad (X - 1)(1 - 2X)^n = \sum_{k \geq 0} \beta_{n,k} X^k,$$

$$(7) \quad (2X - 1)^n + 2 \left(1 + (2X - 1) + \cdots + (2X - 1)^{n-1} \right) = \sum_{k \geq 0} \gamma_{n,k} X^k,$$

we deduce then, for $(n, k) \in \mathbb{N}^2$, the following relations

$$(8) \quad \alpha_{n,k} = (-1)^{k+1} 2^k \binom{n}{k},$$

$$(9) \quad \beta_{n,k} = \alpha_{n,k} - \alpha_{n,k-1},$$

$$(10) \quad \gamma_{n,k} = (-1)^{n+1} \alpha_{n,k} + 2 \sum_{j=0}^{n-1} (-1)^{j+1} \alpha_{j,k}.$$

Theorem 2. *For each $n \geq 0$, we have*

$$(11) \quad T_{2n+1} = \sum_{k=1}^{n+1} \beta_{n,k} X^k T_{2n+1-k},$$

$$(12) \quad U_{2n+1} = \sum_{k=1}^{n+1} \alpha_{n+1,k} X^k U_{2n+1-k}.$$

Theorem 3. *For each $n \geq 0$, we have*

$$(13) \quad T_{2n} = U_{2n} - XU_{2n-1}, \quad n \geq 1,$$

$$(14) \quad U_{2n} = \sum_{k=0}^n \gamma_{n,k} X^k T_{2n-k}.$$

Theorem 4. *For each $n \geq 0$, we have*

$$(15) \quad T_{2n+1} = \sum_{k=1}^{n+1} (\alpha_{n+1,k} - \delta_{k,1}) X^k U_{2n+1-k},$$

$$(16) \quad U_{2n+1} = \sum_{k=1}^{n+1} (\gamma_{n,k-1} + \beta_{n,k}) X^k T_{2n+1-k},$$

where $\delta_{i,j}$ denotes the Kronecker symbol.

The sequences of integers $(\alpha_{n,k})$, $(\beta_{n,k})$ and $(\gamma_{n,k})$ satisfy the following recurrence relation

$$\left\{ \begin{array}{l} \alpha_{n,0} = -1, \text{ for } n \geq 0, \\ \alpha_{0,k} = 0, \text{ for } k \geq 1, \\ \alpha_{n,k} = \alpha_{n-1,k} - 2\alpha_{n-1,k-1}, \text{ for } n, k \geq 1, \end{array} \right.$$

$$\left\{ \begin{array}{l} \beta_{n,0} = -1, \text{ for } n \geq 0, \\ \beta_{0,1} = 1 \text{ and } \beta_{0,k} = 0, \text{ for } k \geq 2, \\ \beta_{n,k} = \beta_{n-1,k} - 2\beta_{n-1,k-1}, \text{ for } n, k \geq 1, \end{array} \right.$$

$$\left\{ \begin{array}{l} \gamma_{n,0} = 1, \text{ for } n \geq 0, \\ \gamma_{0,k} = 0, \text{ for } k \geq 1, \\ \gamma_{n,k} = -\gamma_{n-1,k} + 2\gamma_{n-1,k-1}, \text{ for } n, k \geq 1. \end{array} \right.$$

The following tables give the values of $\alpha_{n,k}$, $\beta_{n,k}$ and $\gamma_{n,k}$ for $0 \leq n \leq 4$

n	$\alpha_{n,0}$	$\alpha_{n,1}$	$\alpha_{n,2}$	$\alpha_{n,3}$	$\alpha_{n,4}$
0	-1				
1	-1	2			
2	-1	4	-4		
3	-1	6	-12	8	
4	-1	8	-24	32	-16

n	$\beta_{n,0}$	$\beta_{n,1}$	$\beta_{n,2}$	$\beta_{n,3}$	$\beta_{n,4}$	$\beta_{n,5}$
0	-1	1				
1	-1	3	-2			
2	-1	5	-8	4		
3	-1	7	-18	20	-8	
4	-1	9	-32	56	-48	16

n	$\gamma_{n,0}$	$\gamma_{n,1}$	$\gamma_{n,2}$	$\gamma_{n,3}$	$\gamma_{n,4}$
0	1				
1	1	2			
2	1	0	4		
3	1	2	-4	8	
4	1	0	8	-16	16

Notice that $\alpha_{n,k} = \gamma_{n,k} = 0$ for $k > n$ and $\beta_{n,k} = 0$ for $k > n + 1$.

According to these tables, one obtains

- Using Theorem 2

$$T_1 = XT_0$$

$$T_3 = 3XT_2 - 2X^2T_1$$

$$T_5 = 5XT_4 - 8X^2T_3 + 4X^3T_2$$

$$T_7 = 7XT_6 - 18X^2T_5 + 20X^3T_4 - 8X^4T_3$$

$$T_9 = 9XT_8 - 32X^2T_7 + 56X^3T_6 - 48X^4T_5 + 16X^5T_4$$

$$U_1 = 2XU_0$$

$$U_3 = 4XU_2 - 4X^2U_1$$

$$U_5 = 6XU_4 - 12X^2U_3 + 8X^3U_2$$

$$U_7 = 8XU_6 - 24X^2U_5 + 32X^3U_4 - 16X^4U_3$$

- Using Theorem 3

$$T_0 = U_0$$

$$T_2 = U_2 - XU_1$$

$$T_4 = U_4 - XU_3$$

$$T_6 = U_6 - XU_5$$

$$T_8 = U_8 - XU_7$$

$$U_0 = T_0$$

$$U_2 = T_2 + 2XT_1$$

$$U_4 = T_4 + 0XT_3 + 4X^2T_2$$

$$U_6 = T_6 + 2XT_5 - 4X^2T_4 + 8X^3T_3$$

$$U_8 = T_8 + 0XT_7 + 8X^2T_6 - 16X^3T_5 + 16X^4T_4$$

- Using Theorem 4

$$T_1 = XU_0$$

$$T_3 = 3XU_2 - 4X^2U_1$$

$$T_5 = 5XU_4 - 12X^2U_3 + 8X^3U_2$$

$$T_7 = 7XU_6 - 24X^2U_5 + 32X^3U_4 - 16X^4U_3$$

$$T_9 = 9XU_8 - 40X^2U_7 + 80X^3U_6 - 80X^4U_5 + 32X^5U_4$$

$$U_1 = 2XT_0$$

$$U_3 = 4XT_2$$

$$U_5 = 6XT_4 - 8X^2T_3 + 8X^3T_2$$

$$U_7 = 8XT_6 - 16X^2T_5 + 16X^3T_4$$

$$U_9 = 10XT_8 - 32X^2T_7 + 64X^3T_6 - 64X^4T_5 + 32X^5T_4$$

2. PROOFS OF THEOREMS

2.1. Proof of Theorem 1

\mathfrak{T}_n and \mathfrak{U}_n are two families of polynomials of $\mathbb{E}_n[X]$ with $\text{card}\mathfrak{T}_n = \text{card}\mathfrak{U}_n = \dim \mathbb{E}_n[X] = \lfloor n/2 \rfloor + 1$. Using the following Lemma, we prove that the determinant of \mathfrak{T}_n and \mathfrak{U}_n relatively to the canonical basis \mathfrak{B}_n of $\mathbb{E}_n[X]$ are not zero. Theorem 1 follows.

Lemma 5. *For any integer $n \geq 0$, by setting $m = \lfloor n/2 \rfloor$, we have*

$$\det_{\mathfrak{B}_n}(\mathfrak{T}_n) = 2^{m(m-1)/2} \quad \text{and} \quad \det_{\mathfrak{B}_n}(\mathfrak{U}_n) = 2^{m(m+1)/2}.$$

Proof. For any integer $m \geq 0$ and for $1 \leq k \leq m+1$, set $V_k^{(m)} = (2X)^{k-1} T_{2m+1-k}$ and $W_k^{(m)} = (2X)^{k-1} U_{2m+1-k}$. Notice that $V_k^{(m)}$ and $W_k^{(m)}$ are polynomials of $\mathbb{E}_{2m}[X]$ with dominant coefficient 2^{2m-1} and 2^{2m} respectively. Using the recurrence equations (1) and (2), we obtain for $m \geq 1$

$$V_{k+1}^{(m)} - V_k^{(m)} = V_k^{(m-1)} \quad \text{and} \quad W_{k+1}^{(m)} - W_k^{(m)} = W_k^{(m-1)}.$$

Let $\Delta_m = \det_{\mathfrak{B}_{2m}}(V_1^{(m)}, V_2^{(m)}, \dots, V_{m+1}^{(m)})$ and

$$D_m = \det_{\mathfrak{B}_{2m}}(W_1^{(m)}, W_2^{(m)}, \dots, W_{m+1}^{(m)}).$$

We have

$$\begin{aligned} \Delta_m &= \det_{\mathfrak{B}_{2m}}(V_1^{(m)}, V_2^{(m)} - V_1^{(m)}, V_3^{(m)} - V_2^{(m)}, \dots, V_{m+1}^{(m)} - V_m^{(m)}) \\ &= \det_{\mathfrak{B}_{2m}}(V_1^{(m)}, V_1^{(m-1)}, V_2^{(m-1)}, \dots, V_m^{(m-1)}) \\ &= 2^{2m-1} \Delta_{m-1} \\ &= 2^{(2m-1)+(2m-3)+\dots+1} \Delta_0 \\ &= 2^{m^2}, \end{aligned}$$

and similarly, we obtain $D_m = 2^{2^m} D_{m-1} = 2^{2^m + (2^{m-2}) + \dots + 2} D_0 = 2^{m(m+1)}$.

For $n = 2m + r$, with $m = \lfloor n/2 \rfloor$ and $r \in \{0, 1\}$, we have

$$\begin{aligned} \det_{\mathfrak{B}_n}(\mathfrak{T}_n) &= \det_{\mathfrak{B}_{2m+r}}(X^r T_{2m}, X^{r+1} T_{2m-1}, \dots, X^{r+m} T_m) \\ &= \det_{\mathfrak{B}_{2m}}(T_{2m}, X T_{2m-1}, \dots, X^m T_m) \\ &= 2^{-(1+2+\dots+m)} \Delta_m \\ &= 2^{m(m-1)/2}, \end{aligned}$$

and similarly $\det_{\mathfrak{B}_n}(\mathfrak{U}_n) = 2^{-(1+2+\dots+m)} D_m = 2^{m(m+1)/2}$. ■

2.2. Proof of Theorems 2, 3 and 4

Let us denote E denote the shift operator on $\mathbb{Q}[X]^{\mathbb{N}}$ defined by

$$E((W_n)_n) = (W_{n+1})_n,$$

or in a more simple form

$$EW_n = W_{n+1}, \quad (n \geq 0).$$

For any $m \geq 0$, define the operators

$$A_m = -(E - 2X)^m,$$

$$B_m = (X - E)(E - 2X)^m,$$

$$C_m = (2X - E)^m + 2 \sum_{k=1}^m E^k (2X - E)^{m-k}.$$

Using relations (5), (6) and (7), we have also

$$A_m = \sum_{k=0}^m \alpha_{m,k} X^k E^{m-k},$$

$$B_m = \sum_{k=0}^{m+1} \beta_{m,k} X^k E^{m+1-k},$$

$$C_m = \sum_{k=0}^m \gamma_{m,k} X^k E^{m-k}.$$

Lemma 6. *For any integer n , we have*

- (a) $(2X - E)^n T_m = T_{m-n}$ and $(2X - E)^n U_m = U_{m-n}$, ($m \geq n \geq 0$).
- (b) $T_n = U_n - XU_{n-1}$ ($n \geq 1$).
- (c) $2T_n = U_n - U_{n-2}$ ($n \geq 2$).
- (d) $U_{2n} = 1 + 2 \sum_{k=1}^n T_{2k}$ ($n \geq 0$).

Proof.

- (a) For $m \geq 1$, one has $(2X - E)T_m = 2XT_m - T_{m+1} = T_{m-1}$ and $(2X - E)U_m = 2XU_m - U_{m+1} = U_{m-1}$. We conclude by induction.
- (b) Letting for $n \geq 1$, $W_n = T_n - U_n + XU_{n-1}$. The sequence $(W_n)_{n \geq 1}$ satisfies the following relation

$$W_n = 2XW_{n-1} - W_{n-2}, \quad (n \geq 3), \quad \text{with } W_1 = W_2 = 0,$$

leading to $W_n = 0$, for $n \geq 1$.

- (c) For $n \geq 2$, one has $2T_n = U_n + (U_n - 2XU_{n-1})$ from (b) and thus $2T_n = U_n - U_{n-2}$ from (2) .
- (d) For $n \geq 0$, one has $U_{2n} = U_0 + \sum_{k=1}^n (U_{2k} - U_{2k-2}) = 1 + 2 \sum_{k=1}^n T_{2k}$ from (c). ■

Proof of Theorem 2. Relations (11) and (12) are respectively equivalent to

$$B_n T_n = 0 \quad \text{and} \quad A_{n+1} U_n = 0, \text{ for } n \geq 0.$$

These last relations follows from Lemma 6 (a). We have, for any integer $n \geq 0$

$$\begin{aligned} B_n T_n &= (-1)^n (X - E) (2X - E)^n T_n \\ &= (-1)^n (X - E) T_0 \\ &= (-1)^n (X T_0 - T_1) = 0, \end{aligned}$$

and

$$\begin{aligned} A_{n+1} U_n &= (-1)^{n+1} (2X - E) (2X - E)^n U_n \\ &= (-1)^{n+1} (2X - E) U_0 \\ &= (-1)^{n+1} (2X U_0 - U_1) = 0. \end{aligned}$$

Proof of Theorem 3. Relation (13) follows from Lemma 6 (b).

Relation (14) is equivalent to $U_{2n} = C_n T_n$, for $n \geq 0$. One has indeed, for any $n \geq 0$

$$\begin{aligned}
C_n T_n &= \left((2X - E)^n + 2 \sum_{k=1}^n E^k (2X - E)^{n-k} \right) T_n \\
&= T_0 + 2 \sum_{k=1}^n T_{2k} \\
&= U_{2n}, \quad (\text{from (d) of Lemma 6}).
\end{aligned}$$

Proof of Theorem 4. For any $n \geq 0$, we have

$$\begin{aligned}
T_{2n+1} &= U_{2n+1} - XU_{2n}, \quad (\text{from (b) of Lemma 6}) \\
&= \sum_{k=1}^{n+1} \alpha_{n+1,k} X^k U_{2n+1-k} - XU_{2n} \quad (\text{from (12)}) \\
&= \sum_{k=1}^{n+1} (\alpha_{n+1,k} - \delta_{k,1}) X^k U_{2n+1-k}.
\end{aligned}$$

We have also, for any $n \geq 0$

$$\begin{aligned}
U_{2n+1} &= XU_{2n} + T_{2n+1}, \quad (\text{from (b) of Lemma 6}) \\
&= \sum_{k=0}^n \gamma_{n,k} X^{k+1} T_{2n-k} + \sum_{k=1}^{n+1} \beta_{n,k} X^k T_{2n+1-k}, \quad (\text{from (11) and (14)}) \\
&= \sum_{k=1}^{n+1} (\gamma_{n,k-1} + \beta_{n,k}) X^k T_{2n+1-k}.
\end{aligned}$$

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