

## COMMUTATIVE DIRECTOIDS WITH SECTIONALLY ANTITONE BIJECTIONS\*

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### Abstract

We study commutative directoids with a greatest element, which can be equipped with antitone bijections in every principal filter. These can be axiomatized as algebras with two binary operations satisfying four identities. A minimal subvariety of this variety is described.

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Join-semilattices whose principal filters are Boolean lattices were used by J.C. Abbott [1] for a characterization of the logic connective implication in the classical propositional logic. These semilattices also have the property that on each principal filter of them an antitone involution is defined.

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Motivated by this observation, the notion of a *join semilattice with sectionally antitone involutions* in [2] and *join semilattice with sectionally antitone bijections* in [5] was defined. In this paper we introduce a further generalization of this concept defining the notion of a *commutative directoid with sectionally antitone bijections*. By means of these directoids we obtain “nice” algebraic structures, forming a variety with “nice” congruence properties.

The concept of directoid was introduced by J. Ježek and R. Quackenbush [8] and independently by V.M. Kopytov and Z.I. Dimitrov [9] in order to axiomatize algebraic structures defined on upward directed ordered sets. In a certain sense directoids generalize semilattices. For the reader’s convenience, we repeat definitions and basic properties of these concepts.

An ordered set  $(A; \leq)$  is *upward directed* if  $U(x, y) \neq \emptyset$  for every  $x, y \in A$ , where  $U(x, y) = \{a \in A; x \leq a \text{ and } y \leq a\}$ . Elements of  $U(x, y)$  are referred to as common upper bounds of  $x, y$ . Of course, if  $(A; \leq)$  has a greatest element then it is upward directed.

Let  $(A; \leq)$  be an upward directed set and  $\sqcup$  denotes a binary operation on  $A$ . The pair  $\mathcal{A} = (A; \sqcup)$  is called a  $\sqcup$ -*directoid* if

- (i)  $x \sqcup y \in U(x, y)$  for all  $x, y \in A$ ;
- (ii) if  $x \leq y$  then  $x \sqcup y = y$  and  $y \sqcup x = y$ .

The following axiomatization of directoids was established in [8]:

**Proposition 1.** *A groupoid  $\mathcal{A} = (A; \sqcup)$  is a  $\sqcup$ -directoid if and only if it satisfies the following identities*

- (D1)  $x \sqcup x = x$ ;
- (D2)  $(x \sqcup y) \sqcup x = x \sqcup y$ ;
- (D3)  $y \sqcup (x \sqcup y) = x \sqcup y$ ;
- (D4)  $x \sqcup ((x \sqcup y) \sqcup z) = (x \sqcup y) \sqcup z$ .

*A binary relation  $\leq$  defined on  $A$  by the rule*

$$x \leq y \quad \text{if and only if} \quad x \sqcup y = y$$

*is an order and  $x \sqcup y \in U(x, y)$  for each  $x, y \in A$ .*

A binary relation  $\leq$  will be called the *induced order* of  $(A; \sqcup)$ . A directoid  $\mathcal{A} = (A; \sqcup)$  is called *commutative* if it satisfies the identity

$$(D5) \quad x \sqcup y = y \sqcup x.$$

It was shown in [8] that commutative directoids are axiomatized by the identities (D1), (D4) and (D5).

Let us denote a greatest element of an ordered set  $(P; \leq)$  by 1. For each  $a \in P$  the interval  $[a, 1]$  will be called a *section*.

**Definition 1.** Let  $(P; \leq)$  be a poset with a greatest element 1.  $(P; \leq)$  is called a *poset with sectionally antitone bijections*, if for each element  $a \in S$  there exists a bijection  $f_a$  of the interval  $[a, 1] \subseteq P$  into itself such that

$$x \leq y \quad \Leftrightarrow \quad f_a(y) \leq f_a(x), \text{ for all } x, y \in [a, 1].$$

Of course, the inverse  $f_a^{-1}$  of  $f_a$  is also an antitone bijection on  $[a, 1]$ . If each  $f_a$  is an *involution*, i.e.,  $f_a^2(x) = x$ , for all  $x \in [a, 1]$ , then  $f_a^{-1} = f_a$  and  $(P; \leq)$  is called a *poset with sectionally antitone involutions*.

**Remark 1.** Any poset  $(P; \leq)$  with a greatest element possesses a commutative  $\sqcup$ -directoid  $(P; \sqcup)$ .

Indeed, take any choice function  $\varphi : 2^P \rightarrow P$ , such that for any nonempty subset  $A \subseteq P$ ,  $\varphi(A) \in A$ . Then define a binary operation  $\sqcup$  on  $P$  as follows

$$x \sqcup y = \begin{cases} \varphi(U(x, y)), & \text{if } x \text{ and } y \text{ are incomparable} \\ y & \text{if } x \leq y \\ x & \text{if } y \leq x. \end{cases}$$

It is evident that this operation satisfies the axioms (D1)–(D5) and that

$$x \leq y \quad \Leftrightarrow \quad x \sqcup y = y.$$

Given a commutative  $\sqcup$ -directoid  $S$  with sectionally antitone bijections, we can introduce two new binary operations on  $S$  as follows:

$$(P) \quad x \circ y = f_y(x \sqcup y), \quad x * y = f_y^{-1}(x \sqcup y).$$

Since  $x \sqcup y \in [y, 1]$ ,  $\circ$  and  $*$  are everywhere defined operations on the set  $S$ . Conversely, one can check immediately that for any  $a \in S$  and  $x \in [a, 1]$

$$(A) \quad f_a(x) = x \circ a, \quad f_a^{-1}(x) = x * a.$$

Clearly, if all the mappings  $f_a$  are involutions, then  $x \circ y = x * y$ , for all  $x, y \in S$ .

**Lemma 1.** *Let  $\mathcal{S}$  be a commutative  $\sqcup$ -directoid with sectionally antitone bijections and  $\circ, *$  be operations defined by (P). Then*

$$a \leq b \Leftrightarrow a \circ b = 1 \Leftrightarrow a * b = 1.$$

**Proof.** Suppose  $a, b \in S$  and  $a \leq b$ . Since  $f_b$  is an antitone bijection on  $[b, 1]$ , we have  $f_b(b) = 1$ . Then

$$(Q) \quad a \circ b = f_b(a \sqcup b) = f_b(b) = 1, \quad a * b = f_b^{-1}(a \sqcup b) = f_b^{-1}(b) = 1.$$

To prove the converse implication we note that for all  $x \in [b, 1]$  it is  $b \leq x$  and thus  $f_b(b) \geq f_b(x)$ . Let  $a \circ b = 1$ . Then  $f_b(a \sqcup b) = 1 = f_b(b)$ , but  $f_b$  is a bijection, hence  $a \sqcup b = b$ , i.e.,  $a \leq b$ . Analogously we obtain  $a * b = 1 \Rightarrow a \leq b$ .  $\blacksquare$

**Theorem 1.** *Let  $\mathcal{S}$  be a commutative  $\sqcup$ -directoid with sectionally antitone bijections and  $\circ, *$  be operations defined by (P). Then*

- (1)  $x \circ x = x * x = 1, x \circ 1 = x * 1 = 1, 1 \circ x = 1 * x = x;$
- (2)  $(x \circ y) * y = (x * y) \circ y = (y \circ x) * x = (y * x) \circ x;$
- (3)  $x \circ (((x \circ y) * y) \circ z) * z = 1;$
- (4)  $(((((x \circ z) * z) \circ y) * y) \circ z) \circ (x \circ z) = ((((((x \circ z) * z) \circ y) * y) * z) \circ (x * z)) = 1.$

Moreover, each of the terms of (2) is equal to  $x \sqcup y$ .

**Proof.**

- (1) By Lemma 1,  $x \circ x = x * x = 1$  and  $x \circ 1 = x * 1 = 1$ . We also conclude  $1 \circ x = f_x(1) = x$  and  $1 * x = f_x^{-1}(1) = x$ . Thus (1) is satisfied.
- (2)  $(x \circ y) * y = f_y^{-1}(f_y(x \sqcup y) \sqcup y) = f_y^{-1}(f_y(x \sqcup y)) = x \sqcup y$  since  $f_y(x \sqcup y) \geq y$ . Analogously, we can check  $(x * y) \circ y = x \sqcup y$ ,  $(y \circ x) * x = x \sqcup y$ , and  $(y * x) \circ x = x \sqcup y$ . Due to the fact that  $\mathcal{S}$  is commutative, we can establish the last two equalities.

$$(3) \quad x \circ (((x \circ y) * y) \circ z) * z = x \circ ((x \sqcup y) \sqcup z) = f_{(x \sqcup y) \sqcup z}(x \sqcup ((x \sqcup y) \sqcup z)) = f_{(x \sqcup y) \sqcup z}((x \sqcup y) \sqcup z) = 1 \text{ by (D4).}$$

$$(4) \quad (((x \circ z) * z) \circ y) * y \circ z = ((x \sqcup z) \sqcup y) \circ z = f_z(((x \sqcup z) \sqcup y) \sqcup z). \text{ Since } f_z \text{ is antitone and } z \leq (x \sqcup z) \sqcup y, x \sqcup z \leq (x \sqcup z) \sqcup y \text{ we have:}$$

$$f_z(((x \sqcup z) \sqcup y) \sqcup z) = f_z((x \sqcup z) \sqcup y) \leq f_z(x \sqcup z) = x \circ z,$$

thus, by (Q):

$$((((x \circ z) * z) \circ y) * y) \circ z \circ (x \circ z) = 1.$$

Analogously we obtain:

$$((((x \circ z) * z) \circ y) * y) * z \circ (x * z) = 1.$$

■

**Theorem 2.** *Let  $\mathcal{A} = (A; \circ, *, 1)$  be an algebra of type  $(2, 2, 0)$  satisfying the identities (1), (2) and (3). Define a binary relation  $\leq$  on  $A$  as follows:*

$$(R) \quad a \leq b \text{ if and only if } a \circ b = 1.$$

*Then  $\leq$  is a partial order on  $A$  with a greatest element 1 and  $(A; \leq)$  is a commutative  $\sqcup$ -directoid  $(A; \sqcup)$  where  $(x \circ y) * y = x \sqcup y$  and the following assertions are equivalent:*

(i) *The algebra  $\mathcal{A}$  satisfies the identity (4);*

(ii) *for any  $x, y, a \in A, x \in [a, 1]$  the following implication holds:*

$$(5) \quad x \leq y \Rightarrow y \circ a \leq x \circ a \text{ and } y * a \leq x * a.$$

*Moreover, for any  $a \in A$  the mappings  $f_a(x) = x \circ a, f_a^{-1}(x) = x * a$  are mutually inverse antitone bijections on  $[a, 1]$ .*

**Proof.** First we prove that the relation  $\leq$  defined by (R) is a partial order.

Due to (1),  $\leq$  is reflexive. Suppose  $x \leq y$  and  $y \leq x$ . Then  $x \circ y = 1$  and  $y \circ x = 1$  hence, by (1) and (2),  $x = 1 * x = (y \circ x) * x = (x \circ y) * y = 1 * y = y$ , thus  $\leq$  is antisymmetrical.

Suppose  $x \leq y$  and  $y \leq z$ . Then  $x \circ y = 1$  and  $y \circ z = 1$ . Hence  $x \circ z = x \circ (1 * z) = x \circ ((y \circ z) * z) = x \circ (((1 * y) \circ z) * z) = x \circ (((x \circ y) * y) \circ z) * z = 1$  by (3). Thus  $\leq$  is transitive, i.e., it is a partial order. Also (1) tells us that  $a \circ 1 = 1$  for all  $a$ , so 1 is the greatest element.

Define  $a \sqcup b = (a \circ b) * b$ . Put  $x = y = a$ ,  $z = b$  in (3). Then, with (1), we can easily derive

$$a \circ ((a \circ b) * b) = 1.$$

Putting  $x = y = b$ ,  $z = a$  in (3) we obtain analogously (using (1) and (2)) that

$$b \circ ((a \circ b) * b) = 1.$$

Thus  $a \leq a \sqcup b$  and  $b \leq a \sqcup b$ .

Suppose now  $a \leq b$ . Then  $a \sqcup b = (a \circ b) * b = 1 * b = b$ . Clearly,  $a \sqcup b = (a \circ b) * b = (b \circ a) * a = b \sqcup a$ , thus  $(A; \sqcup)$  is a commutative  $\sqcup$ -directoid.

(i)  $\Rightarrow$  (ii): Suppose now that also (4) is satisfied and  $x \in [a, 1]$ ,  $x \leq y$ . Then  $a \leq x$  and thus  $(x \circ a) * a = x \sqcup a = x$ . Therefore from (4) for  $z = a$  it follows:  $((x \circ y) * y) \circ a \circ (x \circ a) = 1$ , which yields:  $(y \circ a) \circ (x \circ a) = ((1 * y) \circ a) \circ (x \circ a) = (((x \circ y) * y) \circ a) \circ (x \circ a) = 1$ . Thus  $y \circ a \leq x \circ a$ . Analogously we obtain:  $y * a \leq x * a$ , i.e., the condition (5) holds.

Now we consider  $a \in A$ , and  $f_a, f_a^{-1}$  defined by (A). Assume  $x \in [a, 1]$ . Then

$$f_a^{-1}(f_a(x)) = (x \circ a) * a = x \sqcup a = x,$$

$$f_a(f_a^{-1}(x)) = (x * a) \circ a = x \sqcup a = x,$$

using (3). Since  $x \in [a, 1]$ , by [5], also  $f_a(x) \in [a, 1]$ . Hence  $f_a$  and  $f_a^{-1}$  are bijections on  $[a, 1]$  (each being the inverse of the other).

For  $x \in [a, 1]$  and  $x \leq y$  we have by (5):

$$f_a(y) = y \circ a \leq x \circ a = f_a(x),$$

$$f_a^{-1}(y) = y * a \leq x * a = f_a^{-1}(x),$$

therefore  $f_a$  and  $f_a^{-1}$  are antitone bijections.

(ii)  $\Rightarrow$  (i): By the assumptions  $(A; \sqcup)$  is a commutative  $\sqcup$ -directoid with sectionally antitone bijections. Take  $x, y \in A$ . Since

$$f_y(x \sqcup y) = (x \sqcup y) \circ y = ((x \circ y) * y) \circ y = f_y(f_y^{-1}(x \circ y)) = x \circ y,$$

$$f_y^{-1}(x \sqcup y) = (x \sqcup y) * y = ((x * y) \circ y) * y = f_y^{-1}(f_y(x * y)) = x * y,$$

$\circ$  and  $*$  can be also defined by (P). Applying Theorem 1, we obtain that the algebra  $(A; \circ, *, 1)$  satisfies (4).  $\blacksquare$

**Remark 2.**

(i) According to Theorem 1, to any commutative  $\sqcup$ -directoid  $(P; \leq)$  with sectionally antitone bijections it corresponds an algebra  $(P; \circ, *, 1)$  satisfying the axioms (1)–(4).

(ii) In view of Theorem 2, to any algebra  $\mathcal{A} = (A; \circ, *, 1)$  satisfying the identities (1)–(4) it corresponds a commutative directoid  $(A; \sqcup)$  with sectionally antitone bijections, where  $\leq$  is the induced order of  $(A; \sqcup)$ .

The following example shows, that different algebras satisfying the identities (1)–(4) can be assigned to the same poset  $(P; \leq)$  with sectionally antitone bijections.

A poset  $(P; \leq)$  is called a *complete bipartite* poset if there exists two nonempty subset  $A, B$  of  $P$  such that  $A \cap B = \emptyset$ ,  $A \cup B = P$  and such that for any  $x, y \in P$

$$x \leq y \quad \text{if and only if} \quad \text{either } x = y \text{ or } x \in A \text{ and } y \in B.$$

**Example 1.** Let us consider the finite poset  $P \cup \{1\}$ , where  $(P; \leq)$  is a complete bipartite poset and 1 is an element with the property  $x \leq 1$ , for all  $x \in P$ . Denote the sets  $A$  and  $B$  as follows:  $A = \{a_1, a_2, \dots, a_n\}$ ,  $B = \{b_1, b_2, \dots, b_m\}$ . Clearly  $P \cup \{1\} = \{a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_m, 1\}$  and  $a_i < b_j < 1$  for all  $1 \leq i \leq n$ ,  $1 \leq j \leq m$ , see Figure 1. We show that  $(P \cup \{1\}; \leq)$  is a poset with antitone bijections. Take an arbitrary permutation  $\pi$  of the set  $\{b_1, b_2, \dots, b_m\}$  and let us define an arbitrary bijection for each section  $[p, 1]$ , (where  $p \in P \cup \{1\}$ ) as follows:

$$f_1(1) = 1,$$

$$f_{b_j}(1) = b_j, \quad f_{b_j}(b_j) = 1, \quad 1 \leq j \leq m,$$

$$f_{a_i}(1) = a_i, \quad f_{a_i}(a_i) = 1, \quad 1 \leq i \leq n$$

and  $f_{a_i}(b_j) = \pi(b_j)$  for all  $1 \leq i \leq n$  and  $1 \leq j \leq m$ .

Further, a commutative directoid  $(P \cup \{1\}; \sqcup)$  can be defined on the set  $P \cup \{1\}$  as follows:

Fix an arbitrary element  $b_k$  ( $1 \leq k \leq m$ ). Then define  $\sqcup$  in the following way:

$$x \sqcup y = y \quad \text{iff} \quad x \leq y,$$

for any  $b_i, b_j$ ,  $b_i \neq b_j$  let  $b_i \sqcup b_j = 1$ , for any  $a_i, a_j$ ,  $a_i \neq a_j$  let  $a_i \sqcup a_j = b_k$ .

According to Remark 1,  $(P \cup \{1\}, \sqcup)$  is a commutative directoid with sectionally antitone bijections.

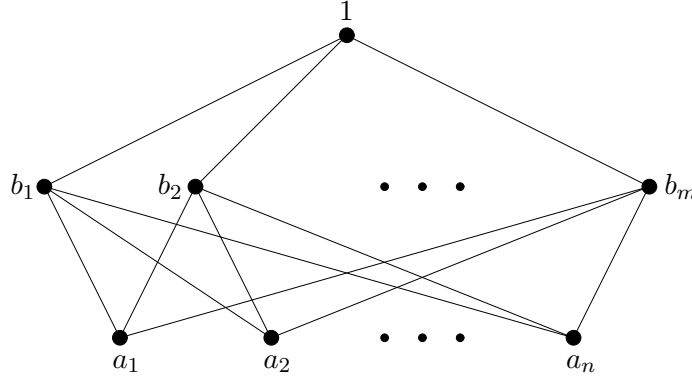


Figure 1

In view of Theorem 1, to any permutation  $\pi$  on the set  $\{b_1, \dots, b_m\}$  and to any  $b_k \in \{b_1, b_2, \dots, b_m\}$  there corresponds an algebra  $\mathcal{A}$  satisfying the identities (1)–(4). In fact we have  $a_i \circ a_i = b_i \circ b_i = a_i \circ b_j = b_j \circ 1 = 1 \circ 1 = 1$  and  $a_i * a_i = b_i * b_i = a_i * b_j = b_j * 1 = 1 * 1 = 1$  for all  $i \neq j$  such that  $1 \leq i \leq n$  and  $1 \leq j \leq m$ . Further, we have  $b_i \circ b_j = b_i * b_j = b_j$ , for all  $i \neq j$  such that  $1 \leq i, j \leq m$ ,  $a_i \circ a_j = \pi(b_k)$ ,  $a_i * a_j = \pi^{-1}(b_k)$ , for all  $1 \leq i, j \leq n$  and  $b_j \circ a_i = \pi(b_j)$ ,  $b_j * a_i = \pi^{-1}(b_j)$ , for all  $1 \leq i \leq m$ ,  $1 \leq j \leq n$ .

Let us recall that an algebra  $(L; \sqcup, \sqcap)$  of type  $(2, 2)$  is called a  $\lambda$ -lattice if its both reducts  $(L; \sqcup)$  and  $(L; \sqcap)$  are commutative directoids and hence the operations are connected by the absorption laws, see e.g. [11].



**Corollary 1.** *Let  $\mathcal{A} = (A; \circ, *, 1)$  be an algebra of type  $(2, 2, 0)$  satisfying the identities (1), (2), (3) and (4) and let  $\leq$  be the induced order. Then  $(A; \leq)$  is a commutative  $\sqcup$ -directoid with 1, where  $x \sqcup y = (x \circ y) * y$  and for each  $a \in A$  the section  $[a, 1]$  becomes a  $\lambda$ -lattice where*

$$x \sqcap_a y = (((x \circ a) \circ (y \circ a)) * (y \circ a)) * a.$$

**Proof.** In view of Theorem 2 (ii), the section  $[a, 1]$  is a commutative  $\sqcup$ -directoid in which  $x \sqcup y = (x \circ y) * y$ . Since  $f_a, f_a^{-1}$  are antitone bijections it follows that letting  $x \sqcap_a y = f_a^{-1}(f_a(x) \sqcup f_a(y))$  we get a commutative  $\sqcap$ -directoid with the same order. Hence  $([a, 1]; \sqcup, \sqcap_a)$  is a  $\lambda$ -lattice.

We have

$$\begin{aligned} x \sqcap_a y &= f_a^{-1}(f_a(x) \sqcup f_a(y)) = ((x \circ a) \sqcup (y \circ a)) * a = \\ &= (((x \circ a) \circ (y \circ a)) * (y \circ a)) * a. \end{aligned}$$

■

When  $\mathcal{A} = (A; \sqcup, \sqcap)$  is a  $\lambda$ -lattice then the induced operation  $\sqcap_a$  in a section  $[a, 1]$  need not coincide with the operation  $\sqcap$ , see e.g. the following:

**Example 2.** Let  $\mathcal{L} = (L; \sqcup, \sqcap)$ ,  $L = \{0, w, z, y, x, 1\}$  be a  $\lambda$ -lattice depicted in Figure 2. Pick  $w = x \sqcap y$  and  $z \sqcup w = x$ .

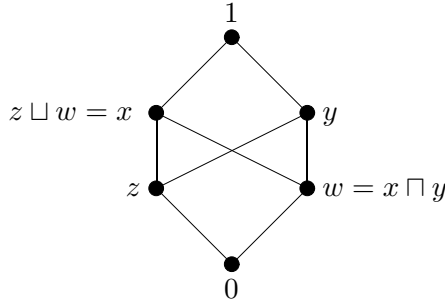


Figure 2

Further, let  $f_z(1) = z, f_z(z) = 1, f_z(x) = x, f_z(y) = y$  and  $f_z^{-1} = f_z$ . Then  $x = x \circ z, y = y \circ z, z = (x \sqcup y) \circ z = (x \circ z) \sqcap_z (y \circ z)$ , i.e.,  $x \sqcap_z y = z$  but  $w = x \sqcap y = (x \circ z) \sqcap (y \circ z)$ , thus  $\sqcap_z$  is not the same as  $\sqcap$ .

An *implication algebra* [1] is an algebra  $(A; \circ)$  satisfying the identities

$$(I1) \quad (x \circ y) \circ x = x;$$

$$(I2) \quad (x \circ y) \circ y = (y \circ x) \circ x;$$

$$(I3) \quad x \circ (y \circ z) = y \circ (x \circ z).$$

It is well-known that any implication algebra  $(A; \circ)$  contains an element  $1 \in A$  such that  $x \circ x = 1$ , for all  $x \in A$  (see e.g. [1]). The algebra  $(A; \circ, \circ, 1)$  of type  $(2,2,0)$  where the operation  $\circ$  is doubled will be called a *double implication algebra*.

It is easy to see that any double implication algebra satisfies the identities (1)–(4).

Indeed, for the algebra  $\mathcal{A} = (A; \circ, \circ, 1)$  the identity (2) is the same as (I2) and we have  $x \circ x = 1$ ,  $1 \circ x = x$  and  $x \circ 1 = (1 \circ x) \circ (1 \circ 1) = 1 \circ ((1 \circ x) \circ 1) = 1 \circ 1 = 1$ , i.e., (1) is also satisfied by  $\mathcal{A}$ . Clearly also (3) holds. According to [1], (R) defines a partial order  $\leq$  with the property (5) on any implication algebra  $(A; \circ)$ . Hence, by Theorem 2, we obtain that  $(A; \circ, \circ, 1)$  satisfies the identity (4), as well.

**Example 3.** Observe that for a two-element chain  $(\{0, 1\}; \leq)$  the algebra  $\mathcal{S}_2 = (\{0, 1\}; \circ, \circ, 1)$  (where, of course,  $f_0(0) = 1$ ,  $f_0(1) = 0$ ) is a double implication algebra. The operation  $\circ$  is given by the table

$\circ$	0	1
0	1	1
1	0	1.

It is proved in [10] that the implication algebras form a minimal quasivariety which is generated by the two-element implication algebra  $(\{0, 1\}; \circ)$ . Hence it is not hard to see that the variety generated by the algebra  $\mathcal{S}_2$  is also a minimal quasivariety and it coincides with the variety of all double implication algebras.

**Proposition 2.** *The variety  $\mathcal{V}$  of the algebras  $(A; \circ, *, 1)$  of type  $(2, 2, 0)$  satisfying (1)–(4) contains a single minimal quasivariety, namely the variety of double implication algebras.*

**Proof.** Let  $\mathcal{W}$  be a nontrivial subquasivariety of  $\mathcal{V}$  and  $\mathcal{A} = (A; \circ, *, 1)$  a nontrivial algebra in  $\mathcal{W}$ . In view of Theorem 1, the corresponding poset  $(A; \leq)$  is a commutative  $\sqcup$ -directoid with sectionally antitone bijections. As  $|A| \geq 2$ , there exists an element  $a \in A$  with  $a \neq 1$ . In view of Theorem 1 we have  $a \circ a = a * a = 1 \circ 1 = 1 * 1 = 1$ ,  $a \circ 1 = a * 1 = 1$  and  $1 \circ a = 1 * a = a$ . Hence  $(\{a, 1\}; \circ, *, 1)$  is a subalgebra of  $(A; \circ, *, 1)$ . Since  $(\{a, 1\}; \leq)$  is a two-element chain, the algebra  $\mathcal{S}_2 = (\{a, 1\}; \circ, *, 1)$  is a double implication algebra with two elements (see Example 3). Denote the variety generated by it as  $\mathcal{V}_2$ . We already shown that  $\mathcal{V}_2$  is the variety of all double implication algebras. Since  $\mathcal{V}_2$  is a minimal quasivariety as well, we have  $\mathcal{V}_2 = Q(\mathcal{S}_2)$ , where  $Q(\mathcal{S}_2)$  denotes the quasivariety generated by the algebra  $\mathcal{S}_2$ . As  $\mathcal{S}_2 \in \mathcal{W}$  we get  $\mathcal{V}_2 = Q(\mathcal{S}_2) \subseteq \mathcal{W}$ , and this proves that  $\mathcal{V}_2$  is the unique minimal quasivariety contained in variety  $\mathcal{V}$ . ■

Let  $\mathcal{A} = (A; \circ, *, 1)$  be an algebra of type  $(2, 2, 0)$ . A nonempty subset  $K \subseteq A$  is called a *congruence kernel* of  $\mathcal{A}$  if  $K = [1]_{\Theta} = \{x \in A; (x, 1) \in \Theta\}$  for some congruence  $\Theta$  of  $\mathcal{A}$ . Recall that  $\mathcal{A}$  is called *3-permutable* if  $\Theta_1 \circ \Theta_2 \circ \Theta_1 = \Theta_2 \circ \Theta_1 \circ \Theta_2$  holds for every  $\Theta_1, \Theta_2 \in \text{Con}\mathcal{A}$ . According to J. Hagemann and A. Mitschke [7], a variety  $\mathcal{V}$  of algebras is 3-permutable if and only if there exist ternary terms  $p_0, p_1, p_2$  and  $p_3$  in  $\mathcal{V}$  such that the following identities hold in  $\mathcal{V}$ :

$$(B) \quad \begin{cases} p_0(x, y, z) = x, p_3(x, y, z) = z, \\ p_i(x, x, y) = p_{i+1}(x, y, y) \text{ for } i \in \{0, 1, 2\}. \end{cases}$$

An algebra  $\mathcal{A}$  with a constant 1 is called *weakly regular* if every  $\Theta \in \text{Con}\mathcal{A}$  is determined by its kernel, i.e., if  $[1]_{\Theta} = [1]_{\Phi}$  implies  $\Theta = \Phi$  for every  $\Phi, \Theta \in \text{Con}\mathcal{A}$ . A variety  $\mathcal{V}$  is weakly regular if every algebra  $\mathcal{A} \in \mathcal{V}$  has this property. The following characterization of weakly regular varieties was established by B. Csákány in [6].

**Proposition 3** [6]. A variety  $\mathcal{V}$  with 1 is weakly regular if and only if there exists  $n \in \mathbb{N}$  and binary terms  $q_1(x, y), q_2(x, y), \dots, q_n(x, y)$  such that

$$(C) \quad q_1(x, y) = q_2(x, y) = \cdots = q_n(x, y) = 1 \Leftrightarrow x = y$$

is satisfied for every algebra  $\mathcal{A} \in \mathcal{V}$ .

For other congruence conditions, the reader is asked to consult e.g. [3].

**Theorem 3.** *The variety  $\mathcal{V}$  of the algebras  $(A; \circ, *, 1)$  of type  $(2, 2, 0)$  satisfying the identities (1)–(4) is weakly regular, 3-permutable, arithmetical at 1 and congruence distributive.*

**Proof.** Consider the terms  $q_1(x, y) = x \circ y$  and  $q_2(x, y) = y \circ x$ . Then  $q_1(x, x) = q_2(x, x) = x \circ x = 1$ . If  $q_1(x, y) = 1$  and  $q_2(x, y) = 1$ , then by (R) we have  $x \leq y$  and  $y \leq x$  thus  $x = y$ . In view of Proposition 3, we conclude that  $\mathcal{V}$  is weakly regular (at 1).

Now, let  $p_0(x, y, z) = x$ ,  $p_3(x, y, z) = z$  and  $p_1(x, y, z) = (z * y) \circ x$ ,  $p_2(x, y, z) = (x * y) \circ z$ . It is easy to see that these terms satisfy the identities (B). Consequently,  $\mathcal{V}$  is congruence 3-permutable.

It is proved in [3] that a variety is arithmetical at 1 if and only if there exists a binary term  $b(x, y)$  of it such that  $b(x, x) = b(1, x) = 1$  and  $b(x, 1) = x$ . Obviously, we can take  $b(x, y) = y \circ x$ .

Since  $\mathcal{V}$  is arithmetical at 1 it is congruence distributive at 1, i.e.,  $[1]_{\Theta \cap (\Phi \vee \Psi)} = [1]_{(\Theta \cap \Phi) \vee (\Theta \cap \Psi)}$ , for all  $\Theta, \Phi, \Psi \in \text{Con}\mathcal{A}$  for  $\mathcal{A} \in \mathcal{V}$ .

As  $\mathcal{V}$  is weakly regular, this equality implies  $\Theta \cap (\Phi \vee \Psi) = (\Theta \cap \Phi) \vee (\Theta \cap \Psi)$  (for all  $\Theta, \Phi, \Psi \in \text{Con}\mathcal{A}$  and  $\mathcal{A} \in \mathcal{V}$ ), thus the variety  $\mathcal{V}$  is congruence distributive. ■

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