

Nd-SOLID VARIETIES

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To the memory of Professor Kazimierz Głazek

Abstract

A non-deterministic hypersubstitution maps any operation symbol of a tree language of type τ to a set of trees of the same type, i.e. to a tree language. Non-deterministic hypersubstitutions can be extended to mappings which map tree languages to tree languages preserving the arities. We define the application of a non-deterministic hypersubstitution to an algebra of type τ and obtain a class of derived algebras. Non-deterministic hypersubstitutions can also be applied to equations of type τ . Formally, we obtain two closure operators which turn out to form a conjugate pair of completely additive closure operators. This allows us to use the theory of conjugate pairs of additive closure operators for a characterization of M -solid non-deterministic varieties of algebras. As an application we consider M -solid non-deterministic varieties of semigroups.

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1. Introduction

Let $(f_i)_{i \in I}$ be an indexed set of operation symbols where f_i is n_i -ary, let $X := \{x_1, \dots, x_n, \dots\}$ be a countably infinite set of variables and for each $n \geq 1$ let $X_n := \{x_1, \dots, x_n\}$ be a finite set of variables. We denote by $W_\tau(X)$ and $W_\tau(X_n)$, respectively the sets of all terms of a finite type $\tau = (n_i)_{i \in I}$ and of all n -ary terms of type τ . We use the well-known Galois connection Id-Mod between sets of identities and classes of algebras of a given type. For any set Σ of identities we denote by $\text{Mod}\Sigma$ the model class of all algebras of type τ which satisfy all identities of Σ ; and for any class K of algebras of the same type we denote by $\text{Id}K$ the set of all identities satisfied by all algebras in K . Classes of the form $\text{Mod}\Sigma$ are called varieties of algebras of type τ . If \mathcal{A} satisfies the equation $s \approx t$ as an identity, we write $\mathcal{A} \models s \approx t$ and if the class K of algebras of type τ satisfies $s \approx t$, we write $K \models s \approx t$. If $\Sigma \subseteq W_\tau(X)^2$ is a set of equations, then $K \models \Sigma$ means that every equation from Σ is satisfied by every algebra from K . Any subset of $W_\tau(X)$, i.e. any element of the power set $\mathcal{P}(W_\tau(X))$ or of $\mathcal{P}(W_\tau(X_n))$ is called a tree language. Our restriction to a finite type is motivated by applications of tree languages in computer science. For tree languages one may define the following superposition operations

$$\hat{S}_m^n : \mathcal{P}(W_\tau(X_n)) \times \mathcal{P}(W_\tau(X_m))^n \rightarrow \mathcal{P}(W_\tau(X_m))$$

inductively by the following steps:

Definition 1.1. Let $m, n \in \mathbb{N}^+ (:= \mathbb{N} \setminus \{0\})$ and let $B \in \mathcal{P}(W_\tau(X_n))$ and $B_1, \dots, B_n \in \mathcal{P}(W_\tau(X_m))$ such that B, B_1, \dots, B_n are non-empty.

- (i) If $B = \{x_j\}$ for $1 \leq j \leq n$, then $\hat{S}_m^n(\{x_j\}, B_1, \dots, B_n) := B_j$.
- (ii) If $B = \{f_i(t_1, \dots, t_{n_i})\}$, and if we assume that $\hat{S}_m^n(\{t_j\}, B_1, \dots, B_n)$ for $1 \leq j \leq n$; are already defined, then $\hat{S}_m^n(\{f_i(t_1, \dots, t_{n_i})\}, B_1, \dots, B_n) := \{f_i(r_1, \dots, r_{n_i}) \mid r_j \in \hat{S}_m^n(\{t_j\}, B_1, \dots, B_n) \text{ for } 1 \leq j \leq n_i\}$.
- (iii) If B is an arbitrary subset of $W_\tau(X_n)$, we define

$$\hat{S}_m^n(B, B_1, \dots, B_n) := \bigcup_{b \in B} \hat{S}_m^n(\{b\}, B_1, \dots, B_n).$$

If one of the sets B, B_1, \dots, B_n is empty, we define $\hat{S}_m^n(B, B_1, \dots, B_n) := \emptyset$. Then we may consider the heterogeneous algebra

$$\mathcal{P} - \text{clone } \tau := ((\mathcal{P}(W_\tau(X_n)))_{n \in \mathbb{N}^+}; (\hat{S}_m^n)_{m, n \in \mathbb{N}^+}, (\{x_i\}_{i \leq n, n \in \mathbb{N}^+}))$$

which is called the power clone of τ ([?]). We mention that $\mathcal{P} - \text{clone } \tau$ satisfies the well-known clone axioms (C1), (C2), (C3) (see e.g. [?, ?]). If $\mathcal{P}_{fin}(W_\tau(X_n))$ is the set of all finite subsets of $W_\tau(X_n)$, then

$$\mathcal{P}_{fin} - \text{clone } \tau := ((\mathcal{P}_{fin}(W_\tau(X_n)))_{n \in \mathbb{N}^+}; (\hat{S}_m^n)_{n \in \mathbb{N}^+}, (\{x_i\}_{i \leq n, n \in \mathbb{N}^+}))$$

is a subalgebra of $\mathcal{P} - \text{clone } \tau$ ([?]).

We mention also that there is a one-based version of $\mathcal{P} - \text{clone } \tau$, the algebra $\mathcal{P}_n - \text{clone } \tau_n := (\mathcal{P}(W_{\tau_n}(X_n)); \hat{S}^n, \{x_1\}, \dots, \{x_n\})$ where τ_n is a finite type consisting of n -ary operation symbols only and where $\hat{S}^n := \hat{S}_n^n$. $\mathcal{P}_n - \text{clone } \tau_n$ is an example of a unitary Menger algebra of rank n (see e.g [?]).

Similar structures can be obtained if one defines a superposition for sets of operations. Let $O^{(n)}(A)$ be the set of all n -ary operations ($n \geq 1$) defined on the set A and let $O(A) := \bigcup_{n \geq 1} O^{(n)}(A)$ be the set of all operations defined on A . Let $e_i^{n,A}$ be an n -ary projection defined on A , i.e., $e_i^{n,A}(a_1, \dots, a_n) := a_i$ for $1 \leq i \leq n$, and let $\mathcal{P}(O^{(n)}(A))$ be the power set of $O^{(n)}(A)$.

Definition 1.2. Let $m, n \in \mathbb{N}^+$ and $B \in \mathcal{P}(O^{(n)}(A)), B_1, \dots, B_n \in \mathcal{P}(O^{(m)}(A))$ such that B, B_1, \dots, B_n are non-empty.

- (i) If $B = \{e_j^{n,A}\}$ for $1 \leq j \leq n$, then $\hat{S}_m^{n,A}(\{e_j^{n,A}\}, B_1, \dots, B_n) := B_j$.
- (ii) If $B = \{f_i^A(t_1^A, \dots, t_{n_i}^A)\}$ with $f_i^A \in O^{(n_i)}(A), t_j^A \in O^{(n)}(A)$ and assume that $\hat{S}_m^{n_i,A}(\{t_j^A\}, B_1, \dots, B_{n_i})$ for $1 \leq j \leq n_i$ are already defined, then $\hat{S}_m^{n,A}(\{f_i^A(t_1^A, \dots, t_{n_i}^A)\}, B_1, \dots, B_n) := \{f_i^A(r_1^A, \dots, r_{n_i}^A) \mid r_j^A \in \hat{S}_m^{n_i,A}(\{t_j^A\}, B_1, \dots, B_{n_i}), 1 \leq j \leq n_i\}$.
- (iii) If $B \in \mathcal{P}(O^{(n)}(A))$ is arbitrary, then we define

$$\hat{S}_m^{n,A}(B, B_1, \dots, B_n) := \bigcup_{b \in B} \hat{S}_m^{n,A}(\{b\}, B_1, \dots, B_n).$$

If one of the sets B, B_1, \dots, B_n is empty, then we define $\hat{S}_m^{n,A}(B, B_1, \dots, B_n) := \emptyset$. In this case we consider the heterogeneous algebra

$$\mathcal{P}_A - \text{clone} := ((\mathcal{P}(O^{(n)}(A)))_{n \in \mathbb{N}^+}; (\hat{S}_m^{n,A})_{m, n \in \mathbb{N}^+}, (\{e_i^{n,A}\})_{i \leq n, n \in \mathbb{N}^+}).$$

Let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be an algebra of type τ . Then we may consider the subclone $\mathcal{P}_A - \text{clone} \mathcal{A}$ of $\mathcal{P}_A - \text{clone}$ which is defined as follows.

Definition 1.3. Let $n \in \mathbb{N}^+$ and $B \in \mathcal{P}(W_\tau(X_n))$. Then we define the set B^A of term operations induced on the algebra $\mathcal{A} = (A; (f_i^A)_{i \in I})$ as follows:

- (i) If $B = \{x_j\}$ for $1 \leq j \leq n$, then $B^A := \{e_j^{n,A}\}$.
- (ii) If $B = \{f_i(t_1, \dots, t_{n_i})\}$ then $B^A = \{f_i^A(t_1^A, \dots, t_{n_i}^A)\}$ where f_i^A is the fundamental operation of \mathcal{A} corresponding to the operation symbol f_i and where t_j^A are term operations on \mathcal{A} which are induced in the usual way by the t_j 's.
- (iii) If B is an arbitrary non-empty subset of $W_\tau(X_n)$, then we define $B^A := \bigcup_{b \in B} \{b\}^A$. If the set B is empty, then we define $B^A := \emptyset$.

Let $\mathcal{P}(W_\tau(X_n))^A$ be the collection of all sets of n -ary term operations induced by sets of n -ary terms of type τ on the algebra $\mathcal{A} = (A; (f_i^A)_{i \in I})$.

From these definitions we obtain the following

Lemma 1.4. Let $B \in \mathcal{P}(W_\tau(X_n))$ and let $B_1, \dots, B_n \in \mathcal{P}(W_\tau(X_m))$. Then

$$[\hat{S}_m^n(B, B_1, \dots, B_n)]^A = \hat{S}_m^{n,A}(B^A, B_1^A, \dots, B_n^A).$$

Proof. If one of the sets B, B_1, \dots, B_n is empty, then one of the sets B^A, B_1^A, \dots, B_n^A is also empty. Thus

$$[\hat{S}_m^n(B, B_1, \dots, B_n)]^A = \emptyset^A = \emptyset = \hat{S}_m^{n,A}(B^A, B_1^A, \dots, B_n^A).$$

Assume now that all of B, B_1, \dots, B_n are different from the empty set. At first we show by induction on the complexity of the term t that for one-element sets $B = \{t\}$ our equation is satisfied.

For $t = x_i$ with $1 \leq i \leq n$, we have $B^A = \{x_i\}^A = \{e_i^{n,A}\}$ and

$$\begin{aligned}
[\hat{S}_m^n(B, B_1, \dots, B_n)]^A &= [\hat{S}_m^n(\{x_i\}, B_1, \dots, B_n)]^A \\
&= B_i^A \\
&= \hat{S}_m^{n,A}(\{e_i^{n,A}\}, B_1^A, \dots, B_n^A) \\
&= \hat{S}_m^{n,A}(\{x_i\}^A, B_1^A, \dots, B_n^A) \\
&= \hat{S}_m^{n,A}(B^A, B_1^A, \dots, B_n^A).
\end{aligned}$$

Let now $t = f_i(t_1, \dots, t_{n_i})$ and assume that for all $1 \leq k \leq n_i$,

$$[\hat{S}_m^n(\{t_k\}, B_1, \dots, B_n)]^A = \hat{S}_m^{n,A}(\{t_k\}^A, B_1^A, \dots, B_n^A).$$

Then

$$\begin{aligned}
&[\hat{S}_m^n(\{f_i(t_1, \dots, t_{n_i})\}, B_1, \dots, B_n)]^A \\
&= \{f_i(r_1, \dots, r_{n_i}) \mid r_k \in \hat{S}_m^n(\{t_k\}, B_1, \dots, B_n), 1 \leq k \leq n_i\}^A \\
&= \{f_i^A(r_1^A, \dots, r_{n_i}^A) \mid r_k \in \hat{S}_m^n(\{t_k\}, B_1, \dots, B_n), 1 \leq k \leq n_i\} \\
&= \{f_i^A(r_1^A, \dots, r_{n_i}^A) \mid r_k^A \in \hat{S}_m^n(\{t_k\}, B_1, \dots, B_n)^A, 1 \leq k \leq n_i\} \\
&= \{f_i^A(r_1^A, \dots, r_{n_i}^A) \mid r_k^A \in \hat{S}_m^{n,A}(\{t_k\}^A, B_1^A, \dots, B_n^A), 1 \leq k \leq n_i\} \\
&= \hat{S}_m^{n,A}(\{f_i^A(t_1^A, \dots, t_{n_i}^A)\}, B_1^A, \dots, B_n^A) \\
&= \hat{S}_m^{n,A}(\{f_i(t_1, \dots, t_{n_i})\}^A, B_1^A, \dots, B_n^A).
\end{aligned}$$

If B is a set of terms consisting of more than one element, then we have

$$\begin{aligned}
[\hat{S}_m^n(B, B_1, \dots, B_n)]^{\mathcal{A}} &= \left[\hat{S}_m^n\left(\bigcup_{b \in B} \{b\}, B_1, \dots, B_n\right) \right]^{\mathcal{A}} \\
&= \left[\bigcup_{b \in B} \hat{S}_m^n(\{b\}, B_1, \dots, B_n) \right]^{\mathcal{A}} \\
&= \bigcup_{b \in B} [\hat{S}_m^n(\{b\}, B_1, \dots, B_n)]^{\mathcal{A}} \\
&= \bigcup_{b \in B} \hat{S}_m^{n, \mathcal{A}}(\{b\}^{\mathcal{A}}, B_1^{\mathcal{A}}, \dots, B_n^{\mathcal{A}}) \\
&= \hat{S}_m^{n, \mathcal{A}}\left(\bigcup_{b \in B} \{b\}^{\mathcal{A}}, B_1^{\mathcal{A}}, \dots, B_n^{\mathcal{A}}\right) \\
&= \hat{S}_m^{n, \mathcal{A}}(B^{\mathcal{A}}, B_1^{\mathcal{A}}, \dots, B_n^{\mathcal{A}}).
\end{aligned}$$

■

Proposition 1.5.

$$\mathcal{P}_A - \text{clone}\mathcal{A} = ((\mathcal{P}(W_\tau(X_n)))^{\mathcal{A}})_{n \in \mathbb{N}^+}; (\hat{S}_m^{n, \mathcal{A}})_{m, n \in \mathbb{N}^+}, (\{e_i^{n, \mathcal{A}}\})_{i \leq n, n \in \mathbb{N}^+})$$

is a subalgebra of $\mathcal{P}_A - \text{clone}$.

Proof. Let $B^{\mathcal{A}} \in \mathcal{P}(W_\tau(X_n))^{\mathcal{A}}$ and let $B_1^{\mathcal{A}}, \dots, B_n^{\mathcal{A}} \in \mathcal{P}(W_\tau(X_m))^{\mathcal{A}}$, then $B \in \mathcal{P}(W_\tau(X_n))$ and $B_1, \dots, B_n \in \mathcal{P}(W_\tau(X_m))$.

From Lemma 1.4 we have that

$$\hat{S}_m^{n, \mathcal{A}}(B^{\mathcal{A}}, B_1^{\mathcal{A}}, \dots, B_n^{\mathcal{A}}) = [\hat{S}_m^n(B, B_1, \dots, B_n)]^{\mathcal{A}} \in \mathcal{P}(W_\tau(X_m))^{\mathcal{A}}.$$

■

If $T^{(n)}(\mathcal{A})$ is the set of all derived n -ary operations of the algebra $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$, then we can also consider the algebra $\mathcal{P}(\mathcal{T}(\mathcal{A})) := ((\mathcal{P}(T^{(n)}(\mathcal{A})))_{n \in \mathbb{N}^+}; (\hat{S}_m^{n, \mathcal{A}})_{m, n \in \mathbb{N}^+}, (\{e_i^{n, \mathcal{A}}\})_{i \leq n, n \in \mathbb{N}^+})$. It is not difficult to prove that $\mathcal{P}_A - \text{clone}\mathcal{A} = \mathcal{P}(\mathcal{T}(\mathcal{A}))$.

Any mapping $\sigma : \{f_i \mid i \in I\} \rightarrow \mathcal{P}(W_\tau(X))$ with $\sigma(f_i) \subseteq W_\tau(X_{n_i})$, for $i \in I$, is called a non-deterministic hypersubstitution (for short nd -hypersubstitution) of type τ . We denote by $Hyp^{nd}(\tau)$ the set of all non-deterministic hypersubstitutions of type τ . Every nd -hypersubstitution can

be extended in the following inductive way to a mapping $\hat{\sigma} : \mathcal{P}(W_\tau(X)) \rightarrow \mathcal{P}(W_\tau(X))$.

- (i) $\hat{\sigma}[\emptyset] := \emptyset$.
- (ii) $\hat{\sigma}[\{x_i\}] := \{x_i\}$ for every variable $x_i \in X$.
- (iii) $\hat{\sigma}[\{f_i(t_1, \dots, t_{n_i})\}] := \hat{S}_n^{n_i}(\sigma(f_i), \hat{\sigma}[\{t_1\}], \dots, \hat{\sigma}[\{t_{n_i}\}])$ if we inductively assume that $\hat{\sigma}[\{t_k\}]$, $1 \leq k \leq n_i$, are already defined.
- (iv) $\hat{\sigma}[B] := \bigcup \{\hat{\sigma}[\{b\}] \mid b \in B\}$ for $B \subseteq W_\tau(X)$.

In the sequel instead of $\hat{\sigma}[\{t\}]$ for a term $t \in W_\tau(X)$ we will simply write $\hat{\sigma}[t]$.

In [?] was proved that for every nd -hypersubstitution σ the mapping $\hat{\sigma}$ is an endomorphism of \mathcal{P} -clone τ . We recall also that the set $Hyp^{nd}(\tau)$ forms a monoid with respect to the operation \circ_{nd} defined by $\sigma_1 \circ_{nd} \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and the identity element $\sigma_{pid} : f_i \mapsto \{f_i(x_1, \dots, x_{n_i})\}$ for every $i \in I$.

In the next section we apply nd -hypersubstitutions to equations and to algebras.

2. The Conjugate Pair $(\chi_{nd}^A, \chi_{nd}^E)$

If $\mathcal{A} = (A; (f_i^A)_{i \in I})$ is an algebra of type τ and if $\sigma \in Hyp^{nd}(\tau)$ is an nd -hypersubstitution, then we define

$$\sigma(\mathcal{A}) := \{(A; (l_i^A)_{i \in I}) \mid l_i \in \sigma(f_i)\}.$$

The set $\sigma(\mathcal{A})$ is called the set of derived algebras. Since for every sequence $(l_i)_{i \in I}$ of terms there is a hypersubstitution mapping f_i to l_i we can write $\sigma(\mathcal{A})$ also in the form $\sigma(\mathcal{A}) = \{\rho(\mathcal{A}) \mid \rho \in Hyp(\tau) \text{ with } \rho(f_i) \in \sigma(f_i) \text{ for } i \in I\}$. For a class K of algebras of type τ we define

$$\sigma(K) := \bigcup_{\mathcal{A} \in K} \sigma(\mathcal{A}).$$

If $M \subseteq Hyp^{nd}(\tau)$ is the universe of a submonoid of $Hyp^{nd}(\tau)$, then we define $\chi_{M-nd}^A[K] := \bigcup_{\sigma \in M} \sigma(K)$. For $M = Hyp^{nd}(\tau)$ we will simply write χ_{nd}^A . We notice that $\chi_{M-nd}^A[K]$ consists of algebras of the same type. For a set $\mathcal{K} \in \mathcal{P}(\mathcal{P}(Alg(\tau)))$ of sets of algebras of type τ and a monoid M

of nd -hypersubstitutions we define $\chi_{M-nd}^A[\mathcal{K}] := \{\sigma(K) \mid K \in \mathcal{K}, \sigma \in M\}$. For $B_1, B_2 \in \mathcal{P}(W_\tau(X))$ we define equations $B_1 \approx B_2$. If $\Sigma \in \mathcal{P}(\mathcal{P}(W_\tau(X)) \times \mathcal{P}(W_\tau(X)))$ and $\sigma \in M \subseteq \text{Hyp}^{nd}(\tau)$ we define

$$\hat{\sigma}[\Sigma] := \{\hat{\sigma}[B_1] \approx \hat{\sigma}[B_2] \mid B_1 \approx B_2 \in \Sigma\}$$

and

$$\chi_{M-nd}^E[\Sigma] := \{\hat{\sigma}[B_1] \approx \hat{\sigma}[B_2] \mid B_1 \approx B_2 \in \Sigma, \sigma \in M\}.$$

For $M = \text{Hyp}^{nd}(\tau)$ we will use simply the notation χ_{nd}^E .

We want to prove that there is a close connection between both operators. Instead of $\chi_{M-nd}^A[\{\{A\}\}]$ we will write $\chi_{M-nd}^A[A]$. For $K \subseteq \chi_{M-nd}^A[A]$ and for a set $B \subseteq W_\tau(X)$ of terms we define the set B^K of induced term operations. For the set $\sigma(\mathcal{A})$ of derived algebras and for a set $B \in \mathcal{P}(W_\tau(X_n))$ of n -ary terms we define the set $B^{\sigma(\mathcal{A})}$ of term operations induced by the set $\sigma(\mathcal{A})$ of derived algebras as follows

Definition 2.1. Let $n \in \mathbb{N}^+$ and $B \in \mathcal{P}(W_\tau(X_n))$, let $\mathcal{A} = (A; (f_i^A)_{i \in I})$ be an algebra of type τ , let $\sigma \in \text{Hyp}^{nd}(\tau)$ be an nd -hypersubstitution and let $\sigma(\mathcal{A}) = \{(A; (l_i^A)_{i \in I}) \mid l_i \in \sigma(f_i)\}$ be the set of derived algebras. Then we define the set $B^{\sigma(\mathcal{A})}$ of term operations induced by the set $\sigma(\mathcal{A})$ of derived algebras as follows:

- (i) If $B := \{x_j\}$ for $1 \leq j \leq n$, then $B^{\sigma(\mathcal{A})} := \{e_j^{n, \rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A})\} = \{e_j^{n, A}\}$.
- (ii) If $B = \{f_i(t_1, \dots, t_{n_i})\}$ then

$$\begin{aligned} B^{\sigma(\mathcal{A})} &:= \{\hat{S}_n^{n_i, A}(\{f_i^{\rho(\mathcal{A})}\} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A}), \{t_1\}^{\sigma(\mathcal{A})}, \dots, \{t_{n_i}\}^{\sigma(\mathcal{A})})\} \\ &= \bigcup_{\rho(\mathcal{A}) \in \sigma(\mathcal{A})} \{\hat{S}_n^{n_i, A}(\{f_i^{\rho(\mathcal{A})}\}, \{t_1\}^{\sigma(\mathcal{A})}, \dots, \{t_{n_i}\}^{\sigma(\mathcal{A})})\} \\ &= \bigcup_{\rho(\mathcal{A}) \in \sigma(\mathcal{A})} \{f_i^{\rho(\mathcal{A})}(r_1, \dots, r_{n_i}) \mid r_k \in \{t_k\}^{\sigma(\mathcal{A})}, \text{ for } 1 \leq k \leq n_i\} \end{aligned}$$

where $f_i^{\rho(\mathcal{A})}$ denotes the fundamental operation of the algebra $\rho(\mathcal{A})$ belonging to the operation symbol f_i and assume that $\{t_k\}^{\sigma(\mathcal{A})}, 1 \leq k \leq n_i$, are already defined.

- (iii) If B is an arbitrary non-empty subset of $W_\tau(X_n)$, then we define $B^{\sigma(\mathcal{A})} := \bigcup_{b \in B} \{b\}^{\sigma(\mathcal{A})}$. If the set B is empty, then we define $B^{\sigma(\mathcal{A})} := \emptyset$.

For any term $t \in W_\tau(X_n)$ and a class G of algebras of type τ we define

$$t^G := \{t\}^G := \{t^{\mathcal{A}} \mid \mathcal{A} \in G\}.$$

Definition 2.2. Let \mathcal{A} be an algebra of type τ and let $K \subseteq \chi_{M-nd}^{\mathcal{A}}[\mathcal{A}]$ and let $n \geq 1$ be an integer. Then we define

- (i) If $B = \{x_j\}$ for $1 \leq j \leq n$, then $B^K = \{e_j^{n,\mathcal{A}}\} \subseteq \mathcal{T}^{(n)}(\mathcal{A})$.
- (ii) If $B = \{f_i(t_1, \dots, t_{n_i})\}$ and let $B_j = t_j^K \subseteq \mathcal{T}^{(n)}(\mathcal{A})$ for $1 \leq j \leq n_i$ are already known, then

$$B^K := \{\hat{S}_n^{n_i,\mathcal{A}}(S, B_1, \dots, B_{n_i}) \mid S = \{\rho(f_i)^{\mathcal{A}} \mid \rho \in Hyp(\tau), \rho(\mathcal{A}) \in K\} \subseteq \mathcal{T}^{(n_i)}(\mathcal{A})\}.$$

Finally for an arbitrary nonempty set $B \in \mathcal{P}(W_\tau(X))$ we set $B^K := \bigcup_{b \in B} \{b\}^K$ and for the empty set B we let $B^K := \emptyset$.

Definition 2.2 contains Definition 2.1 as a special case since for every $\sigma \in Hyp^{nd}(\tau)$ we have $\sigma(\mathcal{A}) \subseteq \chi_{M-nd}^{\mathcal{A}}[\mathcal{A}]$. We have also $\{\mathcal{A}\} \subseteq \chi_{M-nd}^{\mathcal{A}}[\mathcal{A}]$ and $\{\rho(\mathcal{A})\} \subseteq \chi_{M-nd}^{\mathcal{A}}[\mathcal{A}]$ for a hypersubstitution $\rho \in Hyp(\tau)$ and it is easy to see that for a single term $s \in W_\tau(X_n)$ we have $\{\hat{\rho}[s]\}^{\{\mathcal{A}\}} = \hat{\rho}[s]^{\mathcal{A}} = s^{\rho(\mathcal{A})} = \{s\}^{\{\rho(\mathcal{A})\}}$.

Now we prove:

Lemma 2.3. *Let $B \in \mathcal{P}(W_\tau(X_n))$ be an arbitrary set of n -ary terms of type τ , let $\mathcal{A} = (A; (f_i^{\mathcal{A}})_{i \in I})$ be an algebra of type τ and let σ be an nd -hypersubstitution of type τ . Then $\hat{\sigma}[B]^{\mathcal{A}} = B^{\sigma(\mathcal{A})}$.*

Proof. If B is empty, then all is clear. If B is nonempty we will give a proof by induction on the complexity of the terms from the set B .

If $B = \{x_j\}$ for $1 \leq j \leq n$, then $\hat{\sigma}[B]^{\mathcal{A}} = \{x_j\}^{\mathcal{A}} = \{e_j^{n,\mathcal{A}}\}$ by the definition of σ and by Definition 1.3. Further, by Definition 2.1 we have

$$B^{\sigma(\mathcal{A})} = \{x_j\}^{\sigma(\mathcal{A})} = \left\{ e_j^{n,\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A}) \right\} = \{e_j^{n,\mathcal{A}}\}$$

since all algebras $\rho(\mathcal{A})$ have the same universe. Therefore $\hat{\sigma}[B]^{\mathcal{A}} = B^{\sigma(\mathcal{A})}$ for $B = \{x_j\}$ for $1 \leq j \leq n$.

Now let $B = \{f_i(t_1, \dots, t_{n_i})\}$ and assume that $\hat{\sigma}[\{t_k\}]^{\mathcal{A}} = \{t_k\}^{\sigma(\mathcal{A})}$ for $1 \leq k \leq n_i$. Then

$$\begin{aligned} \hat{\sigma}[B]^{\mathcal{A}} &= \hat{\sigma}[\{f_i(t_1, \dots, t_{n_i})\}]^{\mathcal{A}} \\ &= \left[\hat{S}_n^{n_i}(\sigma(f_i), \hat{\sigma}[\{t_1\}], \dots, \hat{\sigma}[\{t_{n_i}\}]) \right]^{\mathcal{A}} \\ &= \hat{S}_n^{n_i,\mathcal{A}} \left(\sigma(f_i)^{\mathcal{A}}, \hat{\sigma}[\{t_1\}]^{\mathcal{A}}, \dots, \hat{\sigma}[\{t_{n_i}\}]^{\mathcal{A}} \right) \\ &= \hat{S}_n^{n_i,\mathcal{A}} \left(\{l_i \mid l_i \in \sigma(f_i)\}^{\mathcal{A}}, \hat{\sigma}[\{t_1\}]^{\mathcal{A}}, \dots, \hat{\sigma}[\{t_{n_i}\}]^{\mathcal{A}} \right) \\ &= \bigcup_{l_i \in \sigma(f_i)} \hat{S}_n^{n_i,\mathcal{A}} \left(\{l_i^{\mathcal{A}}\}, \hat{\sigma}[\{t_1\}]^{\mathcal{A}}, \dots, \hat{\sigma}[\{t_{n_i}\}]^{\mathcal{A}} \right) \\ &= \bigcup_{l_i \in \sigma(f_i)} \hat{S}_n^{n_i,\mathcal{A}} \left(\{l_i^{\mathcal{A}}\}, \{t_1\}^{\sigma(\mathcal{A})}, \dots, \{t_{n_i}\}^{\sigma(\mathcal{A})} \right) \\ &= \hat{S}_n^{n_i,\mathcal{A}} \left(\{f_i^{\rho(\mathcal{A})} \mid \rho(\mathcal{A}) \in \sigma(\mathcal{A})\}, \{t_1\}^{\sigma(\mathcal{A})}, \dots, \{t_{n_i}\}^{\sigma(\mathcal{A})} \right) \\ &= \{f_i(t_1, \dots, t_{n_i})\}^{\sigma(\mathcal{A})} \\ &= B^{\sigma(\mathcal{A})}. \end{aligned}$$

If B is a set of terms consisting of more than one element, then we have

$$\hat{\sigma}[B]^A = \left\{ \bigcup_{b \in B} \hat{\sigma}[\{b\}] \right\}^A = \bigcup_{b \in B} \hat{\sigma}[\{b\}]^A = \bigcup_{b \in B} \{b\}^{\sigma(A)} = B^{\sigma(A)}.$$

■

From Lemma 2.3 we obtain the "conjugate pair property" for the pair $(\chi_{M-nd}^A, \chi_{M-nd}^E)$ of operators. We use the notation $\mathcal{A} \models s \approx t$ if the algebra \mathcal{A} of type τ satisfies the equation $s \approx t$ of type τ as an identity and $K \models s \approx t$ if the class K satisfies $s \approx t$. Moreover, we define

Definition 2.4. Let $B_1, B_2 \subseteq W_\tau(X)$ be sets of terms of type τ and assume that \mathcal{A} is an algebra of type τ and that $K \subseteq \chi_{M-nd}^A[A]$ for a monoid $\mathcal{M} \subseteq \text{Hyp}^{nd}(\tau)$ of non-deterministic hypersubstitution. Then

$$K \models B_1 \approx B_2 \text{ iff } B_1^K = B_2^K.$$

Especially we have $\sigma[A] \models B_1 \approx B_2$ iff $B_1^{\sigma[A]} = B_2^{\sigma[A]}$ and $\{\mathcal{A}\} \models B_1 \approx B_2$ iff $B_1^{\{\mathcal{A}\}} = B_2^{\{\mathcal{A}\}}$ and this means $\mathcal{A} \models B_1 \approx B_2$ iff $B_1^{\mathcal{A}} = B_2^{\mathcal{A}}$.

From Lemma 2.3 we obtain the following conjugate property.

Theorem 2.5. Let \mathcal{A} be an algebra of type τ , and let $B_1 \approx B_2 \in \mathcal{P}(W_\tau(X)) \times \mathcal{P}(W_\tau(X))$ and assume that $\sigma \in \text{Hyp}^{nd}(\tau)$ be a non-deterministic hypersubstitution of type τ . Then

$$\sigma(\mathcal{A}) \models B_1 \approx B_2 \iff \mathcal{A} \models \hat{\sigma}[B_1] \approx \hat{\sigma}[B_2].$$

Proof.

$$\begin{aligned} \sigma(\mathcal{A}) \models B_1 \approx B_2 &\iff B_1^{\sigma(\mathcal{A})} = B_2^{\sigma(\mathcal{A})} \\ &\iff \hat{\sigma}[B_1]^{\mathcal{A}} = \hat{\sigma}[B_2]^{\mathcal{A}} \\ &\iff \mathcal{A} \models \hat{\sigma}[B_1] \approx \hat{\sigma}[B_2]. \end{aligned}$$

■

Let now $M \subseteq Hyp^{nd}(\tau)$ be a monoid of non-deterministic hypersubstitutions. Then we form the set $\bigcup\{\mathcal{P}(\chi_{M-nd}^A[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}$ and consider $\Sigma \subseteq \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$ and $\mathcal{K} \subseteq \bigcup\{\mathcal{P}(\chi_{M-nd}^A[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}$. Definition 2.4 defines a relation between both sets. In the usual way we obtain a Galois connection $(\mathcal{P}Mod; \mathcal{P}Id)$ of non-deterministic models and non-deterministic identities defined by

$$\mathcal{P}Mod\Sigma := \{K \mid K \subseteq \chi_{M-nd}^A[\mathcal{A}] \text{ for some algebra } \mathcal{A} \in Alg(\tau)\}$$

$$\text{and } \forall B_1 \approx B_2 \in \Sigma (K \models B_1 \approx B_2)\}$$

$$\mathcal{P}Id\mathcal{K} := \{B_1 \approx B_2 \mid B_1 \approx B_2 \in \mathcal{P}(W_\tau(X))^2 \text{ and } \forall K \in \mathcal{K} (K \models B_1 \approx B_2)\}$$

By definition, the operators $\chi_{M-nd}^A : \mathcal{P}(\mathcal{P}(Alg(\tau))) \rightarrow \mathcal{P}(\mathcal{P}(Alg(\tau)))$ and $\chi_{M-nd}^E : \mathcal{P}((\mathcal{P}(W_\tau(X)))^2) \rightarrow \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$ are completely additive. This means, for classes $\mathcal{K} \subseteq \mathcal{P}(\mathcal{P}(Alg(\tau)))$ the result of the application of χ_{M-nd}^A to \mathcal{K} is the union of the results obtained by application of χ_{M-nd}^A to the single classes $K \subseteq Alg(\tau) : \chi_{M-nd}^A[\mathcal{K}] = \bigcup_{\sigma \in M} \bigcup_{K \in \mathcal{K}} \sigma(K)$. In a corresponding way for a set $\Sigma \subseteq \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$ and a submonoid $M \subseteq Hyp^{nd}(\tau)$ we have $\chi_{M-nd}^E[\Sigma] = \bigcup_{\sigma \in M} \bigcup_{B_1 \approx B_2 \in \Sigma} \hat{\sigma}[B_1] \approx \hat{\sigma}[B_2]$. Therefore, both operators are monotone, i.e.

$$\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow \chi_{M-nd}^A[\mathcal{K}_1] \subseteq \chi_{M-nd}^A[\mathcal{K}_2]$$

and

$$\Sigma_1 \subseteq \Sigma_2 \Rightarrow \chi_{M-nd}^E[\Sigma_1] \subseteq \chi_{M-nd}^E[\Sigma_2].$$

Since $\sigma_{pid} \in M$ and $\sigma_{pid}(K) = \{K\}$, the operator χ_{M-nd}^A is extensive, i.e. $\mathcal{K} \subseteq \chi_{M-nd}^A[\mathcal{K}]$ for every class $\mathcal{K} \subseteq \mathcal{P}(\mathcal{P}(Alg(\tau)))$. Since $\hat{\sigma}_{pid}[\{B\}] = \{B\}$ for every $B \in \mathcal{P}(W_\tau(X))$, the operator χ_{M-nd}^E is also extensive. It turns out that both operators, χ_{M-nd}^A and χ_{M-nd}^E are closure operators. Altogether, we have

Theorem 2.6. *The pair $(\chi_{M-nd}^A, \chi_{M-nd}^E)$ is a conjugate pair of additive closure operators.*

Proof. From Theorem 2.5, there follows $\chi_{M-nd}^A[K] \models B_1 \approx B_2 \iff K \models \chi_{M-nd}^E[B_1 \approx B_2]$. By the previous remarks it is left to show that the operators χ_{M-nd}^A and χ_{M-nd}^E are idempotent. Extensivity of χ_{M-nd}^A and χ_{M-nd}^E ,

implies $\chi_{M-nd}^A[\mathcal{K}] \subseteq \chi_{M-nd}^A[\chi_{M-nd}^A[\mathcal{K}]]$ and $\chi_{M-nd}^E[\Sigma] \subseteq \chi_{M-nd}^E[\chi_{M-nd}^E[\Sigma]]$ for $\mathcal{K} \in \mathcal{P}(\bigcup\{\mathcal{P}(\chi_{M-nd}^A[\mathcal{A}]) \mid \mathcal{A} \in \text{Alg}(\tau)\})$ and $W \in \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$. We write $\mathcal{K} \models W$ iff $K \models A \approx B$ for all $K \in \mathcal{K}$ and all $B_1 \approx B_2 \in W$. We have to show that the opposite inclusions are satisfied. Let $\mathcal{B} \in \chi_{M-nd}^A[\chi_{M-nd}^A[\mathcal{K}]]$. Then there are nd -hypersubstitutions $\sigma_1, \sigma_2 \in M$ and an algebra $\mathcal{A} \in \mathcal{K}$ such that

$$\begin{aligned}
\mathcal{B} \in \sigma_1[\sigma_2(\mathcal{A})] &= \sigma_1[\{(A; (l_i^A)_{i \in I}) \mid l_i \in \sigma_2(f_i)\}] \\
&= \{\sigma_1(A; (l_i^A)_{i \in I}) \mid l_i \in \sigma_2(f_i)\} \\
&= \{\{(A; (h_i^A)_{i \in I}) \mid h_i \in \hat{\sigma}_1[l_i]\} \mid l_i \in \sigma_2(f_i)\} \\
&= \{(A; (h_i^A)_{i \in I}) \mid h_i \in \hat{\sigma}_1[l_i] \text{ and } l_i \in \sigma_2(f_i)\} \\
&= \{(A; (h_i^A)_{i \in I}) \mid h_i \in \hat{\sigma}_1[\sigma_2(f_i)]\} \\
&= \{(A; (h_i^A)_{i \in I}) \mid h_i \in (\sigma_1 \circ_{nd} \sigma_2)(f_i)\} \\
&= (\sigma_1 \circ_{nd} \sigma_2)(\mathcal{A}) \in \chi_{M-nd}^A[\mathcal{K}].
\end{aligned}$$

This shows $\chi_{M-nd}^A[\chi_{M-nd}^A[\mathcal{K}]] = \chi_{M-nd}^A[\mathcal{K}]$. Now let $B_1 \approx B_2 \in \chi_{M-nd}^E[\chi_{M-nd}^E[\Sigma]]$. Then there is an equation $U \approx V$ in Σ and an nd -hypersubstitution $\sigma_1, \sigma_2 \in M$ such that $B_1 \approx B_2 \in \hat{\sigma}_1[\sigma_2[U]] \approx \hat{\sigma}_1[\sigma_2[V]]$, i.e. $B_1 \approx B_2 \in (\sigma_1 \circ_{nd} \sigma_2)^\wedge[U] \approx (\sigma_1 \circ_{nd} \sigma_2)^\wedge[V] \in \chi_{M-nd}^E[U \approx V] \subseteq \chi_{M-nd}^E[\Sigma]$. \blacksquare

3. M -Nd-Solid Varieties

A solid variety V admits every mapping $\sigma : \{f_i \mid i \in I\} \rightarrow W_\tau(X)$ which maps n_i -ary operation symbols f_i to n_i -ary terms in the sense that every derived algebra $\sigma(\mathcal{A}) = (A; (\sigma(f_i)^A)_{i \in I})$ belongs to V . Equivalently if $s \approx t$ is an identity in a solid variety V , then $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ are also satisfied as identities in V for every hypersubstitution σ . We generalize the definition of a solid variety to M -solid non-deterministic varieties.

Definition 3.1. Let $\mathcal{M} \subseteq \mathcal{Hyp}^{nd}(\tau)$ be a monoid of non-deterministic hypersubstitutions of type τ . A variety V of type τ is said to be an M -solid non-deterministic variety, for short an $M - nd$ -solid variety, if $\{\{\mathcal{A}\} \mid \mathcal{A} \in V\} \models \{\hat{\sigma}[\{s\}] \approx \hat{\sigma}[\{t\}] \mid s \approx t \in IdV, \sigma \in M\}$. In the case that $\mathcal{M} = \mathcal{Hyp}^{nd}(\tau)$ we will speak of a solid non-deterministic variety, for short of an nd -solid variety.

Clearly, the class $Alg(\tau)$ of all algebras of type τ is nd -solid. The trivial variety (consisting only of one-element algebras of type τ) is also nd -solid. The class of all nd -solid varieties of type τ is contained in the class of all solid varieties of this type.

Example 3.2. There is no nontrivial nd -solid variety of semigroups.

Let V be a variety of semigroups. For a proof we consider the nd -hypersubstitutions $\sigma_1, \sigma_2 \in Hyp^{nd}(2)$ defined by $\sigma_1(f) = \{x, xy\}$ and $\sigma_2(f) = \{xy, yx\}$. If V were an nd -solid variety of semigroups, then the application of σ_1 to the associative law gives identities which are satisfied in V . Let $V^* := \{\{\mathcal{A}\} \mid \mathcal{A} \in V\}$, then $V^* \models \{\hat{\sigma}_1[f(x, f(y, z))]\} \approx \{\hat{\sigma}_1[f(f(x, y), z)]\}$ gives $V^* \models \{x, f(x, y), f(x, f(y, z))\} \approx \{x, f(x, y), f(x, z), f(f(x, y), z)\}$. Since every nd -solid variety is solid, this gives especially $V^* \models \{f(x, f(y, z))\} \approx \{f(x, z)\}$. Applying σ_2 to this identity gives $V^* \models \{f(x, f(y, z)), f(x, f(z, y)), f(z, f(y, x)), f(y, f(z, x))\} \approx \{f(x, z), f(z, x)\}$. We use again the fact that every nd -solid variety is solid and the previous identity and obtain $V^* \models \{f(x, z)\} \approx \{f(z, x)\}$ or $V^* \models \{f(x, y)\} \approx \{f(x, z)\}$ or $V^* \models \{f(x, z)\} \approx \{f(y, x)\}$. If we use again the fact that every nd -solid variety must be solid in each of the cases we obtain that V is trivial.

If an identity $s \approx t$ in a variety V is satisfied for all nd -hypersubstitutions we speak of an nd -hyperidentity. More generally we define

Definition 3.3. Let V be a variety of algebras of type τ , let $s \approx t$ be an identity satisfied in V and let $\mathcal{M} \subseteq \mathcal{Hyp}^{nd}(\tau)$ be a monoid of non-deterministic hypersubstitutions. Then $s \approx t$ is an $M - nd$ hyperidentity in V if $V^* \models \chi_{M-nd}^E[\{s\} \approx \{t\}]$ where $V^* = \{\{\mathcal{A}\} \mid \mathcal{A} \in V\}$. In this case we write $V \models_{M-nd-hyp} s \approx t$ and for $M = Hyp^{nd}(\tau)$ we will simply write $V \models_{nd-hyp} s \approx t$ and call $s \approx t$ an nd -hyperidentity in V .

The relation $K \models B_1 \approx B_2$ introduced in Definition 2.4 defines the Galois connection $(\mathcal{P}Mod, \mathcal{P}Id)$ with the operations

$$\mathcal{P}Mod : \mathcal{P}((\mathcal{P}(W_\tau(X)))^2) \rightarrow \mathcal{P}\left(\bigcup\{\mathcal{P}(\chi_{M-nd}^A[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}\right),$$

$$\mathcal{P}Id : \mathcal{P}\left(\bigcup\{\mathcal{P}(\chi_{M-nd}^A[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}\right) \rightarrow \mathcal{P}((\mathcal{P}(W_\tau(X)))^2).$$

The relation $\models_{M-nd-hyp}$ defines one more Galois connection

$$(H_{M-nd}\mathcal{P}Mod, H_{M-nd}\mathcal{P}Id)$$

for sets $\Sigma \subseteq \mathcal{P}((\mathcal{P}(W_\tau(X)))^2)$ and classes $\mathcal{K} \subseteq \bigcup\{\mathcal{P}(\chi_{M-nd}^A[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}$ as follows

$$H_{M-nd}\mathcal{P}Mod : \mathcal{P}((\mathcal{P}(W_\tau(X)))^2) \rightarrow \mathcal{P}\left(\bigcup\{\mathcal{P}(\chi_{M-nd}^A[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}\right),$$

$$H_{M-nd}\mathcal{P}Id : \mathcal{P}\left(\bigcup\{\mathcal{P}(\chi_{M-nd}^A[\mathcal{A}]) \mid \mathcal{A} \in Alg(\tau)\}\right) \rightarrow \mathcal{P}((\mathcal{P}(W_\tau(X)))^2).$$

The products $\mathcal{P}Mod\mathcal{P}Id$, $\mathcal{P}Id\mathcal{P}Mod$, $H_{M-nd}\mathcal{P}IdH_{M-nd}\mathcal{P}Mod$, $H_{M-nd}\mathcal{P}ModH_{M-nd}\mathcal{P}Id$ are closure operators and their fixed points are complete lattices. The lattice of all $M - nd$ -solid varieties arises if we restrict the operator $H_{M-nd}\mathcal{P}ModH_{M-nd}\mathcal{P}Id$ to classes of the form V^* where V is a variety of algebras of type τ . Moreover we have the conjugate pair $(\chi_{M-nd}^A, \chi_{M-nd}^E)$ of additive closure operators. Their fixed points form two more complete lattices. Now we may apply the theory of conjugate pairs of additive closure operators (see e.g. [?]) and obtain the following propositions:

Lemma 3.4. *Let $K \subseteq Alg(\tau)$ be a class of algebras and let $\Sigma \subseteq (\mathcal{P}W_\tau(X))^2$ be a set of equations. Then the following properties hold:*

- (i) $H_{M-nd}\mathcal{P}Id(K^*) = \mathcal{P}Id\chi_{M-nd}^A[K^*]$,
- (ii) $H_{M-nd}\mathcal{P}Id(K^*) \subseteq \mathcal{P}Id(K^*)$,
- (iii) $\chi_{M-nd}^E[H_{M-nd}\mathcal{P}Id(K^*)] = H_{M-nd}\mathcal{P}Id(K^*)$,
- (iv) $\chi_{M-nd}^A[\mathcal{P}Mod(H_{M-nd}\mathcal{P}Id(K^*))] = \mathcal{P}Mod(H_{M-nd}\mathcal{P}Id(K^*))$,
- (v) $H_{M-nd}\mathcal{P}Id(H_{M-nd}\mathcal{P}Mod(\Sigma)) = \mathcal{P}Id(\mathcal{P}Mod(\chi_{M-nd}^E[\Sigma]))$; and dually

- (i)' $H_{M-nd}\mathcal{P}Mod(\Sigma) = \mathcal{P}Mod\chi_{M-nd}^E(\Sigma)$,
- (ii)' $H_{M-nd}\mathcal{P}Mod(\Sigma) \subseteq \mathcal{P}Mod(\Sigma)$,
- (iii)' $\chi_{M-nd}^A[H_{M-nd}\mathcal{P}Mod(\Sigma)] = H_{M-nd}\mathcal{P}Mod(\Sigma)$,
- (iv)' $\chi_{M-nd}^E[\mathcal{P}Id(H_{M-nd}\mathcal{P}Mod(\Sigma))] = \mathcal{P}Id(H_{M-nd}\mathcal{P}Mod(\Sigma))$,
- (v)' $H_{M-nd}\mathcal{P}Mod[H_{M-nd}\mathcal{P}Id(K^*)] = \mathcal{P}Mod(\mathcal{P}Id(\chi_{M-nd}^A[K^*]))$.

Using these propositions one obtains the following characterization of $M - nd$ -solid varieties.

Theorem 3.5. *Let V be a variety of type τ and let Σ be an equational theory of type τ (i.e. $IdMod(\Sigma) = \Sigma$). Further we assume that $\mathcal{M} \subseteq \mathcal{Hyp}^{nd}(\tau)$ is a monoid of non-deterministic hypersubstitutions of type τ .*

Then the following propositions are equivalent:

- (i) $H_{M-nd}\mathcal{P}ModH_{M-nd}\mathcal{P}Id(V^*) = V^*$,
- (ii) $\chi_{M-nd}^A[V^*] = V^*$ (i.e. V^* is $M - nd$ solid),
- (iii) $\mathcal{P}Id(V^*) = H_{M-nd}\mathcal{P}Id(V^*)$ (i.e. every identity in V^* is satisfied as a non-deterministic hyperidentity),
- (iv) $\chi_{M-nd}^E[\mathcal{P}IdV^*] = \mathcal{P}IdV^*$.

4. $M - Nd$ -Solid Varieties of Semigroups

We consider some examples of $M - nd$ -solid varieties of semigroups and use the following notation for varieties of semigroups;

$B = Mod\{x(yz) \approx (xy)z, x^2 \approx x\}$ – the variety of bands,

$RB = Mod\{x(yz) \approx (xy)z \approx xz, x^2 \approx x\}$ – the variety of rectangular bands

$SL = Mod\{x(yz) \approx (xy)z, x^2 \approx x, xy \approx yx\}$ – the variety of semilattices, bands,

$LZ = Mod\{xy \approx x\}$ – the variety of left-zero bands.

Let $M = \{\sigma_{pid}, \sigma_1, \sigma_2\}$ with $\sigma_1(f) = \{x\}$ and $\sigma_2(f) = \{y\}$. Then M forms a monoid and the multiplication \circ_{nd} is given by the following table:

\circ_{nd}	σ_{pid}	σ_1	σ_2
σ_{pid}	σ_{pid}	σ_1	σ_2
σ_1	σ_1	σ_1	σ_2
σ_2	σ_2	σ_1	σ_2

We will prove the following proposition:

Proposition 4.1. *Let $M = \{\sigma_{pid}, \sigma_1, \sigma_2\}$ as defined before. A non-trivial variety V of semigroups is M – nd-solid iff $RB \subseteq V$.*

Proof. It is well-known that $IdRB$ is the set of all outermost equations of type $\tau = (2)$, i.e. the set of all equations $s \approx t$ such that the first variables in s and in t and the last variables in s and in t agree. Therefore $RB \subseteq V$ means that all identities in V are outermost and for any $s \approx t \in Id$ we have $\hat{\sigma}_1[s] = \{\text{first variable in } s\} = \{\text{first variable in } t\} = \hat{\sigma}_1[t]$ and $\hat{\sigma}_2[s] = \{\text{last variable in } s\} = \{\text{last variable in } t\} = \hat{\sigma}_2[t]$. Clearly $s \approx t$ is closed under σ_{pid} .

Conversely, let V be a nontrivial M – nd-solid variety. Then $\sigma_1, \sigma_2 \in M$ requires $RB \subseteq V$. ■

Let $var(B)$ be the set of all variables occurring in the set B of terms. Now let

$$M' = \{\sigma \in Hyp^{nd}(\tau) \mid var(\sigma(f)) = \{x\}\}.$$

Clearly $M' \cup \{\sigma_{pid}\}$ forms a submonoid of $Hyp^{nd}(\tau)$. Then we have

Proposition 4.2. *A non-trivial variety V of semigroups is M' – nd-solid iff $LZ \subseteq V \subseteq B$.*

Proof. It is well-known that $IdLZ$ is the set of all equations $s \approx t$ of type $\tau = (2)$ such that the first variable in s is equal to the first variable in t . Because of $var(\sigma(f)) = \{x\}$ the terms in $\hat{\sigma}[s]$ and the terms in $\hat{\sigma}[t]$ can be written as x^r and as x^l for some $r, l \in \mathbb{N}^+$. Since $V \subseteq B$ by the idempotent law all equations of the form $x^r \approx x^l$ are satisfied in V . This shows that V is M' – nd-solid.

Conversely, let V be a nontrivial M' – nd -solid variety of semigroups. If we apply σ with $\sigma(f) = \{x, x^2\}$ to the identity $f(x, y) \approx f(x, y)$ we obtain $x \approx x^2$, i.e. $V \subseteq B$. If we apply σ' with $\sigma'(f) = \{x\}$ we get $leftmost(s) \approx leftmost(t) \in IdV$ and this means $LZ \subseteq V$. Altogether, we have $LZ \subseteq V \subseteq B$. ■

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