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RETRACTS AND Q-INDEPENDENCE

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Dedicated to the memory of Professor Kazimierz Głazek

Abstract

A non-empty set X of a carrier A of an algebra $\mathbf{A}$ is called Q-*independent* if the equality of two term functions f and g of the algebra $\mathbf{A}$ on any finite system of elements $a_1, a_2, \ldots, a_n$ of X implies $f(p(a_1), p(a_2), \ldots, p(a_n)) = g(p(a_1), p(a_2), \ldots, p(a_n))$ for any mapping $p \in Q$. An algebra $\mathbf{B}$ is a *retract* of $\mathbf{A}$ if $\mathbf{B}$ is the image of a *retraction* (i.e. of an idempotent endomorphism of $\mathbf{B}$). We investigate Q-independent subsets of algebras which have a retraction in their set of term functions.

Keywords: general algebra, term function, Q-independence, M, I, S, S_0, A_1, G-independence, t-independence, retraction, retract, Stone algebra, skeleton and set of dense element of Stone algebra, Glivenko congruence.

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1. Introduction

A set X of elements of an algebra $\mathbf{A}$ is M-independent if the subalgebra generated by X is free over the equational class generated by $\mathbf{A}$ (see [12]). This definition (in an equivalent form) is due to E. Marczewski [13], who observed that many different concepts of independence used in various branches of mathematics are special cases of it. However this scheme of independence was not wide enough to cover stochastic independence,

independence in projective spaces and some others. As a common way of defining almost all known notions of independence E. Marczewski in [16] introduced the notion of independence with respect to a family Q of mappings.

More details and the best general reference can be found in K. Głazek [5, 7] and [8].

In [4] we investigated Q-independent subsets in Stone algebras for some specified families Q of mappings (e.g. M, S, S_0, G, I, and A_1), using the well-known triple representation of Stone algebras. Now, we summarize without the proofs these results and generalize them. We indicate connections between Q-independence in an abstract algebra and Q-independence in its subalgebras, reducts and retracts.

For a fixed algebra $\mathbf{A} = (A; \mathbb{F})$ we denote by $\mathbb{T}^{(n)}(\mathbf{A})$ $(n = 1, 2, ...)$ the class of all *n-ary term functions* of $\mathbf{A}$, i.e. the smallest class of functions satisfying the following conditions:

(i) $e_i^n \in \mathbb{T}^{(n)}(\mathbf{A})$, i.e. projections $e_i^n(x_1, x_2, \ldots, x_n) = x_i$ (for $i = 1, 2, \ldots, n$) are n-ary term functions;

(ii) if $g_1, g_2, \ldots, g_k \in \mathbb{T}^{(n)}(\mathbf{A})$, $f \in \mathbb{F}$ is a k-ary fundamental operation, then

$$\hat{f}(g_1, g_2, \ldots, g_k)(x_1, x_2, \ldots, x_n) = f(g_1(x_1, x_2, \ldots, x_n), \ldots, g_k(x_1, x_2, \ldots, x_n))$$

belongs to $\mathbb{T}^{(n)}(\mathbf{A})$.

$\mathbb{T}^{(0)}(\mathbf{A})$ denotes the set of all (algebraic) constant functions of the algebra $\mathbf{A}$. It is convenient to identify a constant function with it's value.

Let $\mathbf{A} = (A; \mathbb{F})$ be an algebra. Denote by $M(A)$ the family of all mappings $p : X \to A$ from every nonempty subset $X \subseteq A$ to A, and by $H(A)$ the set of all mappings $p : X \to A$ $(X \subseteq A)$ which possess an extension to a homomorphism $\overline{p}$ from $\langle X \rangle_{\mathbf{A}}$ (the subalgebra generated by X) to A $(\overline{p}|_X = p)$.

A nonempty set $X \subseteq A$ is said to be *independent with respect to the family* $Q \subseteq M(A)$ in algebra $\mathbf{A}$ (Q-independent or $X \in Ind(\mathbf{A}, Q)$, for short) if $Q \cap A^X \subseteq H(A)$, or equivalently

$$(\forall n \in \mathbb{N}, n \leq card(X))(\forall f, g \in \mathbb{T}^{(n)}(\mathbf{A}))(\forall p : X \to A)(\forall a_1, \ldots, a_n \in X)$$

$$[f(a_1, \ldots, a_n) = g(a_1, \ldots, a_n) \Rightarrow f(p(a_1), \ldots, p(a_n)) = g(p(a_1), \ldots, p(a_n))].$$

If we put $Q = M = \bigcup\{A^X \mid X \subseteq A\}$, we obtain *M-independence* (defined by E. Marczewski). For $Q = S = \bigcup\{\langle X\rangle_{\mathbf{A}}^X \mid X \subseteq A\}$, we get *S-independence* (*local independence* introduced by J. Schmidt in [18]). If $Q = S_0 = \bigcup\{X^X \mid X \subseteq A\}$, we have S_0-*independence* (*weak independence* in sense of S. Świerczkowski, [19]). For $Q = G = \bigcup\{p|_X \mid p \in A^A$ is diminishing, $X \subseteq A\}$ we get *G-independence* (*weak independence* in sense of G. Grätzer, [10]), where a mapping p is called *diminishing* if $(\forall f, g \in \mathbb{T}^{(1)}(\mathbf{A}))\ (\forall a \in A)\ [f(a) = g(a) \Rightarrow f(p(a)) = g(p(a))]$. Another notion of independence may be obtain by putting $Q = A_1 = \{f_{|X} \mid f \in \mathbb{T}^{(1)}(\mathbf{A}), X \subseteq A\}$ (introduced by K. Głazek in [6]). And for $Q = I = \bigcup\{p \mid p \in A^X$ injective, $X \subseteq A\}$, we get *I-independence* (defined as *R-independence* by K. Głazek, [6]). Let us recall that

$$
\begin{gathered}
Ind(\mathbf{A}, M) \subseteq Ind(\mathbf{A}, Q) \text{ for all } Q \subseteq M, \\
Ind(\mathbf{A}, S) \subseteq Ind(\mathbf{A}, S_0) \text{ and } Ind(\mathbf{A}, S) \subseteq Ind(\mathbf{A}, A_1).
\end{gathered} \tag{1}
$$

Another kind of independence, the so-called *t-independence*, was introduced by J. Płonka and W. Poguntke (see [17]). A set $X \subseteq A$ is called *t-independent* ($X \in Ind_t(\mathbf{A})$) in algebra $\mathbf{A} = (A; \mathbb{F})$ if for any finite system of different elements $a_1, \ldots, a_n \in X$ and for any n-ary term function f which is not a projection, we have $f(a_1, \ldots, a_n) \neq a_i$ for all $i = 1, \ldots, n$. It is easy to show that $Ind(\mathbf{A}, M) \subset Ind_t(\mathbf{A})$ for every $\mathbf{A}$. K. Głazek proved (see [6]) that for every family J of subsets of A such that $Ind(\mathbf{A}, M) \subset J$ there exists a family of mappings $Q \subset M$ satisfying the equality $Ind(\mathbf{A}, Q) = J$. So there exists a family of mappings Q such that $Ind_t(\mathbf{A}) = Ind(\mathbf{A}, Q)$, but the problem of defining this family for any algebra is still open.

An algebra $\mathbf{B}$ is a *retract* of $\mathbf{A}$ if $\mathbf{B}$ is the image of some *retraction* g (i.e. $g \in End(\mathbf{A})$ and $g(g(x)) = g(x)$ for all $x \in A$). Clearly $g(\mathbf{A}) = (g(A); \mathbb{F})$ is a subalgebra of $\mathbf{A}$. For $a \in g(\mathbf{A})$, we denote by $F_a = \{x \in A \mid g(x) = g(a) = a\}$ the equivalence class of a modulo kernel of the retraction g.

The remaining notions and notations used are rather standard, and for them the reader is referred to [2] and [12].

2. Q-independent subsets in Stone algebras

A *Stone algebra* is an algebra $\mathbf{L} = (L; \vee, \wedge, ^*, \mathbf{0}, \mathbf{1})$ of type $(2, 2, 1, 0, 0)$ such that $(L; \vee, \wedge, \mathbf{0}, \mathbf{1})$ is a distributive lattice with the least element $\mathbf{0}$ and the greatest element $\mathbf{1}$, * is a unary operation on L such that $a \wedge x = \mathbf{0}$ iff $x \leq a^*$

and the following *Stone identity* holds $x^* \vee x^{**} = \mathbf{1}$. We assume that the reader is familiar with the basic properties of Stone algebras, as presented in [1] or [11].

Two significant subsets of a Stone algebra $\mathbf{L}$ are the set of *dense* elements $D(L) = \{x \in L \mid x^* = \mathbf{0}\}$ and the *skeleton* $S(L) = \{x \in L \mid x^{**} = x\}$. Let $\mathbf{F}(D(L))$ denote the family of all filters of $D(L)$. The relationship between elements of $S(L)$ and $D(L)$ is expressed by the homomorphism $\varphi_L : S(L) \to \mathbf{F}(D(L))$ defined by $\varphi_L(a) = \{x \in D(L) \mid x \geq a^*\}$. C. C. Chen and G. Grätzer (see [3]) proved that the triple $(S(L), D(L), \varphi_L)$ characterizes $\mathbf{L}$ up to isomorphism.

It is easy to check that $g(x) = x^{**}$ is a retraction of $\mathbf{L}$ and $S(L) = g(L)$ is a retract. The kernel of this retraction is exactly the so-called *Glivenko congruence* θ and $[\mathbf{1}]_\theta = D(L)$. Every θ-class F_a contains exactly one element of $S(L)$, which is the greatest element in this class. Moreover, $\mathbf{F_a} = (F_a; \vee, \wedge)$ is a subalgebra of the reduct $\mathbf{L}_D = (L; \vee, \wedge)$ of the algebra $\mathbf{L}$, which is a distributive lattice. Define a mapping $\phi : L \to D(L)$ by $\phi(x) = x \vee x^*$. Then $\phi|_{F_a}$ is a lattice-isomorphism from F_a onto $\varphi_L(a)$ for every $a \in S(L)$.

Since the families of Q-independence sets in distributive lattices were precisely characterized (by G. Szász [20], E. Marczewski [14], J. Płonka, W. Poguntke [17], and A. Chwastyk, K. Głazek [4]), the next result establishes connections between Q-independence in distributive lattices and Q-independence in Stone algebras. For the proofs of the next two theorems we refer the reader to [4].

Theorem 1. *Let* $\mathbf{L} = (L; \vee, \wedge, ^*, \mathbf{0}, \mathbf{1})$ *be a Stone algebra. Then*

1) $(\forall a \in S(L))(\forall X \subseteq F_a)\ [X \in \mathit{Ind}(\mathbf{L}, S_0) \Leftrightarrow X \in \mathit{Ind}(\mathbf{F_a}, S_0)]$;

2) $(\forall X \subseteq D(L))[X \in \mathit{Ind}(\mathbf{L}, G) \Leftrightarrow X \setminus \{\mathbf{1}\} \in \mathit{Ind}(\mathbf{D}(\mathbf{L}), M)]$;

3) $(\forall X \subseteq L)[X \in \mathit{Ind}(\mathbf{L}, Q) \Rightarrow \phi(X) \in \mathit{Ind}(\mathbf{D}(\mathbf{L}), Q)]$ *for* $Q = M, S$ *or* S_0.

It is easily seen that the retract $\mathbf{S}(\mathbf{L}) = (S(L); \vee, \wedge, ^*, \mathbf{0}, \mathbf{1})$ is a Boolean algebra. The next theorem shows how the resent results of Q-independence in Boolean algebras (E. Marczewski [15], K. Głazek [6], K. Golema-Hartman [9]) may be used to investigate Q-independence in Stone algebras.

Theorem 2. *Let* $\mathbf{L} = (L; \vee, \wedge, ^*, \mathbf{0}, \mathbf{1})$ *be a Stone algebra and* $X \subseteq L$. *Then*

1) $X \in Ind_t(\mathbf{L}) \Leftrightarrow |X| = |g(X)| \ \wedge \ g(X) \in Ind(\mathbf{S}(\mathbf{L}), M)$;

2) $S(L) \supseteq X \in Ind(\mathbf{L}, Q) \Leftrightarrow X \in Ind(\mathbf{S}(\mathbf{L}), M)$ *for* $Q = S, S_0$ *or* G;

3) $X \in Ind(\mathbf{L}, Q) \Rightarrow g(X) \in Ind(\mathbf{S}(\mathbf{L}), Q)$ *for* $Q = M, S, S_0, I, G$.

3. Subalgebras, reducts, retractions and Q-independence

Now, we will formulate connections between Q-independence in algebras and Q-independence in their subalgebras and reducts.

Theorem 3. *Let* $\mathbf{A} = (A; \mathbb{F})$ *be an algebra,* $X \subseteq B \subseteq A$ *and* $\mathbb{F}' \subseteq \mathbb{F}$. *If* $\mathbf{B}' = (B; \mathbb{F}')$ *is a subalgebra of the reduct* $(A; \mathbb{F}')$ *of the algebra* $\mathbf{A}$, *then*

$$X \in Ind(\mathbf{A}, S_0) \Rightarrow X \in Ind(\mathbf{B}', S_0). \tag{2}$$

Moreover, if $\mathbf{B} = (B; \mathbb{F})$ *is a subalgebra of* $\mathbf{A}$ *and* $Q = S, S_0$ *or* A_1, *then*

$$X \in Ind(\mathbf{A}, Q) \Leftrightarrow X \in Ind(\mathbf{B}, Q). \tag{3}$$

Proof. Suppose that $X \subseteq B \subseteq A$ and $X \in Ind(\mathbf{A}, S_0)$. Let f_1, f_2 be n-ary term functions on some reduct $(A, \mathbb{F}')$ of the algebra $\mathbf{A}$ and $f_1(a_1, \ldots, a_n) = f_2(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in X$, $n \in N$. These term functions corresponding to some terms, which can be realized in the algebra $\mathbf{A}$ as n-ary term functions f_3, f_4. Then $f_3(a_1, \ldots, a_n) = f_1(a_1, \ldots, a_n) = f_2(a_1, \ldots, a_n) = f_4(a_1, \ldots, a_n)$ implies $f_3(p(a_1), \ldots, p(a_n)) = f_4(p(a_1), \ldots, p(a_n))$ for every $p : X \to X$. As $p(a_i) \in X \subseteq B$ $(i = 1, \ldots, n)$ we have $f_1(p(a_1), \ldots, p(a_n)) = f_2(p(a_1), \ldots, p(a_n))$. Thus $X \in Ind(\mathbf{B}', S_0)$.

In the case where $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ the implication (2) holds also for S and A_1-independence, because $q(a_i) \in B$ for all mappings $q : X \to \langle X \rangle_{\mathbf{B}}$ or $q = f_0|_X$ $(f_0 \in \mathbb{T}^{(1)}(\mathbf{A}))$.

For the converse implication, choose $X \in Ind(\mathbf{B}, Q)$, $Q = S, S_0$ or A_1 and $f_5(b_1, \ldots, b_n) = f_6(b_1, \ldots, b_n)$ for some $f_5, f_6 \in \mathbb{T}^{(n)}(\mathbf{A})$, $a_1, \ldots, a_n \in X$.

Since $\mathbf{B}$ is the subalgebra of $\mathbf{A}$, we conclude that $f_i(a_1, \ldots, a_n) \in B$ and $f_i|_B \in \mathbb{T}^{(n)}(\mathbf{B})$ for $i = 5, 6$. Hence $f_5(p(a_1), \ldots, p(a_n)) = f_6(p(a_1), \ldots, p(a_n))$ for every $p \in X^X$, $p \in \langle X \rangle_{\mathbf{A}}^X = \langle X \rangle_{\mathbf{B}}^X$ or $p = f_0|_X$, $f_0 \in \mathbb{T}^{(1)}(\mathbf{A})$, and the proof is complete. ■

The next result shows relations between Q-independence in algebras which have a retraction in their set of term functions and Q-independence in their retracts.

Theorem 4. *Let $\mathbf{A} = (A; \mathbb{F})$ be an algebra and $X \subseteq A$. If there exists a retraction g of $\mathbf{A}$ such that $g \in \mathbb{T}^{(1)}(\mathbf{A}) \setminus \mathbb{T}^{(0)}(\mathbf{A})$ and g is not the projection, then*

(a) $X \in Ind(\mathbf{A}, Q) \cup Ind_t(\mathbf{A}) \Rightarrow X \cap g(A) = \emptyset$ *for* $Q = M$ *or* I;

(b) $X \in Ind(\mathbf{A}, Q) \Rightarrow [X \subseteq g(A) \ \vee \ X \cap g(A) = \emptyset]$ *for* $Q = S_0$ *or* S;

(c) $X \in Ind(\mathbf{A}, Q) \Rightarrow g(X) \in Ind(g(\mathbf{A}), Q)$ *for* $Q = M$ *or* A_1;

(d) $g(A) \supseteq X \in Ind(g(\mathbf{A}), Q) \Rightarrow X \in Ind(\mathbf{A}, Q)$ *for* $Q = A_1, S, S_0, G$;

(e) $X \in Ind(\mathbf{A}, Q) \Rightarrow |g(X)| = |X|$ *for* $Q = M$ *or* I;

(f) $X \in Ind(\mathbf{A}, Q) \Rightarrow [|g(X)| = |X| \ \vee \ (\exists a \in g(A)) X \subseteq F_a]$ *for* $Q = S_0$ *or* S.

Proof.

(a) Suppose that there exists $a \in X \cap g(A)$. Then $e_1^1(a) = a = g(a)$ and, by assumption, $e_1^1 \neq g \in \mathbb{T}^{(1)}(\mathbf{A})$, so $\{a\} \notin Ind(\mathbf{A}, M)$ which is equivalent (see [6]) to $\{a\} \notin Ind(\mathbf{A}, I)$. From t-independence definition, we see that $\{a\} \notin Ind_t(\mathbf{A})$. Since the families of M, I and t-independent subsets are hereditary, we obtain a contradiction.

(b) Choose $X \in Ind(\mathbf{A}, S_0)$ and $a \in X \cap g(A)$, $b \in X$. Consider the term functions $f_3(x, y) = g(x)$, $e_1^2(x, y) = x$, (obviously, $f_3 \neq e_1^2$) and a mapping $p_3 : X \to X$ given by $p_3(x) = b$. Then $f_3(a, b) = g(a) = a = e_1^2(a, b)$ and $f_3(p_3(a), p_3(b)) = e_1^2(p_3(a), p_3(b))$, by S_0-independence. Hence $f_3(b, b) = e_1^2(b, b)$, which implies $g(b) = b$. We thus get $b \in g(A)$ and, in consequence, $X \subseteq g(A)$. According to (1), this implication holds also for S-independence.

(c) Let $f_1(a_1,\dots,a_n) = f_2(a_1,\dots,a_n)$ for some $f_1, f_2 \in \mathbb{T}^{(n)}(\mathbf{A})$, $a_1,\dots,a_n \in g(X)$. Certainly, $a_i = g(b_i)$ $(i = 1,\dots,n)$ for some $b_1,\dots,b_n \in X$. Then $f_1(g(b_1),\dots,g(b_n)) = f_2(g(b_1),\dots,g(b_n))$, so $g(f_1(b_1,\dots,b_n)) = g(f_2(b_1,\dots,b_n))$ and $\hat{g}(f_1), \hat{g}(f_2) \in \mathbb{T}^{(n)}(\mathbf{A})$.

Suppose now that $X \in Ind(\mathbf{A}, M)$, then $\hat{g}(f_1) = \hat{g}(f_2)$ in the algebra $\mathbf{A}$ (see [13]). Hence $\hat{g}(f_1)(c_1,\dots,c_n) = \hat{g}(f_2)(c_1,\dots,c_n)$ for every $c_1,\dots,c_n \in g(A)$. Since $c_i = g(c_i)$ $(i = 1,\dots,n)$, we get $f_1(c_1,\dots,c_n) = f_2(c_1,\dots,c_n)$, so $f_1 = f_2$ in $g(\mathbf{A})$. Consequently, $g(X) \in Ind(g(\mathbf{A}), M)$.

Taking $X \in Ind(\mathbf{A}, A_1)$, we obtain $g(f_1(p(a_1),\dots,p(a_n))) = g(f_2(p(a_1),\dots,p(a_n)))$ for every $p = f_0|_X$, $f_0 \in \mathbb{T}^{(1)}(\mathbf{A})$, this means that $f_1(g(p(a_1)),\dots,g(p(a_n))) = f_1(p(g(a_1)),\dots,p(g(a_n))) = f_1(p(b_1),\dots,p(b_n)) = f_2(p(b_1),\dots,p(b_n))$. Therefore $g(X) \in Ind(g(\mathbf{A}), A_1)$.

(d) Let $g(A) \supseteq X \in Ind(g(\mathbf{A}), Q)$ for $Q = A_1, S, S_0$ or G and $p \in Q \cap A^X$. It is easy to see that $p : X \to g(A)$. Moreover, g is an endomorphism, so $\langle X\rangle_{\mathbf{A}} = \langle X\rangle_{g(\mathbf{A})}$. Then p possess the extension to a homomorphism $\bar{p} : \langle X\rangle_{\mathbf{A}} \to g(A) \subseteq A$, which yields $X \in Ind(\mathbf{A}, Q)$.

(e) On the contrary, suppose that $g(a) = g(b)$ for some $a, b \in X$. Define two binary term functions $f_3(x,y) = g(x)$ and $f_4(x,y) = g(y)$. As g is not a constant function, we have $g(c) \neq g(a)$ for some $c \in A$. Considering an injective mapping $p_1 : \{a,b\} \to A$ defined by $p_1(a) = a$, $p_1(b) = c$, we get $f_3(a,b) = g(a) = g(b) = f_4(a,b)$, but $f_3(p_1(a), p_1(b)) = g(p_1(a)) = g(a) \neq g(c) = g(p_1(b)) = f_4(p_1(a), p_1(b))$, which shows that $\{a,b\}$ is not I-independent in algebra $\mathbf{A}$, so it is not M-independent, by (1). In consequence, $X \notin Ind(\mathbf{A}, Q)$ for $Q = M, I$.

(f) This follows by the same method as in (e). ∎

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