

LATTICES OF RELATIVE COLOUR-FAMILIES AND ANTIVARIETIES

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Abstract

We consider general properties of lattices of relative colour-families and antivarieties. Several results generalise the corresponding assertions about colour-families of undirected loopless graphs, see [1]. Conditions are indicated under which relative colour-families form a lattice. We prove that such a lattice is distributive. In the class of lattices of antivarieties of relation structures of finite signature, we distinguish the most complicated (universal) objects. Meet decompositions in lattices of colour-families are considered. A criterion is found for existence of irredundant meet decompositions. A connection is found between meet decompositions and bases for anti-identities.

Keywords: colour-family, antivariety, lattice of antivarieties, meet decomposition, basis for anti-identities.

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1. PRELIMINARY FACTS

Throughout the article, by a *structure* we mean a relation structure of a fixed signature $\sigma = (r_j)_{j \in J}$. A structure is said to be *finite* if its universe is a finite set. A *homomorphism* from a structure \mathcal{A} into a structure \mathcal{B} is a map $\varphi : A \rightarrow B$ such that $(\varphi(a_1), \dots, \varphi(a_n)) \in r_j^{\mathcal{B}}$ for all $j \in J$ and

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$a_1, \dots, a_n \in A$ with $(a_1, \dots, a_n) \in r_j^A$. If there exists a homomorphism from \mathcal{A} into \mathcal{B} then we write $\mathcal{A} \rightarrow \mathcal{B}$; otherwise, we write $\mathcal{A} \nrightarrow \mathcal{B}$.

For every class \mathbf{K} , let \mathbf{K}_f denote the set of isomorphism types of finite structures in \mathbf{K} . On \mathbf{K}_f , define an equivalence relation \equiv as follows: $\mathcal{A} \equiv \mathcal{B}$ if and only if $\mathcal{A} \rightarrow \mathcal{B}$ and $\mathcal{B} \rightarrow \mathcal{A}$. The relation \rightarrow induces a partial order \leq on the quotient set \mathbf{K}_f / \equiv . Let $\text{Core}(\mathbf{K})$ denote the resulting partially ordered set.

In the sequel, it is convenient to consider an isomorphic partially ordered set whose universe is the set of cores of finite structures in \mathbf{K} . Recall [2, Section 2] that a finite structure is a *core* if all its endomorphisms are automorphisms. A structure \mathcal{A} is a *core of a structure* \mathcal{B} if \mathcal{A} is a minimal retract of \mathcal{B} (with respect to set inclusion). Simple properties of cores can be found, for example, in [2, Proposition 2.1]. It is easy to see that, in every coset \mathcal{G} / \equiv , there exists a unique (up to isomorphism) core. We denote this core by $\text{Core}(\mathcal{G})$. The map defined by the rule $\mathcal{G} / \equiv \mapsto \text{Core}(\mathcal{G})$ is an isomorphism.

Let \mathbf{K} be a class of structures. For every $\mathcal{A} \in \mathbf{K}$, let

$$[\mathbf{K} \rightarrow \mathcal{A}] = \{\mathcal{B} \in \mathbf{K} : \mathcal{B} \rightarrow \mathcal{A}\}.$$

If there is no ambiguity or \mathbf{K} is the class of all structures of a given signature then we write $[\rightarrow \mathcal{A}]$ instead of $[\mathbf{K} \rightarrow \mathcal{A}]$. For every set $\mathbf{A} \subseteq \mathbf{K}$, let $[\mathbf{K} \rightarrow \mathbf{A}] = \bigcup_{\mathcal{A} \in \mathbf{A}} [\mathbf{K} \rightarrow \mathcal{A}]$. If \mathbf{A} is a finite set of finite structures then $[\mathbf{K} \rightarrow \mathbf{A}]$ is called a *\mathbf{K} -colour-family*. If $|\mathbf{A}| = 1$ then the \mathbf{K} -colour-family $[\mathbf{K} \rightarrow \mathbf{A}]$ is said to be *principal*. Let $L_0(\mathbf{K})$ denote the partially ordered set of all \mathbf{K} -colour-families with respect to set inclusion. (If $L_0(\mathbf{K})$ has no greatest element then by $L_0(\mathbf{K})$ we mean the set of all \mathbf{K} -colour-families with a new greatest element $1_{\mathbf{K}}$).

We recall the definition of operations with structures from [3]. Let \mathcal{A} and \mathcal{B} be structures and let $(\mathcal{A}_i)_{i \in I}$ be a family of structures.

On the disjoint union of the universes of \mathcal{A} and \mathcal{B} , define a structure $\mathcal{A} + \mathcal{B}$ of signature σ as follows: $(a_1, \dots, a_n) \in r^{\mathcal{A} + \mathcal{B}}$ if and only if either $(a_1, \dots, a_n) \in r^{\mathcal{A}}$ or $(a_1, \dots, a_n) \in r^{\mathcal{B}}$. The resulting structure is called the *sum* of \mathcal{A} and \mathcal{B} . We have

$$(1) \quad \mathcal{A} + \mathcal{B} \rightarrow \mathcal{C} \iff \mathcal{A} \rightarrow \mathcal{C} \text{ and } \mathcal{B} \rightarrow \mathcal{C}$$

for every structure \mathcal{C} . A structure \mathcal{A} is said to be *connected* if it cannot be represented in the form $\mathcal{A}_1 + \mathcal{A}_2$, where $\mathcal{A} \nrightarrow \mathcal{A}_i$, $i = 1, 2$.

On the Cartesian product of the universes of \mathcal{A}_i , $i \in I$, define a structure $\prod_{i \in I} \mathcal{A}_i$ of signature σ as follows: $(a_1, \dots, a_n) \in r^{\prod_{i \in I} \mathcal{A}_i}$ if and only if $(a_1(i), \dots, a_n(i)) \in r^{\mathcal{A}_i}$ for every $i \in I$. The resulting structure is called the *product* of the family $(\mathcal{A}_i)_{i \in I}$. We have

$$(2) \quad \mathcal{C} \rightarrow \prod_{i \in I} \mathcal{A}_i \iff \mathcal{C} \rightarrow \mathcal{A}_i \text{ for all } i \in I$$

for every structure \mathcal{C} .

On the set A^B of all functions from B into A , define a structure \mathcal{A}^B of signature σ as follows: $(f_1, \dots, f_n) \in r^{\mathcal{A}^B}$ if and only if $(f_1(b_1), \dots, f_n(b_n)) \in r^{\mathcal{A}}$ for all $b_1, \dots, b_n \in B$ with $(b_1, \dots, b_n) \in r^B$. The resulting structure is called the *exponent* of \mathcal{A} by B . We have

$$(3) \quad \mathcal{C} \rightarrow \mathcal{A}^B \iff \mathcal{B} \times \mathcal{C} \rightarrow \mathcal{A}$$

for every structure \mathcal{C} .

2. LATTICES OF COLOUR-FAMILIES

In this section, we show that partially ordered sets of \mathbf{K} -colour-families are usually lattices and study their lattice-theoretical properties.

Lemma 1. *If \mathbf{K} is a class satisfying the condition*

$$(4) \quad \mathcal{A} \times \mathcal{B} \in \mathbf{K} \text{ for all finite } \mathcal{A}, \mathcal{B} \in \mathbf{K}$$

then $L_0(\mathbf{K})$ is a distributive lattice with respect to the set-theoretical operations.

Proof. Let $\mathbf{K}_1 = [\mathbf{K} \rightarrow (\mathcal{A}_i)_{i < n}]$, $\mathbf{K}_2 = [\mathbf{K} \rightarrow (\mathcal{B}_j)_{j < m}]$. It is clear that $\mathbf{K}_1 \cup \mathbf{K}_2$ is a \mathbf{K} -colour-family, i.e., $\mathbf{K}_1 \vee \mathbf{K}_2 = \mathbf{K}_1 \cup \mathbf{K}_2$.

Let $\mathcal{A} \in \mathbf{K}_1 \cap \mathbf{K}_2$. Then $\mathcal{A} \in \mathbf{K}$ and there exist $i < n$ and $j < m$ such that $\mathcal{A} \rightarrow \mathcal{A}_i$ and $\mathcal{A} \rightarrow \mathcal{B}_j$. Hence, $\mathcal{A} \rightarrow \mathcal{A}_i \times \mathcal{B}_j$, cf. (2). By (4), we have $\mathcal{A}_i \times \mathcal{B}_j \in \mathbf{K}$. Conversely, if $\mathcal{A} \rightarrow \mathcal{A}_i \times \mathcal{B}_j$, where $\mathcal{A} \in \mathbf{K}$, $i < n$, and $j < m$, then we have $\mathcal{A} \rightarrow \mathcal{A}_i$ and $\mathcal{A} \rightarrow \mathcal{B}_j$. Hence, $\mathbf{K}_1 \cap \mathbf{K}_2 = \left[\mathbf{K} \rightarrow (\mathcal{A}_i \times \mathcal{B}_j)_{\substack{i < n, \\ j < m}} \right]$. In view of (4), we have $\mathcal{A}_i \times \mathcal{B}_j \in \mathbf{K}$ for all $i < n$ and $j < m$. ■

Remark 2. Another lattice of (principal) colour-families was considered in [4, 5]. In fact, the universe of that lattice is the set of cores, the meet operation coincides with the meet operation in $L_0(\mathbf{K})$, while the join operation corresponds to the sum of relation structures. That lattice is distributive too.

Let L be a lattice and let $a, b \in L$. By a *relative pseudocomplement* of a with respect to b we mean an element $a * b$ such that

$$a \wedge x \leq b \iff x \leq a * b$$

for all $x \in L$. If a relative pseudocomplement exists for every pair of elements of L then L is said to be a *relatively pseudocomplemented lattice*.

Lemma 3. *If \mathbf{K} is a class satisfying (4) and the condition*

$$(5) \quad \mathcal{A}^{\mathcal{B}} \in \mathbf{K} \text{ for all finite } \mathcal{A}, \mathcal{B} \in \mathbf{K} \text{ with } \mathcal{B} \twoheadrightarrow \mathcal{A}$$

then $L_0(\mathbf{K})$ is a relatively pseudocomplemented lattice.

Proof. Let $\mathbf{K}_1 = [\mathbf{K} \rightarrow (\mathcal{A}_i)_{i < n}]$ and let $\mathbf{K}_2 = [\mathbf{K} \rightarrow (\mathcal{B}_j)_{j < m}]$. We introduce the notation $\mathbf{K}^i = [\mathbf{K} \rightarrow (\mathcal{B}_j^{\mathcal{A}_i})_{j < m}]$. By (5), we have $\mathcal{B}_j^{\mathcal{A}_i} \in \mathbf{K}$ provided $\mathcal{A}_i \twoheadrightarrow \mathcal{B}_j$. If $\mathcal{A}_i \twoheadrightarrow \mathcal{B}_j$ then $\mathcal{A}_i \times \mathcal{C} \twoheadrightarrow \mathcal{B}_j$ for every $\mathcal{C} \in \mathbf{K}$. By (3), we obtain $\mathcal{C} \twoheadrightarrow \mathcal{B}_j^{\mathcal{A}_i}$, i.e., $[\mathbf{K} \rightarrow \mathcal{B}_j^{\mathcal{A}_i}] = \mathbf{K}$ is the greatest element of $L_0(\mathbf{K})$. Therefore, $[\mathbf{K} \rightarrow \mathcal{B}_j^{\mathcal{A}_i}] \in L_0(\mathbf{K})$ for all $i < n$ and $j < m$.

By Lemma 1, we have $[\mathbf{K} \rightarrow \mathcal{A}_i] \cap \mathbf{K}^i = [\mathbf{K} \rightarrow (\mathcal{B}_j^{\mathcal{A}_i} \times \mathcal{A}_i)_{j < m}]$. It is immediate from (3) that $\mathcal{B}_j^{\mathcal{A}_i} \times \mathcal{A}_i \twoheadrightarrow \mathcal{B}_j$. Hence, $[\mathbf{K} \rightarrow \mathcal{A}_i] \cap \mathbf{K}^i \subseteq \mathbf{K}_2$. Let $\mathbf{K}_3 = [\mathbf{K} \rightarrow (\mathcal{C}_k)_{k < l}]$ be a \mathbf{K} -colour-family such that $[\mathbf{K} \rightarrow \mathcal{A}_i] \cap \mathbf{K}_3 \subseteq \mathbf{K}_2$. By Lemma 1, we have $[\mathbf{K} \rightarrow \mathcal{A}_i] \cap \mathbf{K}_3 = [\mathbf{K} \rightarrow (\mathcal{A}_i \times \mathcal{C}_k)_{k < l}]$. Therefore, for every $k < l$, there exists a $j < m$ such that $\mathcal{A}_i \times \mathcal{C}_k \twoheadrightarrow \mathcal{B}_j$. By definition, $\mathcal{C}_k \twoheadrightarrow \mathcal{B}_j^{\mathcal{A}_i}$. Thus, $\mathbf{K}_3 \subseteq \mathbf{K}^i$.

We have proven that \mathbf{K}^i is a pseudocomplement of $[\mathbf{K} \rightarrow \mathcal{A}_i]$ with respect to \mathbf{K}_2 . For every distributive lattice L and elements $a, b, c \in L$, if $a * c$ and $b * c$ exist then so does $(a \vee b) * c$ and $(a \vee b) * c = (a * c) \wedge (b * c)$, cf. [6, Theorem 9.2.3]. Hence, $\mathbf{K}_1 * \mathbf{K}_2 = \bigcap_{i < n} \mathbf{K}^i$. \blacksquare

3. LATTICES OF ANTIVARIETIES

Recall [2] that a \mathbf{K} -antivariety is a class defined in \mathbf{K} by some (possibly, empty) set of *anti-identities*, i.e., sentences of the form

$$\forall x_1 \dots \forall x_n (\neg R_1(\bar{x}) \vee \dots \vee \neg R_m(\bar{x})),$$

where each $R_i(\bar{x})$ is an atomic formula. By [2, Theorem 1.2], for every universal Horn class \mathbf{K} , a subclass \mathbf{K}' is a \mathbf{K} -antivariety if and only if $\mathbf{K}' = \mathbf{K} \cap \mathbf{H}^{-1}\mathbf{SP}_u^*(\mathbf{K}')$, where \mathbf{H}^{-1} , \mathbf{S} , and \mathbf{P}_u^* are operators for taking homomorphic pre-images, substructures, and nontrivial ultraproducts. In particular, each \mathbf{K} -colour-family is a \mathbf{K} -antivariety. Let $L(\mathbf{K})$ denote the partially ordered set of all \mathbf{K} -antivarieties with respect to set inclusion.

Lemma 4. *For every universal Horn class \mathbf{K} , the partially ordered set $L(\mathbf{K})$ is a distributive lattice with respect to the set-theoretical operations.*

Proof. It is clear that $\mathbf{K}_1 \wedge \mathbf{K}_2 = \mathbf{K}_1 \cap \mathbf{K}_2$ and $\mathbf{K}_1 \cup \mathbf{K}_2 \subseteq \mathbf{K}_1 \vee \mathbf{K}_2$ for all $\mathbf{K}_1, \mathbf{K}_2 \in L(\mathbf{K})$. Since \mathbf{K}_1 and \mathbf{K}_2 are elementary classes, the class $\mathbf{K}_1 \cup \mathbf{K}_2$ is elementary too. In particular, $\mathbf{P}_u^*(\mathbf{K}_1 \cup \mathbf{K}_2) \subseteq \mathbf{K}_1 \cup \mathbf{K}_2$. By [2, Theorem 1.2], we have $\mathbf{K}_1 \vee \mathbf{K}_2 = \mathbf{K} \cap \mathbf{H}^{-1}\mathbf{SP}_u^*(\mathbf{K}_1 \cup \mathbf{K}_2)$. Hence, for every $\mathcal{A} \in \mathbf{K}_1 \vee \mathbf{K}_2$, there exists a $\mathcal{B} \in \mathbf{K}_1 \cup \mathbf{K}_2$ such that $\mathcal{A} \rightarrow \mathcal{B}$. Since \mathbf{K}_1 and \mathbf{K}_2 are closed under $\mathbf{H}^{-1}\mathbf{S}$ in \mathbf{K} , we obtain $\mathcal{A} \in \mathbf{K}_1 \cup \mathbf{K}_2$ and, consequently, $\mathbf{K}_1 \vee \mathbf{K}_2 = \mathbf{K}_1 \cup \mathbf{K}_2$. ■

The proof of the following lemma is similar to that of [1, Lemma 3.4]

Lemma 5. *For every universal Horn class \mathbf{K} and \mathbf{K} -antivariety \mathbf{X} , we have $\mathbf{X} = \mathbf{H}^{-1}\mathbf{SP}_u^*(\text{Core}(\mathbf{X})) \cap \mathbf{K}$.*

If σ is a finite signature then the partially ordered set $\text{Core}(\mathbf{K})$ and the lattices $L_0(\mathbf{K})$ and $L(\mathbf{K})$ are related as follows.

Lemma 6. *For every universal Horn class \mathbf{K} of finite signature, the lattice $L(\mathbf{K})$ is isomorphic to the ideal lattice $I(L_0(\mathbf{K}))$ of the lattice $L_0(\mathbf{K})$ and to the lattice $I_o(\text{Core}(\mathbf{K}))$ of order ideals of the partially ordered set $\text{Core}(\mathbf{K})$.*

This lemma generalises [1, Theorem 3.6] to the case of arbitrary relation structures of finite signature. The proof follows the lines of the proof in [1].

Proof. Put $\varphi(\mathbf{K}') = \{\mathbf{A} \in L_0(\mathbf{K}) : \mathbf{A} \subseteq \mathbf{K}'\}$ for every $\mathbf{K}' \in L(\mathbf{K})$ and put $\psi(J) = \bigvee \{\mathbf{A} \in L_0(\mathbf{K}) : \mathbf{A} \in J\}$ for every ideal J of $L_0(\mathbf{K})$. It is easy to see that φ is a map from $L(\mathbf{K})$ into $I(L_0(\mathbf{K}))$ and ψ is a map from $I(L_0(\mathbf{K}))$ into $L(\mathbf{K})$; moreover, both maps are monotone. We prove that $\varphi = \psi^{-1}$, which implies that φ and ψ are isomorphisms.

It is clear that $\varphi\psi(J) \supseteq J$ and $\psi\varphi(\mathbf{K}') \subseteq \mathbf{K}'$ for all $J \in I(L_0(\mathbf{K}))$ and $\mathbf{K}' \in L(\mathbf{K})$.

Let $\mathcal{A} \in \mathbf{K}'$ be a finite structure. Then $[\mathbf{K} \rightarrow \mathcal{A}] \subseteq \mathbf{K}'$ because \mathbf{K}' is a \mathbf{K} -antivariety. We have $\mathcal{A} \in [\mathbf{K} \rightarrow \mathcal{A}] \subseteq \bigcup \{\mathbf{A} \in L_0(\mathbf{K}) : \mathbf{A} \subseteq \mathbf{K}'\}$. Since the \mathbf{K} -antivariety $\psi\varphi(\mathbf{K}')$ is generated by its finite structures, we obtain $\psi\varphi(\mathbf{K}') = \mathbf{K}'$.

Let $\mathbf{A} \in \varphi\psi(J)$, i.e., $\mathbf{A} \in L_0(\mathbf{K})$ and $\mathbf{A} \subseteq \bigvee_{\mathbf{B} \in J} \mathbf{B}$. Then $\mathbf{A} = [\mathbf{K} \rightarrow (\mathcal{A}_i)_{i < n}]$, where each finite structure \mathcal{A}_i belongs to $\bigvee_{\mathbf{B} \in J} \mathbf{B}$. By [2, Theorem 1.2], for every $i < n$, there exist a family $(\mathcal{B}_{ij})_{j \in J_i} \subseteq \bigcup_{\mathbf{B} \in J} \mathbf{B}$ and an ultrafilter U_i over J_i such that $\mathcal{A}_i \rightarrow \prod_{j \in J_i} \mathcal{B}_{ij}$. Since \mathcal{A}_i is a finite structure of finite signature, from [1, Lemma 3.2] it follows that there exists a $j(i) \in J_i$ such that $\mathcal{A}_i \rightarrow \mathcal{B}_{j(i)}$. Hence, $\mathbf{A} \subseteq [\mathbf{K} \rightarrow (\mathcal{B}_{j(i)})_{i < n}]$. Since $\mathcal{B}_{j(i)} \in \bigcup_{\mathbf{B} \in J} \mathbf{B}$, we obtain $[\mathbf{K} \rightarrow \mathcal{B}_{j(i)}] \in J$. Since J is an ideal, we conclude that $\bigvee_{i < n} [\mathbf{K} \rightarrow \mathcal{B}_{j(i)}] \in J$. Thus, $\mathbf{A} \in J$ and, consequently, $\varphi\psi(J) = J$.

For proving the fact that $L(\mathbf{K})$ and $I_o(\text{Core}(\mathbf{K}))$ are isomorphic put

$$\varphi_0(\mathbf{K}') = \{\text{Core}(\mathcal{A}) : \mathcal{A} \in \mathbf{K}'\}, \quad \psi_0(J) = \mathbf{H}^{-1}\mathbf{SP}_u^*(J) \cap \mathbf{K}.$$

By Lemma 5 and [1, Lemma 3.2], we have $\varphi_0^{-1} = \psi_0$. It is clear that φ_0 and ψ_0 are monotone. Therefore, the lattices $L(\mathbf{K})$ and $I_o(\text{Core}(\mathbf{K}))$ are isomorphic. ■

Corollary 7. *For every universal Horn class \mathbf{K} of finite signature, the lattice $L(\mathbf{K})$ is relatively pseudocomplemented. For all $\mathbf{K}_1, \mathbf{K}_2 \in L(\mathbf{K})$, the following equality holds: $\mathbf{K}_1 * \mathbf{K}_2 = \mathbf{H}^{-1}\mathbf{SP}_u^*\{\mathcal{A} \in \mathbf{K}_f : \mathcal{A} \times \mathcal{B} \in \mathbf{K}_2 \text{ for all } \mathcal{B} \in (\mathbf{K}_1)_f\} \cap \mathbf{K}$.*

Proof. By [7, Corollary II.1.4], we have $I * J = \{a \in L : a \wedge i \in J \text{ for all } i \in I\}$ for every distributive lattice L and ideals I and J of L . This equality, together with Lemma 6, yields the required assertion. ■

4. COMPLEXITY OF LATTICES OF ANTIVARIETIES

In this section, we introduce the notion of a universal (the most complicated) lattice among the lattices of antivarieties of relation structures of finite signature and give examples of universal lattices.

Let \mathbf{K} be a class of structures. By the category \mathbf{K} we mean the category whose objects are structures in \mathbf{K} and morphisms are homomorphisms. A one-to-one functor Φ from a category \mathbf{K}_1 into a category \mathbf{K}_2 is called a *full embedding* if, for every morphism $\alpha : \Phi(\mathcal{A}) \rightarrow \Phi(\mathcal{B})$ in \mathbf{K}_2 , there exists a morphism $\beta : \mathcal{A} \rightarrow \mathcal{B}$ in \mathbf{K}_1 such that $\Phi(\beta) = \alpha$. For more information about categories, the reader is referred to [8]. By \mathbf{G} we denote the class (and the category) of undirected loopless graphs.

Lemma 8. *Let \mathbf{K} be a class of structures and let there exist a full embedding $\Phi : \mathbf{G} \rightarrow \mathbf{K}$ such that, for every finite graph \mathcal{G} , the structure $\Phi(\mathcal{G})$ is finite. Then there exists an embedding $\varphi : \text{Core}(\mathbf{G}) \rightarrow \text{Core}(\mathbf{K})$.*

Proof. Put $\varphi(\mathcal{G}) = \text{Core}(\Phi(\mathcal{G}))$ for every $\mathcal{G} \in \text{Core}(\mathbf{G})$. It is clear that φ is a map from $\text{Core}(\mathbf{G})$ into $\text{Core}(\mathbf{K})$. We show that φ is an embedding.

Let $\mathcal{G} \leq \mathcal{H}$, where $\mathcal{G}, \mathcal{H} \in \text{Core}(\mathbf{G})$, and let ψ be the corresponding homomorphism. Then the composition

$$\varphi(\mathcal{G}) = \text{Core}(\Phi(\mathcal{G})) \xrightarrow{e} \Phi(\mathcal{G}) \xrightarrow{\Phi(\psi)} \Phi(\mathcal{H}) \xrightarrow{r} \text{Core}(\Phi(\mathcal{H})) = \varphi(\mathcal{H})$$

is a homomorphism from $\varphi(\mathcal{G})$ into $\varphi(\mathcal{H})$. Hence, $\varphi(\mathcal{G}) \leq \varphi(\mathcal{H})$.

Let $\varphi(\mathcal{G}) \leq \varphi(\mathcal{H})$ for some $\mathcal{G}, \mathcal{H} \in \text{Core}(\mathbf{G})$ and let ψ be the corresponding homomorphism. Then the composition

$$\Phi(\mathcal{G}) \xrightarrow{r} \text{Core}(\Phi(\mathcal{G})) = \varphi(\mathcal{G}) \xrightarrow{\psi} \varphi(\mathcal{H}) = \text{Core}(\Phi(\mathcal{H})) \xrightarrow{e} \Phi(\mathcal{H})$$

is a homomorphism from $\Phi(\mathcal{G})$ into $\Phi(\mathcal{H})$. Denote this homomorphism by α . Since Φ is a full embedding, we have $\alpha = \Phi(\beta)$ for some homomorphism $\beta : \mathcal{G} \rightarrow \mathcal{H}$. Hence, $\mathcal{G} \leq \mathcal{H}$.

It remains to show that φ is a one-to-one map. If $\varphi(\mathcal{G}) = \varphi(\mathcal{H})$ then $\varphi(\mathcal{G}) \leq \varphi(\mathcal{H})$ and $\varphi(\mathcal{H}) \leq \varphi(\mathcal{G})$. By the above, $\mathcal{G} \leq \mathcal{H}$ and $\mathcal{H} \leq \mathcal{G}$. Hence, $\mathcal{G} = \mathcal{H}$. ■

A category \mathbf{K} satisfying the conditions of Lemma 8 is said to be *finite-to-finite universal*. As is known [9] (see also [10, Theorem 2.10]), the partially ordered set $\text{Core}(\mathbf{G})$ is ω -universal, i.e., each countable partially ordered set

is embeddable into $\text{Core}(\mathbf{G})$. By Lemma 8, for every finite-to-finite universal category \mathbf{K} , the partially ordered set $\text{Core}(\mathbf{K})$ is ω -universal.

Recall that a lattice L is called a *factor* of a lattice K if L is a homomorphic image of a suitable sublattice of K . We say that $L(\mathbf{K})$ is a *universal lattice* if, for every universal Horn class \mathbf{K}' of relation structures of finite signature, the lattice $L(\mathbf{K}')$ is a factor of the lattice $L(\mathbf{K})$.

Theorem 9. *Let \mathbf{K} be a universal Horn class of relation structures of finite signature. If \mathbf{K} is a finite-to-finite universal category then $L(\mathbf{K})$ is a universal lattice.*

Proof. By Lemma 8, for every universal Horn class \mathbf{K}' of relation structures of finite signature, there exists an embedding $\varphi : \text{Core}(\mathbf{K}') \rightarrow \text{Core}(\mathbf{K})$.

By Lemma 6, we may consider the lattice of order ideals of the partially ordered set of cores instead of the lattice of antivarieties. For every $I \in \text{I}_o(\text{Core}(\mathbf{K}'))$, put

$$\psi(I) = \{\mathcal{H} \in \text{Core}(\mathbf{K}) : \mathcal{H} \leq \varphi(\mathcal{G}) \text{ for some } \mathcal{G} \in I\}.$$

It is easy to verify that, for every order ideal I of $\text{Core}(\mathbf{K}')$, the set $\psi(I)$ is an order ideal of $\text{Core}(\mathbf{K})$.

We prove that ψ is one-to-one. Let $\psi(I) = \psi(J)$. For every $\mathcal{H} \in I$, we have $\varphi(\mathcal{H}) \in \psi(I) = \psi(J)$, i.e., there exists an element $\mathcal{G} \in J$ such that $\varphi(\mathcal{H}) \leq \varphi(\mathcal{G})$. Since φ is an embedding, we have $\mathcal{H} \leq \mathcal{G}$, i.e., $\mathcal{H} \in J$. We have proven that $I \subseteq J$. The proof of the converse inclusion is similar.

We prove that ψ is a join homomorphism. Consequently, the join semilattice of $\text{I}_o(\text{Core}(\mathbf{K}'))$ is embeddable into the join semilattice of $\text{I}_o(\text{Core}(\mathbf{K}))$. The inclusion $\psi(I) \vee \psi(J) \subseteq \psi(I \vee J)$ is obvious. Conversely, let $\mathcal{H} \in \psi(I \vee J)$. Then there exists a $\mathcal{G} \in I \vee J$ such that $\mathcal{H} \leq \varphi(\mathcal{G})$. Since $I \vee J = I \cup J$, we obtain $\mathcal{H} \in \psi(I) \cup \psi(J) = \psi(I) \vee \psi(J)$.

Let L be the sublattice of $\text{I}_o(\mathbf{K})$ generated by the set $\{\psi(I) : I \in \text{I}_o(\text{Core}(\mathbf{K}'))\}$. Then, for every $X \in L$, there exist a lattice term $t(v_0, \dots, v_{n-1})$ and order ideals J_0, \dots, J_{n-1} of $\text{Core}(\mathbf{K}')$ such that $X = t(\psi(J_0), \dots, \psi(J_{n-1}))$.

We prove that, for every lattice term $t(v_0, \dots, v_{n-1})$ and order ideals J_0, \dots, J_{n-1} of $\text{Core}(\mathbf{K}')$, the equality

$$(6) \quad t(\psi(J_0), \dots, \psi(J_{n-1})) \cap \varphi(\text{Core}(\mathbf{K}')) = \varphi(t(J_0, \dots, J_{n-1}))$$

holds, where $\varphi(M) = \{\varphi(m) : m \in M\}$ for every set M .

We use induction on the length of the term. Let $t(v_0, \dots, v_{n-1}) = v_i$. Then the right-hand side of (6) is $\varphi(J_i)$ and the left-hand side of (6) is $\psi(J_i) \cap \varphi(\text{Core}(\mathbf{K}'))$. It is clear that $\varphi(J_i) \subseteq \psi(J_i) \cap \varphi(\text{Core}(\mathbf{K}'))$. Conversely, let $\mathcal{H} \in \psi(J_i) \cap \varphi(\text{Core}(\mathbf{K}'))$. Then $\mathcal{H} = \varphi(\mathcal{G})$ for some $\mathcal{G} \in \text{Core}(\mathbf{K}')$. Let J be the least ideal of $\text{Core}(\mathbf{K}')$ containing $J_i \cup \{\mathcal{G}\}$. We have $\psi(J) = \psi(J_i) \cup \{\mathcal{A} \in \text{Core}(\mathbf{K}') : \mathcal{A} \leq \varphi(\mathcal{G})\}$. Since $\varphi(\mathcal{G}) = \mathcal{H} \in \psi(J_i)$, we obtain $\psi(J) = \psi(J_i)$. Since ψ is a one-to-one map, we have $J = J_i$, i.e., $\mathcal{G} \in J_i$. Therefore, $\mathcal{H} \in \varphi(J_i) \subseteq \psi(J_i)$.

Assume that $t = t_1 \wedge t_2$ or $t = t_1 \vee t_2$ for some terms t_1 and t_2 . We introduce the notation

$$Y_i = t_i(\psi(J_0), \dots, \psi(J_{n-1})), \quad X_i = t_i(J_0, \dots, J_{n-1}),$$

where $i = 1, 2$. By induction, $Y_i \cap \text{Core}(\mathbf{K}') = \varphi(X_i)$, $i = 1, 2$.

If $t = t_1 \wedge t_2$ then $t(\psi(J_0), \dots, \psi(J_{n-1})) = Y_1 \cap Y_2$, $t(J_0, \dots, J_{n-1}) = X_1 \cap X_2$. By induction, $Y_1 \cap Y_2 \cap \varphi(\text{Core}(\mathbf{K}')) = \varphi(X_1) \cap \varphi(X_2) \supseteq \varphi(X_1 \cap X_2)$. For every $\mathcal{A} \in \varphi(X_1) \cap \varphi(X_2)$, there exist $\mathcal{A}_i \in X_i$, $i = 1, 2$, such that $\mathcal{A} = \varphi(\mathcal{A}_1) = \varphi(\mathcal{A}_2)$. Since φ is a one-to-one map, we obtain $\mathcal{A}_1 = \mathcal{A}_2 \in X_1 \cap X_2$.

If $t = t_1 \vee t_2$ then $t(\psi(J_0), \dots, \psi(J_{n-1})) = Y_1 \cup Y_2$, $t(J_0, \dots, J_{n-1}) = X_1 \cup X_2$. By induction, $(Y_1 \cup Y_2) \cap \varphi(\text{Core}(\mathbf{K}')) = (Y_1 \cap \varphi(\text{Core}(\mathbf{K}'))) \cup (Y_2 \cap \varphi(\text{Core}(\mathbf{K}'))) = \varphi(X_1) \cup \varphi(X_2) \subseteq \varphi(X_1 \vee X_2)$. The converse inclusion is an easy consequence of the equality $X_1 \vee X_2 = X_1 \cup X_2$.

Since the operations of the lattice of order ideals are the set-theoretical operations, the union and the intersection, from (6) we obtain

$$(7) \quad \varphi^{-1}(t(\psi(J_0), \dots, \psi(J_{n-1})) \cap \varphi(\text{Core}(\mathbf{K}'))) = t(J_0, \dots, J_{n-1}).$$

Let $X = t(\psi(J_0), \dots, \psi(J_{n-1})) \in L$. Put $\alpha(X) = \varphi^{-1}(X \cap \varphi(\text{Core}(\mathbf{K}')))$. It is immediate from (7) that α is a map from L onto $I_o(\text{Core}(\mathbf{K}'))$. By (6), α is a homomorphism. ■

We present an example showing that the converse to Theorem 9 is not true. Namely, we indicate a quasivariety \mathbf{K} of loopless digraphs such that \mathbf{K} is not a finite-to-finite universal category but $L(\mathbf{K})$ is a universal lattice.

Example 10. Let σ consist of one binary relation symbol r . Denote by \mathbf{K} the quasivariety of structures of the signature σ defined by the quasi-identities

$$\forall x \forall y (r(x, x) \rightarrow x \approx y),$$

$$\forall x \forall y \forall z (r(x, y) \& r(x, z) \rightarrow y \approx z),$$

$$\forall x \forall y \forall z (r(y, x) \& r(z, x) \rightarrow y \approx z).$$

Let \mathcal{C}_n , $n \geq 2$, denote the *cycle* of length n , i.e., the structure whose universe is $C_n = \{0, 1, \dots, n-1\}$ and $(i, j) \in r^{\mathcal{C}_n}$ if and only if $i+1 \equiv j \pmod{n}$. It is easy to see that, for every $n \geq 2$, we have $\mathcal{C}_n \in \mathbf{K}$.

Let \mathbb{P} denote the set of prime numbers. Denote by a_p , $p \in \mathbb{P}$, the \mathbf{K} -colour-family $[\mathbf{K} \rightarrow \mathcal{C}_p]$. Let L be the sublattice of $L_0(\mathbf{K})$ generated by the elements $(a_p)_{p \in \mathbb{P}}$.

We show that the distributive lattice L is freely generated by the set $(a_p)_{p \in \mathbb{P}}$. Since $|\mathbb{P}| = \omega$, this means that the free distributive lattice $F_{\mathbf{D}}(\omega)$ of countable rank is embedded into $L_0(\mathbf{K})$. We use [7, Theorem II.2.3]. It suffices to verify that, for all finite nonempty subsets $I, J \subseteq \mathbb{P}$, from $\bigwedge_{i \in I} a_i \leq \bigvee_{j \in J} a_j$ it follows that $I \cap J \neq \emptyset$.

Let I and J be finite and nonempty. Assume that $\bigwedge_{i \in I} a_i \leq \bigvee_{j \in J} a_j$. By Lemma 1, we have $\bigwedge_{i \in I} a_i = [\mathbf{K} \rightarrow \prod_{i \in I} \mathcal{C}_i]$ and $\bigvee_{j \in J} a_j = [\mathbf{K} \rightarrow (\mathcal{C}_j)_{j \in J}]$. Let $k = \prod_{i \in I} i$. It is easy to see that $\prod_{i \in I} \mathcal{C}_i \simeq \mathcal{C}_k$ (cf., for example, [11]). Since $\bigwedge_{i \in I} a_i \leq \bigvee_{j \in J} a_j$, there exists a prime $j \in J$ with $\mathcal{C}_k \in [\mathbf{K} \rightarrow \mathcal{C}_j]$. We have $\mathcal{C}_k \rightarrow \mathcal{C}_j$ if and only if j divides k (cf., for example, [12]). Since j is prime and k is a product of distinct primes, we conclude that $j \in I$. Thus, $I \cap J \neq \emptyset$.

We show that the ideal lattice $I(F_{\mathbf{D}}(\omega))$ of the free distributive lattice of countable rank is embeddable into $L(\mathbf{K})$. Let L and K be distributive lattices and let $\varphi : L \rightarrow K$ be an embedding. Define a map $\psi : I(L) \rightarrow I(K)$ by the following rule: $\psi(I)$ is the ideal of K generated by $\varphi(I)$. Using the definition of an ideal generated by a set, we easily find that ψ is an embedding. In particular, $I(F_{\mathbf{D}}(\omega))$ is embeddable into $I(L_0(\mathbf{K}))$. The latter lattice is isomorphic to $L(\mathbf{K})$ in view of Lemma 6.

We show that the lattice $L(\mathbf{G})$ of antivarieties of undirected loopless graphs is a homomorphic image of the lattice $I(F_{\mathbf{D}}(\omega))$. Since $L_0(\mathbf{G})$ is a countable distributive lattice, there exists a homomorphism from $F_{\mathbf{D}}(\omega)$ onto $L_0(\mathbf{G})$. As above, this homomorphism induces a homomorphism between the corresponding ideal lattices. It remains to use Lemma 6.

Therefore, $L(\mathbf{G})$ is a factor of $L(\mathbf{K})$. We conclude that $L(\mathbf{K})$ is a universal lattice. The class of rigid objects in the category \mathbf{K} consists of trivial structures and finite directed chains only (cf. [8, Exercise IV.1.6]). Therefore, the category \mathbf{K} is not universal and, consequently, is not finite-to-finite universal.

5. IRREDUNDANT MEET DECOMPOSITIONS IN LATTICES OF COLOUR-FAMILIES

Recall that \mathbf{G} denotes the universal Horn class and the category of undirected loopless graphs. The study of the lattice $L_0(\mathbf{G})$ was initiated in [1]. It was proven that this lattice possesses neither completely join irreducible nor completely meet irreducible nonzero elements. A simple description for join irreducible colour-families was found. The question on meet irreducible elements turned to be closely connected with a well-known problem in the graph theory, Hedetniemi's conjecture [13].

Here, we consider meet decompositions of \mathbf{K} -colour-families with the help of Lemma 3, which says that the lattice of \mathbf{K} -colour-families is relatively pseudocomplemented. A similar approach was first used in [4].

We present necessary definitions. By a *meet decomposition* of an element $x \in L$, where L is an arbitrary lattice, we mean a representation

$$(8) \quad x = \bigwedge_{i \in I} m_i,$$

where $(m_i)_{i \in I}$ is a family of meet irreducible elements, i.e., for each $i \in I$, we have $m_i \neq 1$ and from $m_i = a \wedge b$ it follows that either $m_i = a$ or $m_i = b$. A meet decomposition (8) is *irredundant* if $x < \bigwedge_{i \in J} m_i$ for every proper subset $J \subsetneq I$. For distributive relatively pseudocomplemented lattices, the following criterion for meet irreducibility of elements is known [14].

Proposition 11. *Let L be a distributive relatively pseudocomplemented lattice and let $m \in L$. The element m is meet irreducible if and only if $x * m = m$ for every $x \in L$ with $x \not\leq m$.*

Throughout this section, we assume that L is an arbitrary distributive relatively pseudocomplemented lattice. Let \vee , \wedge , and $*$ denote the operations of L . For every $x \in L$, let $\text{Reg}(x) = \{y * x : y \in L\}$, i.e., let $\text{Reg}(x)$ denote the

set of regular elements of the principal filter $[x]$ of L . For all $u, v \in \text{Reg}(x)$, put

$$u + v = ((u \vee v) * x) * x, \quad u \cdot v = u \wedge v, \quad 0 = x, \quad 1 = 1_L, \quad u' = u * x.$$

The set $\text{Reg}(x)$ with the operations $+$, \cdot , and $'$ and constants 1 and 0 is a Boolean algebra; moreover, the map r from L to $\text{Reg}(x)$ defined by the rule $r(y) = (y * x) * x$ is a homomorphism between Heyting algebras [6, Theorem 8.4.3].

We mention the following relationship between meet irreducible elements of L and dual atoms of $\text{Reg}(x)$ [4, Theorem 6].

Proposition 12. *Let $x, y \in L$ and let $x < y$. The element y is a dual atom of the Boolean algebra $\text{Reg}(x)$ if and only if $y * x > x$ and y is meet irreducible in L .*

Recall [15] that a Boolean algebra A is *atomic* if, for every nonzero element $a \in A$, there exists an atom b such that $b \leq a$. An element a is said to be *atomless* if $a \neq 0$ and there is no atom b such that $b \leq a$.

Theorem 13. *Let L be a distributive pseudocomplemented lattice and let $a \in L$. The element a admits an irredundant meet decomposition in L if and only if the Boolean algebra $\text{Reg}(a)$ is atomic.*

Proof. Let $a = \bigwedge_{i \in I} m_i$ be an irredundant meet decomposition. We prove that $m_i * a > a$ for every $i \in I$. In view of Proposition 12, this means that each m_i , $i \in I$, is a dual atom of $\text{Reg}(a)$. Since $a = \bigwedge_{i \in I} m_i$, we have $m_i * a = m_i * (\bigwedge_{i \in I} m_i) = \bigwedge_{m_j \not\leq m_i} m_j = \bigwedge_{j \neq i} m_j$ (cf. [16, IV.7.2 (8)]). Since the meet decomposition is irredundant, we have $m_i * a = \bigwedge_{j \neq i} m_j > a$.

Assume that there exists an atomless element $b \in \text{Reg}(a)$. Since the complement of a dual atom is an atom, we conclude that $m_i * a \not\leq b$ for all $i \in I$. Hence, $b \wedge (m_i * a) = a$ for all $i \in I$. By the definition of a relative pseudocomplement, we have $b \leq (m_i * a) * a = m_i$ for all $i \in I$. Therefore, $b \leq \bigwedge_{i \in I} m_i = a$. This proves that $\text{Reg}(a)$ possesses no atomless element, i.e., $\text{Reg}(a)$ is an atomic Boolean algebra.

Conversely, assume that $\text{Reg}(a)$ is an atomic Boolean algebra. Let $(a_i)_{i \in I}$ be the set of atoms. If $|\text{Reg}(a)| = 2$ then the element a is meet irreducible in view of Proposition 11. In the sequel, we assume that $|\text{Reg}(a)| > 2$. We denote $m_i = a_i * a$, $i \in I$. For every $i \in I$, the element m_i is a dual atom of $\text{Reg}(a)$; moreover, each dual atom is of the form m_i , $i \in I$.

By Proposition 12, we have $m_i * a > a$ and m_i is meet irreducible for every $i \in I$. It remains to show that $a = \bigwedge_{i \in I} m_i$ (in the lattice L). It is clear that a is a lower bound for $(m_i)_{i \in I}$. If a is not the greatest lower bound then there exists a lower bound $b_0 \in L$ for $(m_i)_{i \in I}$ such that $b_0 \not\leq a$. Consider the element $(b_0 * a) * a \in \text{Reg}(a)$. Since $b_0 \leq m_i$, we conclude that $(b_0 * a) * a \leq (m_i * a) * a = m_i$, $i \in I$. Hence, $b = (b_0 * a) * a \in \text{Reg}(a)$ is a lower bound for $(m_i)_{i \in I}$. Since $b_0 \not\leq a$ and $b_0 \leq b$, we find that $b \neq a$, i.e., $b > a$. Since $\text{Reg}(a)$ is an atomic Boolean algebra, there exists an atom a_j such that $a_j \leq b$. Passing to the complements, we find that $b' \leq m_j$. Since $b \leq m_i$ for all $i \in I$, we obtain $b + b' = 1 \leq m_j + m_j = m_j < 1$, a contradiction. ■

Similar questions for undirected loopless graphs were considered in [17], where the notion of the *level of nonmultiplicativity* of a graph was introduced. In our terminology, the level of nonmultiplicativity of a graph \mathcal{G} is the number of dual atoms of the Boolean algebra $\text{Reg}([\mathbf{G} \rightarrow \mathcal{G}])$. In [17], the following conjecture is stated: *The level of nonmultiplicativity of each finite graph is finite.* In connection with Theorem 13, we formulate the following

Problem 14. Let \mathbf{K} be a universal Horn class of relation structures of finite signature. Is this true that, for every \mathbf{K} -colour-family \mathbf{A} , the following conditions are equivalent:

- (1) there exists an irredundant meet decomposition $\mathbf{A} = \bigwedge_{i \in I} \mathbf{M}_i$,
- (2) there exists a finite meet decomposition $\mathbf{A} = \bigwedge_{i < n} \mathbf{M}_i$, $n < \omega$,
- (3) the Boolean algebra $\text{Reg}(\mathbf{A})$ is finite?

In the next section, we find a connection between this problem and existence of independent bases for anti-identities.

6. ANTI-IDENTITIES OF FINITE STRUCTURES

Recall that a set Σ of anti-identities is a *basis for anti-identities* of a class \mathbf{K} if \mathbf{K} is the class of structures in which all anti-identities of Σ are valid, i.e., $\mathbf{K} = \text{Mod}(\Sigma)$. By a basis for anti-identities of a structure \mathcal{A} we mean a basis for anti-identities of the antivariety generated by \mathcal{A} . A basis Σ is said to be *independent* if, for every $\varphi \in \Sigma$, the proper inclusion $\text{Mod}(\Sigma) \subsetneq \text{Mod}(\Sigma \setminus \{\varphi\})$ holds.

A structure \mathcal{A} is said to be *weakly atomic compact* if every locally consistent in \mathcal{A} set of atomic formulas is consistent in \mathcal{A} .

We reduce the question on existence of an independent basis for anti-identities of a finite relation structure of finite signature to Problem 14.

Let \mathcal{A} be a finite relation structure of finite signature and let $\Sigma = (\varphi_i)_{i \in I}$ be an independent basis for anti-identities of \mathcal{A} . With each anti-identity φ_i , $i \in I$, we associate a finitely presented structure \mathcal{B}_i as follows:

Let $\varphi_i \Leftarrow \forall \bar{x} (\neg \psi_1(\bar{x}) \vee \dots \vee \neg \psi_n(\bar{x}))$; then \mathcal{B}_i is the structure defined by generators \bar{x} and relations $\psi_1(\bar{x}), \dots, \psi_n(\bar{x})$.

It is easy to see that the antivariety defined by Σ coincides with the class

$$\bigcap_{i \in I} [\mathcal{B}_i \twoheadrightarrow] = \{\mathcal{B} : \mathcal{B}_i \twoheadrightarrow \mathcal{B} \text{ for all } i \in I\}.$$

Since Σ is an independent basis, we have $\mathcal{B}_i \rightarrow \mathcal{B}_j$ if and only if $i = j$.

Lemma 15. *The following two conditions are equivalent:*

- (1) $[\rightarrow \mathcal{A}] = \bigcap_{i \in I} [\mathcal{B}_i \twoheadrightarrow]$,
- (2) *there exists a family of finite structures $(\mathcal{A}_i)_{i \in I}$ such that $[\mathcal{B}_i \twoheadrightarrow] = [\rightarrow \mathcal{A}_i]$ for all $i \in I$ and $[\rightarrow \mathcal{A}] = [\rightarrow \prod_{i \in I} \mathcal{A}_i]$.*

Proof. It is clear that (2) implies (1). Indeed, if such a family exists then, for every structure \mathcal{C} , we have

$$\begin{aligned} \mathcal{C} \in [\rightarrow \mathcal{A}] = \left[\rightarrow \prod_{i \in I} \mathcal{A}_i \right] &\iff \mathcal{C} \rightarrow \mathcal{A}_i \text{ for all } i \in I \iff \\ &\iff \mathcal{C} \in [\mathcal{B}_i \twoheadrightarrow] \text{ for all } i \in I \iff \\ &\iff \mathcal{C} \in \bigcap_{i \in I} [\mathcal{B}_i \twoheadrightarrow]. \end{aligned}$$

We prove that (1) implies (2).

Notice that each structure \mathcal{B}_i , $i \in I$, is connected. Assume the contrary, i.e., let there exist an element $i \in I$ such that \mathcal{B}_i is not connected. Then $\mathcal{B}_i = \mathcal{B}_i^1 + \mathcal{B}_i^2$ for some structures \mathcal{B}_i^k with $\mathcal{B}_i \twoheadrightarrow \mathcal{B}_i^k$, $k = 1, 2$. Since $\mathcal{B}_j \twoheadrightarrow \mathcal{B}_i$ provided $j \neq i$, we have $\mathcal{B}_i^k \in [\mathcal{B}_j \twoheadrightarrow]$ for all $j \neq i$ and $k = 1, 2$. Therefore, $\mathcal{B}_i^k \in \bigcap_{i \in I} [\mathcal{B}_i \twoheadrightarrow] = [\rightarrow \mathcal{A}]$, $k = 1, 2$. Thus, $\mathcal{B}_i \in [\rightarrow \mathcal{A}] \subseteq [\mathcal{B}_i \twoheadrightarrow]$, which is a contradiction.

For an arbitrary $i \in I$, consider the interval $[\mathcal{A}, \mathcal{A} + \mathcal{B}_i]$ of the partially ordered set of cores. If there exists a core \mathcal{C} such that $\mathcal{A} \rightarrow \mathcal{C} \rightarrow \mathcal{A} + \mathcal{B}_i$ and $\mathcal{B}_i \twoheadrightarrow \mathcal{C} \twoheadrightarrow \mathcal{A}$ then, by (1), we obtain $\mathcal{C} \notin \bigcap_{i \in I} [\mathcal{B}_i \twoheadrightarrow]$. Hence, there exists a $j \in I$ such that $\mathcal{B}_j \rightarrow \mathcal{C}$. It is easy to see that $i \neq j$. Since $\mathcal{B}_j \rightarrow \mathcal{C} \rightarrow \mathcal{A} + \mathcal{B}_i$ and \mathcal{B}_j is connected, we conclude that $\mathcal{B}_j \rightarrow \mathcal{B}_i$, where $i \neq j$. Since Σ is an independent basis, we arrive at a contradiction. Thus, $\mathcal{A} + \mathcal{B}_i$ covers \mathcal{A} in the partially ordered set of cores (in symbols: $\mathcal{A} \prec \mathcal{A} + \mathcal{B}_i$). Since this is a distributive lattice (cf. Remark 2), we conclude that $\mathcal{A} \times \mathcal{B}_i \prec \mathcal{B}_i$. We denote $\mathcal{C}_i = \mathcal{A} \times \mathcal{B}_i$, $i \in I$. By (2) and (3), we obtain $\mathcal{C}_i \rightarrow \mathcal{A} \rightarrow \mathcal{C}_i^{\mathcal{B}_i}$ for all $i \in I$. By [5, Lemma 2.5], for every $i \in I$, the equality $[\rightarrow \mathcal{C}_i^{\mathcal{B}_i}] = [\mathcal{B}_i \twoheadrightarrow]$ holds. Put $\mathcal{A}_i = \mathcal{C}_i^{\mathcal{B}_i}$. We prove that $\bigcap_{i \in I} [\mathcal{B}_i \twoheadrightarrow] = [\rightarrow \prod_{i \in I} \mathcal{A}_i]$. We deduce

$$\begin{aligned} \mathcal{D} \in \bigcap_{i \in I} [\mathcal{B}_i \twoheadrightarrow] &\iff \mathcal{D} \in [\mathcal{B}_i \twoheadrightarrow] \text{ for all } i \in I \iff \\ &\iff \mathcal{D} \in [\rightarrow \mathcal{A}_i] \text{ for all } i \in I \iff \\ &\iff \mathcal{D} \in \left[\rightarrow \prod_{i \in I} \mathcal{A}_i \right]. \end{aligned}$$

Since $\bigcap_{i \in I} [\mathcal{B}_i \twoheadrightarrow] = [\rightarrow \mathcal{A}]$, we obtain $[\rightarrow \mathcal{A}] = [\rightarrow \prod_{i \in I} \mathcal{A}_i]$. Moreover, if the structure \mathcal{A} has no trivial substructure then the structure $\prod_{i \in I} \mathcal{A}_i$ has no trivial substructure either. \blacksquare

In the sequel, we assume that (equivalent) conditions (1) and (2) of Lemma 15 are satisfied. Without loss of generality, we may assume that \mathcal{A} is a core. Since \mathcal{A} is finite, the class $[\rightarrow \mathcal{A}]$ is elementary. By [2, Proposition 2.2], the structure $\prod_{i \in I} \mathcal{A}_i$ is weakly atomic compact and $[\rightarrow \prod_{i \in I} \mathcal{A}_i]$ is the antivariety generated by $\prod_{i \in I} \mathcal{A}_i$. In view of [2, Corollary 2.6], there exists a unique (up to isomorphism) core \mathcal{A}^* of $\prod_{i \in I} \mathcal{A}_i$; moreover, the antivarieties generated by \mathcal{A}^* and $\prod_{i \in I} \mathcal{A}_i$ coincide. Therefore, the antivarieties generated by \mathcal{A} and \mathcal{A}^* coincide. By [2, Corollary 2.5], the structures \mathcal{A} and \mathcal{A}^* are isomorphic.

Since \mathcal{A}^* is a finite structure of finite signature, there exists a finite subset $F \subseteq I$ such that \mathcal{A}^* is embeddable into $\prod_{i \in F} \mathcal{A}_i$.

We suggest the following

Conjecture 16. The equality $[\rightarrow \mathcal{A}] = [\rightarrow \prod_{i \in F'} \mathcal{A}_i]$ holds for some finite subset $F' \subseteq I$ with $F \subseteq F'$.

If this conjecture is true then Σ is a finite basis, which means that every finite relation structure of finite signature having no finite basis for its anti-identities possesses no independent basis for its anti-identities.

We return to Problem 14. Let \mathbf{K}_i be the principal colour-family generated by \mathcal{A}_i , $i \in I$. Then

$$[\rightarrow \mathcal{A}] = \bigwedge_{i \in I} \mathbf{K}_i$$

is an irredundant meet decomposition of $[\rightarrow \mathcal{A}]$ (in the lattice of colour-families). Therefore, if the answer to the question in Problem 14 is positive then Conjecture 16 is true.

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