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MAXIMAL SUBMONOIDS OF MONOIDS OF HYPERSUBSTITUTIONS

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Abstract

For a monoid M of hypersubstitutions, the collection of all M-solid varieties forms a complete sublattice of the lattice $\mathcal{L}(\tau)$ of all varieties of a given type τ. Therefore, by the study of monoids of hypersubstitutions one can get more insight into the structure of the lattice $\mathcal{L}(\tau)$. In particular, monoids of hypersubstitutions were studied in [9] as well as in [5]. We will give a complete characterization of all maximal submonoids of the monoid $Reg(n)$ of all regular hypersubstitutions of type $\tau = (n)$ (introduced in [4]). The concept of a transformation hypersubstitution, introduced in [1], gives a relationship between monoids of hypersubstitutions and transformation semigroups. In the present paper, we apply the recent results about transformation semigroups by I. Guydzenov and I. Dimitrova ([11], [12]) to describe monoids of transformation hypersubstitutions.

Keywords: regular hypersubstitutions, maximal monoids of hypersubstitutions, transformation semigroups.

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1. Introduction

A number of fairly natural examples of submonoids of the monoid $Hyp(\tau)$ of all hypersubstitutions of a given type τ is listed in [9]. In particular, the monoid $Reg(n)$ of all so-called regular hypersubstitutions of type (n),

$1 \leq n \in \mathbb{N}$, is studied. This monoid was first introduced by K. Denecke and J. Koppitz in [4] (see also [5], [6] and [9]). Properties of several monoids of hypersubstitutions of type (n) are studied by Th. Changphas ([2], [3]). For example, the monoid of all so-called full hypersubstitutions of type (n), i.e. hypersubstitutions σ where $\sigma(f)$ is a full term, is considered in [3]. The concept of a full term was introduced in [7]. On the other hand one can consider transformation hypersubstitutions ([1], [2]). A hypersubstitution σ of type $\tau = (n)$ is called a transformation hypersubstitution if $\sigma(f) = f(x_{s(1)}, \ldots, x_{s(n)})$ for some mapping $s : \overline{n} \to \overline{n}$, where $\overline{n} := \{1, \ldots, n\}$ ([1]). In the present paper, we will introduce particular submonoids of the monoid $TR(n)$ of all transformation hypersubstitutions of type $\tau = (n)$. K. Denecke and M. Reichel established a Galois-connection between monoids of hypersubstitutions of a given type τ and varieties of the same type, showing that for any monoid M of hypersubstitutions of type τ, the collection of all M-solid varieties of type τ forms a complete sublattice of the lattice of all varieties of type τ ([8]). It is a general goal of research in this area to study monoids of hypersubstitutions of a given type τ. In particular, it is of some interest to know what a monoid of hypersubstitutions looks like. In the present paper, we want to give a contribution to the research on monoids of hypersubstitutions. We will describe the monoid $Reg(n)$, $2 \leq n \in \mathbb{N}$, by characterization of its maximal submonoids. On the other hand we will consider submonoids of $TR(n)$ based on transformation semigroups. Using the recent results about isotone transformations with defect ≥ 2 ([12]) and monotone transformations ([13]), we are able to describe the appropriate monoids of transformation hypersubstitutions by characterization of their maximal submonoids.

In Section 2 we set out some notations concerning hypersubstitutions and introduce our new definitions. Section 3 works out all maximal submonoids of $Reg(n)$, $2 \leq n \in \mathbb{N}$. In Section 4 we describe particular submonoids of $TR(n)$ by characterization of their maximal submonoids.

2. Hypersubstitutions, terms and transformations

We fix a natural number $n \geq 1$ and an n-ary operation symbol f. Let $W_n(X)$ be the set of all terms of type (n) over some fixed alphabet $X = \{x_1, x_2, \ldots\}$. Terms in $W_n(X_k)$ with $X_k = \{x_1, \ldots, x_k\}$, $k \geq 1$, are called k-ary. For any term $s \in W_n(X_k)$ and $t_1, \ldots, t_k \in W_n(X)$, the term $s(t_1, \ldots, t_k)$ arises by substitution of terms, i.e. in the term s one replaces the variables $x_1, \ldots, x_k$ by the terms $t_1, \ldots, t_k$, respectively. The concept of a hypersubstitution will be a crucial one. A mapping $\sigma : \{f\} \to W_n(X_n)$ which assigns to the n-ary operation symbol f an n-ary term of type (n) will be called a hypersubstitution of type $\tau = (n)$. Here we have only one n-ary operation symbol f in our type and any hypersubstitution σ is completely determined by the image $\sigma(f)$. Thus we will denote a hypersubstitution σ by σ_t if $\sigma(f) = t$. Any hypersubstitution σ can be uniquely extended to a mapping $\widehat{\sigma} : W_n(X) \to W_n(X)$, inductively as follows:

(i) $\widehat{\sigma}[w] := w$ for $w \in X$;

(ii) $\widehat{\sigma}[f(t_1, \ldots, t_n)] := \sigma(f)(\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])$ for $t_1, \ldots, t_n \in W_n(X)$ where $\widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]$ will be assumed to be already defined.

We define a product $\circ_h$ of hypersubstitutions σ_1, σ_2 by $\sigma_1 \circ_h \sigma_2 := \widehat{\sigma}_1 \circ \sigma_2$, where $\circ$ is the usual composition of functions. Then the set $Hyp(n)$ of all hypersubstitutions of type $\tau = (n)$ forms a monoid $(Hyp(n); \circ_h, \sigma_{id})$, where σ_{id} is the identity hypersubstitution, defined by

$$\sigma_{id}(f) := f(x_1, \ldots, x_n).$$

Since $Hyp(n) = \{\sigma_t \mid t \in W_n(X_n)\}$,

$$\varphi_n : Hyp(n) \to W_n(X_n) \text{ with } \varphi_n(\sigma) = \sigma(f)$$

is a bijection. Let us define a binary operation $\diamond$ on $W_n(X_n)$ by setting

$$s \diamond t := \widehat{\sigma}_s[t].$$

Then one can verify that $(W_n(X_n); \diamond, \sigma_{id}(f))$ forms a monoid which is isomorphic to $(Hyp(n); \circ_h, \sigma_{id})$. If $\emptyset \neq X \subseteq W_n(X_n)$ then the carry set of the subsemigroup of $(W_n(X_n); \diamond)$ generated by X is denoted by $\langle X \rangle$.

Proposition 1. *Let* $1 \leq n \in \mathbb{N}$. *Then the monoid* $(Hyp(n); \circ_h, \sigma_{id})$ *is isomorphic to* $(W_n(X_n); \diamond, \sigma_{id}(f))$.

Proof. We want to show that the bijection φ_n is an isomorphism. Let us mention that $\varphi_n(\sigma_{id}) = \sigma_{id}(f)$ by definition of φ_n. Moreover, for $\sigma_s, \sigma_t \in Hyp(n)$ it holds $\varphi_n(\sigma_s \circ_h \sigma_t) = (\sigma_s \circ_h \sigma_t)(f) = \widehat{\sigma}_s[\sigma_t(f)] = s \diamond \sigma_t(f) = \sigma_s(f) \diamond \sigma_t(f) = \varphi_n(\sigma_s) \diamond \varphi_n(\sigma_t)$. ∎

Thus $(W_n(X_n); \diamond, \sigma_{id}(f))$ forms a monoid, which is isomorphic to the monoid of all hypersubstitutions of type $\tau = (n)$. This suggests the idea to study properties of the monoid $W_n(X_n)$ and its submonoids. We will use the following concepts and notation in the next statements and their proofs. For a term $t \in W_n(X_n)$ we put

$vb(t)$ – the total number of occurrences of variables in t(including multiplicities)

$op(t)$ – the number of occurrence of the operation symbol f in t

$vb_i(t)$ – the number of occurrence of x_i in t for $i \in \overline{n}$

$var(t)$ – the set of all variables occurring in t.

Notation 2. Let $1 \leq n \in \mathbb{N}$. Then we put $W_n^{reg} := \{t \mid t \in W_n(X_n)$ and $var(t) = X_n\}$.

The set W_n^{reg} corresponds to the set $Reg(n) = \{\sigma \mid \sigma \in Hyp(n)$ and $\sigma(f) \in W_n^{reg}\}$ of all regular hypersubstitutions of type $\tau = (n)$ which forms a monoid (see [5]). Clearly, $\sigma_{id}(f) \in W_n^{reg}$ and the monoid $(Reg(n); \circ_h, \sigma_{id})$ is isomorphic to $(W_n^{reg}; \diamond, \sigma_{id}(f))$ by the isomorphism φ_n restricted to $Reg(n)$, i.e. W_n^{reg} forms a monoid which is isomorphic to the monoid of all regular hypersubstitutions of type $\tau = (n)$. Our first aim is to determine all maximal submonoids of $(W_n^{reg}; \diamond, \sigma_{id}(f))$. This will be done in the next section.

A second kind of terms is determined by transformations on the set $\overline{n}$. Let T_n be the set of all transformations on the set $\overline{n}$, i.e. T_n is the set of all mappings $h : \overline{n} \to \overline{n}$. Then one gets a monoid $(T_n; \circ, \varepsilon_n)$, where $\circ$ is the usual composition of functions and ε_n is the identity mapping on $\overline{n}$. The natural number $n_h := n - |\{h(a) \mid a \in \overline{n}\}|$ is called the defect of a given transformation h. A transformation h is called isotone if the following implication holds for all $a, b \in \overline{n}$:

$$a \leq b \Rightarrow h(a) \leq h(b).$$

For $1 \leq k < n$ let $I_{n,k}$ be the set of all isotone transformations on $\overline{n}$ with defect $\geq k$. The set $O_n := I_{n,1}$ forms a semigroup and each of the set $I_{n,k}$, $2 \leq k < n$, forms an ideal of O_n ([14]). A transformation h is called antitone if the following implication holds for all $a, b \in \overline{n}$:

$$a \leq b \Rightarrow h(a) \geq h(b).$$

A transformation is called monotone if it is isotone or antitone. The set M_n of all monotone transformations on $\overline{n}$ with defect ≥ 1 forms a semigroup, too ([11]). Moreover, it is well-known that the set S_n of all bijective transformations on $\overline{n}$, i.e. permutations on $\overline{n}$, forms a subgroup of T_n.

For any transformation h on $\overline{n}$, we can consider the term $f(x_{h(1)}, \ldots, x_{h(n)})$. So we get terms defined by transformations and it is very natural to define hypersubstitutions by transformations. For any transformation h, we will denote the hypersubstitution σ with $\sigma(f) = f(x_{h(1)}, \ldots, x_{h(n)})$ by σ_h. For any set $A \subseteq T_n$, we put $A^{hyp} := \{\sigma_h \mid h \in A\}$ and $W_A := \{f(x_{h(1)}, \ldots, x_{h(n)}) \mid h \in A\}$. In particular, we put $P_n := W_{S_n}$. Clearly, the mapping $\rho_n : T_n \to T_n^{hyp}$ defined by

$$\rho_n(h) := \sigma_h$$

is a bijection. In particular, ρ_n is an anti-isomorphism (dual isomorphism).

Proposition 3. *Let* $1 \leq n \in \mathbb{N}$. *Then* $(T_n; \circ, \varepsilon_n)$ *is anti-isomorphic to* $(T_n^{hyp}; \circ_h, \sigma_{id})$.

Proof. We have to show that $\rho_n(\varphi \circ \pi) = \rho_n(\pi) \circ_h \rho_n(\varphi)$ for all $\varphi, \pi \in A$. Let $\varphi, \pi \in A$. Then we have $\rho_n(\pi) \circ_h \rho_n(\varphi) = \sigma_\pi \circ_h \sigma_\varphi = \widehat{\sigma}_\pi \circ \sigma_\varphi$ and $\rho_n(\varphi \circ \pi) = \sigma_{\varphi\circ\pi}$. Further we have

$$\begin{aligned}
\sigma_{\varphi\circ\pi}(f) &= f(x_{(\varphi\circ\pi)(1)}, \ldots, x_{(\varphi\circ\pi)(n)}) \\
&= f(x_{\pi(1)}, \ldots, x_{\pi(n)})(x_{\varphi(1)}, \ldots, x_{\varphi(n)}) \\
&= \widehat{\sigma}_{f(x_{\pi(1)}, \ldots, x_{\pi(n)})}[f(x_{\varphi(1)}, \ldots, x_{\varphi(n)})] \\
&= \widehat{\sigma}_\pi[f(x_{\varphi(1)}, \ldots, x_{\varphi(n)})] \\
&= \widehat{\sigma}_\pi[\sigma_\varphi(f)] \\
&= (\widehat{\sigma}_\pi \circ \sigma_\varphi)(f).
\end{aligned}$$

This shows that $\rho_n(\pi) \circ_h \rho_n(\varphi) = \rho_n(\varphi \circ \pi)$. ∎

In particular, then each of the sets O_n^{hyp}, M_n^{hyp} and $I_{n,2}^{hyp}$ forms a semigroup. We will consider these semigroups in Section 4. Moreover, the mapping $\gamma_n : T_n \to W_{T_n}$ defined by $\gamma_n(h) := f(x_{h(1)}, \ldots, x_{h(n)})$ is evidently an anti-isomorphism by Proposition 1 and Proposition 3. This gives:

Corollary 4. *Let* $1 \leq n \in \mathbb{N}$. *Then* $(T_n; \circ, \varepsilon_n)$ *is anti-isomorphic to* $(W_{T_n}; \diamond, \sigma_{id}(f))$.

3. THE MAXIMAL SUBMONOIDS OF $Reg(n)$

To characterize all maximal subsemigroups of $(W_n^{reg}; \diamond)$ for a given natural number $n \geq 2$, we need some technical lemmas. The following facts were proved in [10]:

Lemma 5. *Let* $s \in W_n^{reg}$ *and* $t, t_1, \ldots, t_n \in W_n(X_n)$ *with* $t = f(t_1, \ldots, t_n)$. *Then*

(a) $vb(s \diamond t) \geq vb(t)$;

(b) $vb(s \diamond t) = \sum_n^{i=1} vb_i(s) vb(s \diamond t_i)$;

(c) $vb(s(t_1, \ldots, t_n)) = \sum_{i=1}^n vb_i(s) vb(t_i)$.

Corollary 6. *For* $s, t \in W_n^{reg}$ *it holds:*

(a) *If* $s \notin P_n$ *then* $vb(t) < vb(s \diamond t)$.

(b) *If* $t \notin P_n$ *then* $vb(s) < vb(s \diamond t)$.

Proof. We have $vb(s \diamond t) \geq \sum_{i=1}^n vb_i(s) vb(t_i)$ by Lemma 5 and $vb_i(s) \neq 0$ for $i \in \overline{n}$ since $var(s) = X_n$.

(a) If $s \notin P_n$ then there is a $j \in \overline{n}$ with $vb_j(s) \geq 2$ and thus

$$\sum_{i=1}^n vb_i(s) vb(t_i) \geq vb(t_j) + \sum_{i=1}^n vb(t_i) > \sum_{i=1}^n vb(t_i) = vb(t).$$

(b) If $t \notin P_n$ then there is a $j \in \overline{n}$ with $vb(t_j) \geq 2$ and thus

$$\sum_{i=1}^n vb_i(s) vb(t_i) \geq 1 + \sum_{i=1}^n vb_i(s) > \sum_{i=1}^n vb_i(s) = vb(s).$$

■

Lemma 7. *For $s \in W_n^{reg}$ and $t \in P_n$ we have $vb(s \diamond t) = vb(t \diamond s) = vb(s)$.*

Proof. There is a $\pi \in S_n$ such that $t = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$. Further, there are $s_1, \ldots, s_n \in W_n(X_n)$ such that $s = f(s_1, \ldots, s_n)$. Then we have $vb(s \diamond t) = vb(\widehat{\sigma}_s[t]) = vb(s(x_{\pi(1)}, \ldots, x_{\pi(n)})) = \sum_{i=1}^n vb_i(s) vb(x_{\pi(i)}) = \sum_{i=1}^n vb_i(s) = vb(s)$ by Lemma 5c). We show now by induction that $vb(\widehat{\sigma}_t[r]) = vb(r)$ for all $r \in W_n(X_n)$. Clearly, $\widehat{\sigma}_t[r] = r$ for $r \in X_n$. Let $r = f(r_1, \ldots, r_n)$, $r_1, \ldots, r_n \in W_n(X_n)$, and suppose that $vb(\widehat{\sigma}_t[r_i]) = vb(r_i)$ for $i \in \overline{n}$. Then

$$\begin{aligned} vb(\widehat{\sigma}_t[r]) &= vb(f(x_{\pi(1)}, \ldots, x_{\pi(n)})(\widehat{\sigma}_t[r_1], \ldots, \widehat{\sigma}_t[r_n])) \\ &= vb(f(\widehat{\sigma}_t[r_{\pi(1)}], \ldots, \widehat{\sigma}_t[r_{\pi(n)}])) = \sum_{i=1}^n vb(\widehat{\sigma}_t[r_i]) = \sum_{i=1}^n vb(r_i) = vb(r). \end{aligned}$$

In particular, $vb(t \diamond s) = vb(\widehat{\sigma}_t[s]) = vb(s)$. ■

Now we consider the Green's relation $\mathcal{J}$ on the semigroup $(W_n^{reg}; \diamond)$ which is defined by $s\mathcal{J}t$ if there are $s_1, s_2, t_1, t_2 \in W_n^{reg}$ such that $s = t_1 \diamond t \diamond t_2$ and $t = s_1 \diamond s \diamond s_2$. For $t \in W_n^{reg}$, we denote the J-class containing t by J_t, i.e.

$$J_t := \{s \mid s \in W_n^{reg} \text{ and } s\mathcal{J}t\}.$$

The relation $\mathcal{J}$ on W_n^{reg} can be characterized as follows:

Lemma 8. *Let $s, t \in W_n^{reg}$. Then there holds $s\mathcal{J}t$ iff there are $s_1, s_2, t_1, t_2 \in P_n$ such that $s = t_1 \diamond t \diamond t_2$ and $t = s_1 \diamond s \diamond s_2$.*

Proof. One direction is clear. Conversely, let $s\mathcal{J}t$. Then there are $s_1, s_2, t_1, t_2 \in W_n^{reg}$ such that $s = t_1 \diamond t \diamond t_2$ and $t = s_1 \diamond s \diamond s_2$. We will show that $s_1, s_2, t_1, t_2 \in P_n$. We have $s = (t_1 \diamond s_1) \diamond s \diamond (s_2 \diamond t_2)$. Assume that $(t_1 \diamond s_1) \notin P_n$. Then $vb(s) < vb((t_1 \diamond s_1) \diamond s)$ by Corollary 6. Further, we have $vb((t_1 \diamond s_1) \diamond s)) \leq vb((t_1 \diamond s_1) \diamond s \diamond (s_2 \diamond t_2))$ by Corollary 6 and Lemma 7, respectively. This gives $vb(s) < vb(s)$, a contradiction. Assume that $(s_2 \diamond t_2) \notin P_n$. Then $vb((t_1 \diamond s_1) \diamond s) < vb((t_1 \diamond s_1) \diamond s \diamond (s_2 \diamond t_2)) = vb(s)$ by Corollary 6. But since $(t_1 \diamond s_1) \in P_n$ we have $vb((t_1 \diamond s_1) \diamond s) = vb(s)$ by Lemma 7. This gives $vb(s) < vb(s)$, a contradiction. So, both terms $(t_1 \diamond s_1)$ and $(s_2 \diamond t_2)$ belong to P_n. This provides $vb(s_1), vb(t_1) \geq vb(t_1 \diamond s_1) = 1$ and $vb(s_2), vb(t_2) \geq vb(t_2 \diamond s_2) = 1$. But this is only possible if $s_1, s_2, t_1, t_2 \in P_n$, by Corollary 6. ■

Proposition 9. *Let $s, t \in W_n^{reg}$. Then there holds $s\mathcal{J}t$ iff there are $s_1, s_2 \in P_n$ such that $t = s_1 \diamond s \diamond s_2$.*

Proof. One direction is clear by Lemma 8. Conversely, let $s_1, s_2 \in P_n$ such that $t = s_1 \diamond s \diamond s_2$. Then there are $\rho, \pi \in S_n$ with $s_1 = f(x_{\pi(1)}, \ldots, x_{\pi(n)})$ and $s_2 = f(x_{\rho(1)}, \ldots, x_{\rho(n)})$. Then there are $\rho^{-1}, \pi^{-1} \in S_n$ with $\rho^{-1} \circ \rho = \pi \circ \pi^{-1} = \varepsilon_n$ and we get $f(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}) \diamond t \diamond f(x_{\rho^{-1}(1)}, \ldots, x_{\rho^{-1}(n)}) = f(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}) \diamond f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \diamond s \diamond f(x_{\rho(1)}, \ldots, x_{\rho(n)}) \diamond f(x_{\rho^{-1}(1)}, \ldots, x_{\rho^{-1}(n)})$. Then Proposition 1 provides $\sigma_{\pi^{-1}} \circ_h \sigma_t \circ_h \sigma_{\rho^{-1}} = \sigma_{\pi^{-1}} \circ_h \sigma_\pi \circ_h \sigma_s \circ_h \sigma_\rho \circ_h \sigma_{\rho^{-1}}$ where $\sigma_{\pi^{-1}} \circ_h \sigma_\pi \circ_h \sigma_s \circ_h \sigma_\rho \circ_h \sigma_{\rho^{-1}} = \sigma_{(\pi \circ \pi^{-1})} \circ_h \sigma_s \circ_h \sigma_{(\rho^{-1} \circ \rho)} = \sigma_{\varepsilon_n} \circ_h \sigma_s \circ_h \sigma_{\varepsilon_n} = \sigma_s$ by Proposition 3. This gives $f(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}) \diamond t \diamond f(x_{\rho^{-1}(1)}, \ldots, x_{\rho^{-1}(n)}) = s$ again by Proposition 1. Altogether, this shows that $s\mathcal{J}t$ using Lemma 8. ■

Notation 10. A term $t \in W_n^{reg}$ is called a proper $\diamond$-product if there are $r, s \in W_n^{reg} \setminus P_n$ such that $t = r \diamond s$. Let W_n^{dec} denote the set of all proper $\diamond$-products of W_n^{reg}.

Now we are able to characterize all maximal subsemigroups of $(W_n^{reg}; \diamond)$, i.e. all subsets $W \subseteq W_n^{reg}$ with $\langle W \cup \{a\} \rangle = W_n^{reg}$ for all $a \in W_n^{reg} \setminus W$.

Theorem 11. *A set $W \subseteq W_n^{reg}$ forms a maximal subsemigroup of $(W_n^{reg}; \diamond)$ iff one of the following statements is satisfied:*

(i) *There is a $t \in W_n^{reg} \setminus (W_n^{dec} \cup P_n)$ such that $W = W_n^{reg} \setminus J_t$.*

(ii) *There is a maximal subgroup S of S_n such that $W = (W_n^{reg} \setminus P_n) \cup W_S$.*

Proof. Suppose that (i) is satisfied, i.e. there is a $t \in W_n^{reg} \setminus (W_n^{dec} \cup P_n)$ such that $W = W_n^{reg} \setminus J_t$. We show that W forms a subsemigroup of $(W_n^{reg}; \diamond)$. For this let $a, b \in W_n^{reg} \setminus J_t$. Then $a \diamond b \in W_n^{reg}$. Assume that $a \diamond b \in J_t$. Then there are $s_1, s_2 \in P_n$ such that $t = (s_1 \diamond a) \diamond (b \diamond s_2)$ by Proposition 9. Since $t \notin W_n^{dec}$ we have $(s_1 \diamond a) \in P_n$ or $(b \diamond s_2) \in P_n$. Without loss of generality let $(s_1 \diamond a) \in P_n$. Then we get $b \in J_t$ by Proposition 9, a contradiction. Thus $a \diamond b \in W_n^{reg} \setminus J_t = W$. This shows that $(W; \diamond)$ is a subsemigroup of $(W_n^{reg}; \diamond)$. Now we show that W is maximal. First, we show that $P_n \subseteq W$. Assume that $P_n \nsubseteq W$. Then there are an $s \in P_n \cap J_t$ and $s_1, s_2 \in P_n$ such that $t = s_1 \diamond s \diamond s_2$. Then Lemma 7 provides $vb(t) = vb(s) = n$, i.e. $t \in P_n$, a contradiction. Hence $P_n \subseteq W$.

Let now $s \in W_n^{reg} \setminus W$, i.e. $s \in J_t$. Then there are $s_1, s_2 \in P_n$ such that $t = s_1 \diamond s \diamond s_2$. Hence $t \in \langle W \cup \{s\} \rangle$ and thus $\{s_1 \diamond t \diamond s_2 \mid s_1, s_2 \in P_n\} \subseteq \langle W \cup \{s\} \rangle$. Further, we have $J_t = \{s_1 \diamond t \diamond s_2 \mid s_1, s_2 \in P_n\}$ by Proposition 9, hence $J_t \subseteq \langle W \cup \{s\} \rangle$. This shows that W is maximal.

Suppose that (ii) is satisfied, i.e. there is a maximal subgroup S of S_n such that $W = (W_n^{reg} \setminus P_n) \cup W_S$. We show that W forms a subsemigroup of $(W_n^{reg}; \diamond)$. For this let $a, b \in W$. If $a \notin P_n$ or $b \notin P_n$ then $vb(b) < vb(a \diamond b)$ and $vb(a) < vb(a \diamond b)$, respectively, by Corollary 6. Thus $vb(a \diamond b) > 1$. This shows that $a \diamond b \notin P_n$, i.e. $a \diamond b \in W$. We consider now the case that $a, b \in P_n$, i.e. $a, b \in W_S$. Since S is a subgroup of S_n, we have $a \diamond b \in W_S$ by Corollary 4. This shows that $(W; \diamond)$ is a subsemigroup of $(W_n^{reg}; \diamond)$. Now we conclude that $(W_n^{reg} \setminus P_n) \cup W_S$ is maximal since $(W_S; \diamond)$ is a maximal subgroup of $(P_n; \diamond)$ by Corollary 4.

Conversely, let $(W; \diamond)$ be a maximal subsemigroup of $(W_n^{reg}; \diamond)$. We put $M := W_n^{reg} \setminus W$. Then we have $M \cap P_n = \emptyset$ or $M \cap P_n \neq \emptyset$. Suppose that $M \cap P_n \neq \emptyset$. Let us consider the set $S := \{\pi \in S_n \mid f(x_{\pi(1)}, \ldots, x_{\pi(n)}) \in W\}$. Then we have $W \cap P_n = W_S$. Since both sets W and P_n form semigroups, $W \cap P_n = W_S$ forms a subsemigroup of $(P_n; \diamond)$. Then S is a proper subsemigroup of S_n by Proposition 3. Since S_n is finite, each subsemigroup of S_n is a group. Hence there is a maximal subgroup T of S_n containing S and we have $W \subseteq (W_n^{reg} \setminus P_n) \cup W_T$ where $(W_n^{reg} \setminus P_n) \cup W_T$ forms a subsemigroup of $(W_n^{reg}; \diamond)$ by the previous considerations. Since $(W; \diamond)$ is a maximal subsemigroup of $(W_n^{reg}; \diamond)$, we can conclude that $W = (W_n^{reg} \setminus P_n) \cup W_T$.

Suppose that $M \cap P_n = \emptyset$. Let $t \in M$. Then $t \notin P_n$. Assume that $t \in W_n^{dec}$. Then there are $t_1, t_2 \in W_n^{reg} \setminus P_n$ such that $t = t_1 \diamond t_2$. Since $t \notin W$, one of the terms t_1 and t_2 does not belong to W. Without loss of generality let $t_1 \notin W$. Then Corollary 6 implies $vb(t_1) < vb(t_1 \diamond t_2) = vb(t)$. Let $a_1, \ldots, a_k \in W_n^{reg}$ for some natural number $k > 0$ with $a_j = t$ for some $j \in \{1, \ldots, k\}$. Then $vb(a_1 \diamond \ldots \diamond a_k) \geq vb(t)$ by Corollary 6 and Lemma 7. Thus $t_1 \notin \langle W \cup \{t\} \rangle$, i.e. $\langle W \cup \{t\} \rangle \neq W_n^{reg}$. Because of the maximality of $(W; \diamond)$ we get $t \in W$, a contradiction. Hence $t \notin W_n^{dec}$ and altogether $t \in W_n^{reg} \setminus (W_n^{dec} \cup P_n)$. Now we show $J_t \subseteq M$. Otherwise there is an $s \in J_t$ with $s \in W$. Then there are $s_1, s_2 \in P_n$ such that $t = s_1 \diamond s \diamond s_2$. Since $P_n \subseteq W$ we get $t \in W$, a contradiction. Now we have $W \subseteq W_n^{reg} \setminus J_t$, where $W_n^{reg} \setminus J_t$ forms a semigroup (see the previous considerations). Since $(W; \diamond)$ is a maximal subsemigroup of $(W_n^{reg}; \diamond)$, we obtain $W = W_n^{reg} \setminus J_t$. ■

Remark 12. Clearly, each maximal submonoid of $(W_n^{reg};\diamond)$ contains the identity element $\sigma_{id}(f)$. Therefore, Theorem 11 characterizes all maximal submonoids of $(W_n^{reg};\diamond,\sigma_{id}(f))$.

Since the only maximal subgroup of S_2 as well as the four maximal subgroups of S_3 are well known, we can formulate Theorem 11 for $n=2$ and $n=3$, respectively, in the following way.

Proposition 13. *A set $W\subseteq W_2^{reg}$ forms a maximal subsemigroup of $(W_2^{reg};\diamond)$ iff $W=W_2^{reg}\setminus\{f(x_2,x_1)\}$ or $W=W_2^{reg}\setminus J_t$ for some $t\in W_2^{reg}\setminus(W_2^{dec}\cup P_2)$.*

Proposition 14. *A set $W\subseteq W_3^{reg}$ forms a maximal subsemigroup of $(W_3^{reg};\diamond)$ iff $W=W_3^{reg}\setminus J_t$ for some $t\in W_3^{reg}\setminus(W_3^{dec}\cup P_3)$ or W coincides with one of the following four sets:*

(a) $W_3^{reg}\setminus\{f(x_1,x_3,x_2),f(x_3,x_2,x_1),f(x_2,x_1,x_3)\}$

(b) $W_3^{reg}\setminus\{f(x_1,x_3,x_2),f(x_3,x_2,x_1),f(x_2,x_3,x_1),f(x_3,x_1,x_2)\}$

(c) $W_3^{reg}\setminus\{f(x_1,x_3,x_2),f(x_2,x_1,x_3),f(x_2,x_3,x_1),f(x_3,x_1,x_2)\}$

(d) $W_3^{reg}\setminus\{f(x_2,x_1,x_3),f(x_3,x_2,x_1),f(x_2,x_3,x_1),f(x_3,x_1,x_2)\}$.

To determine all maximal subsemigroups of W_n^{reg} we need the knowledge of all maximal subgroups of S_n and of all elements of the set $W_n^{reg}\setminus(W_n^{dec}\cup P_n)$. The O'Nan Scott-Theorem gives a classification of all maximal subgroups of S_n (e.g. [15]). But the characterization of all proper $\diamond$-products seems to be a too complex problem. Therefore we restrict ourselves to study only necessary or only sufficient properties of the elements of W_n^{dec}. First, we show that a term $t\in W_n^{reg}$ does not belong to W_n^{dec} if $op(t)$ is a prime number.

Lemma 15. *Let $s\in W_n^{reg}$ and $t\in W_n(X_n)$. Then there is a natural number $k\geq op(t)$ such that $op(s\diamond t)=k\cdot op(s)$.*

Proof. If $t\in X_n$ then $op(t)=0$ and thus $op(s\diamond t)=op(\widehat{\sigma}_s[t])=op(t)=0=0\cdot op(s)$. Suppose that $t=f(t_1,\dots,t_n)$ with $t_1,\dots,t_n\in W_n(X_n)$ and $op(s\diamond t_i)=k_i\cdot op(s)$ with $k_i\geq op(t_i)$ for $i\in\overline{n}$. We have $op(s\diamond t)=op(s)+\sum_{i=1}^n vb_i(s)\cdot op(\widehat{\sigma}_s[t_i])$ (see [10]). Further, it holds

$$op(s) + \sum_{i=1}^{n} vb_i(s) \cdot op(\widehat{\sigma}_s[t_i])$$

$$= op(s) + \sum_{i=1}^{n} vb_i(s) \cdot k_i \cdot op(s)$$

$$= op(s) \cdot \left(1 + \sum_{i=1}^{n} vb_i(s) \cdot k_i\right).$$

Since $vb_i(s) \neq 0$ (because of $var(s) = X_n$) and $k_i \geq op(t_i)$ for $i \in \overline{n}$ we have $\sum_{i=1}^{n} vb_i(s) \cdot k_i \geq \sum_{i=1}^{n} vb_i(s) \cdot op(t_i) \geq \sum_{i=1}^{n} op(t_i) = op(t) - 1$, i.e. $1 + \sum_{i=1}^{n} vb_i(s) \cdot k_i \geq op(t)$. ∎

Proposition 16. *If $t \in W_n^{reg}$ such that $op(t)$ is a prime number then $t \notin W_n^{dec} \cup P_n$.*

Proof. Since $op(t)$ is a prime number, $op(t) \geq 2$. Thus $t \notin P_n$. Assume that $t \in W_n^{dec}$. Then there are $r, s \in W_n^{reg} \setminus P_n$ such that $t = r \diamond s$. This provides $op(t) = op(r \diamond s) = k \cdot op(r)$ for some $k \geq op(s)$ by Lemma 15. Since $r, s \notin P_n$ we have $op(r)$, $op(s) \geq 2$ and thus $k \geq op(s) \geq 2$. Hence $op(t) = k \cdot op(r)$ is not a prime number, a contradiction. This shows that $t \notin W_n^{dec}$. ∎

An element of W_n^{dec} has the following structure:

Proposition 17. *For any $t \in W_n^{dec}$ there are an $s \in W_n^{reg} \setminus P_n$ and $t_1, \ldots, t_n \in W_n(X_n)$ with $t_j \notin X_n$ for some $j \in \overline{n}$ such that $t = s(s \diamond t_1, \ldots, s \diamond t_n)$.*

Proof. Since $t \in W_n^{dec}$ there are $r, s \in W_n^{reg} \setminus P_n$ such that $t = r \diamond s = \widehat{\sigma}_r[s]$. Further, there are $s_1, \ldots, s_n \in W_n(X_n)$ such that $s = f(s_1, \ldots, s_n)$ and we obtain $\widehat{\sigma}_r[s] = r(\widehat{\sigma}_r[s_1], \ldots, \widehat{\sigma}_r[s_n]) = r(r \diamond s_1, \ldots, r \diamond s_n)$. Since $s \notin P_n$ there is a $j \in \overline{n}$ with $s_j \notin X_n$. ∎

Example 18. We consider the case $n = (3)$ and the term

$$t = f(x_1, f(x_1, x_1, x_1), f(x_1, x_1, f(x_1, x_2, x_3))).$$

Although $op(t)$ is not a prime number, t does not belong to W_3^{dec}. Indeed, assume that there are an $s \in W_3^{reg} \setminus P_3$ and $t_1, t_2, t_3 \in W_3(X_3)$ with $t_i \notin X_3$ for some $i \in \{1, 2, 3\}$ such that $t = s(s \diamond t_1, s \diamond t_2, s \diamond t_3)$. Then $op(s) \geq 2$ and $op(s \diamond t_i) \geq 4$ by Lemma 15. This provides that $op(t) \geq 4 + op(s) > 4$, a contradiction.

4. Transformation hypersubstitutions

A list of all maximal subsemigroups of the ideal O_n of all isotone transformations on $\overline{n}$ with defect ≥ 1 is given in [16]. I. Guydzenov and I. Dimitrova have determined all maximal subsemigroups of M_n as well as of the ideal $I_{n,2}$ of all isotone transformations on $\overline{n}$ with defect ≥ 2, see [12] and [13], respectively. These results can be regarded as generalizations of the results in [16] concerning O_n. We want to use the mentioned results to characterize the maximal submonoids of particular monoids of transformation hypersubstitutions. It is easy to verify that each of the sets $O_n^{hyp} \cup \{\sigma_{id}\}$, $M_n^{hyp} \cup \{\sigma_{id}\}$ and $I_{n,2}^{hyp} \cup \{\sigma_{id}\}$ forms a submonoid of $TR(n)$. We can use Proposition 3 to characterize the maximal submonoids of each of these monoids.

Lemma 19. *Let $1 \leq n \in \mathbb{N}$ and let $(A; \circ)$ be a transformation semigroup on $\overline{n}$ with $\varepsilon_n \notin A$. Then a set $M \subseteq TR(n)$ forms a maximal submonoid of $(A^{hyp} \cup \{\sigma_{id}\}; \circ_h, \sigma_{id})$ iff there is a maximal subsemigroup $(B; \circ)$ of $(A; \circ)$ such that $M = B^{hyp} \cup \{\sigma_{id}\}$.*

Proof. Suppose that $(M; \circ_h, \sigma_{id})$ is a maximal submonoid of $(A^{hyp} \cup \{\sigma_{id}\}; \circ_h, \sigma_{id})$. Then there is a set $B \subseteq A$ such that $B^{hyp} \cup \{\sigma_{id}\} = M$. Since $\varepsilon_n \notin A$, $\sigma_{id} \notin A^{hyp}$ and thus $(A^{hyp}; \circ_h)$ forms a semigroup. Hence $M \setminus \{\sigma_{id}\}$ forms a semigroup, in particular, $(M \setminus \{\sigma_{id}\}; \circ_h)$ is a maximal subsemigroup of $(A^{hyp}; \circ_h)$. This implies that $(B; \circ)$ is a maximal subsemigroup of $(A; \circ)$ by Proposition 3. Conversely, suppose that $(B; \circ)$ is a maximal subsemigroup of $(A; \circ)$ such that $M = B^{hyp} \cup \{\sigma_{id}\}$. Then $(B^{hyp}; \circ_h)$ is a maximal subsemigroup of $(A^{hyp}; \circ_h)$ by Proposition 3. Thus $\langle M \cup \{\sigma\} \rangle = \langle B^{hyp} \cup \{\sigma, \sigma_{id}\} \rangle = \langle B^{hyp} \cup \{\sigma\} \rangle \cup \{\sigma_{id}\} = A^{hyp} \cup \{\sigma_{id}\}$ for all $\sigma \in (A^{hyp} \cup \{\sigma_{id}\}) \setminus M$. This shows that $(M; \circ_h, \sigma_{id})$ is a maximal submonoid of $(A^{hyp} \cup \{\sigma_{id}\}; \circ_h, \sigma_{id})$. ■

For the set $TR_2(n) := I_{n,2}^{hyp} \cup \{\sigma_{id}\}$ we have

Corollary 20. *Let $1 \leq n \in \mathbb{N}$. Then the following monoids are all maximal submonoids of $(TR_2(n); \circ_h, \sigma_{id})$: $(A^{hyp} \cup \{\sigma_{id}\}; \circ_h, \sigma_{id})$, where $(A; \circ)$ is a maximal subsemigroup of $(I_{n,2}; \circ)$.*

The maximal subsemigroups of $(I_{n,2};\circ)$ are listed in [12]. For example, let us consider the case $n=4$.

Example 21. Let $n=4$. Then we have $TR_2(4)=\{\sigma_{f(x_i,x_i,x_i,x_i)} \mid 1\leq i\leq 4\}\cup\{\sigma_{f(x_i,x_i,x_i,x_j)} \mid 1\leq i<j\leq 4\}\cup\{\sigma_{f(x_i,x_i,x_j,x_j)} \mid 1\leq i<j\leq 4\} \cup \{\sigma_{f(x_i,x_j,x_j,x_j)} \mid 1\leq i<j\leq 4\}\cup\{\sigma_{id}\}$. There are eleven maximal submonoids of $(TR_2(4);\circ_h,\sigma_{id})$, namely

$$A_{i,j}=TR_2(4)\backslash\{\sigma_{f(x_i,x_i,x_i,x_j)},\sigma_{f(x_i,x_i,x_j,x_j)},\sigma_{f(x_i,x_j,x_j,x_j)}\} \text{ for } 1\leq i<j\leq 4$$

$$A_1 \ =TR_2(4)\backslash\{\sigma_{f(x_i,x_j,x_j,x_j)} \mid 1\leq i<j\leq 4 \ \text{ and } \ i+j\neq 3\}$$

$$A_2 \ =TR_2(4)\backslash\{\sigma_{f(x_i,x_i,x_i,x_j)} \mid 1\leq i<j\leq 4 \ \text{ and } \ i+j\neq 7\}$$

$$A_3 \ =TR_2(4)\backslash(\{\sigma_{f(x_1,x_1,x_1,x_j)} \mid 2\leq j\leq 4\}\cup\{\sigma_{f(x_1,x_1,x_j,x_j)} \mid 2\leq j\leq 4\})$$

$$A_4 \ =TR_2(4)\backslash(\{\sigma_{f(x_i,x_j,x_j,x_j)} \mid 1\leq i<j\leq 4 \ \text{ and } \ i+j\neq 3,7\})\cup$$

$$\{\sigma_{f(x_i,x_i,x_i,x_j)} \mid 1\leq i<j\leq 4 \ \text{ and } \ i+j\neq 3,7\}$$

$$A_5 \ =TR_2(4)\backslash\{\sigma_{f(x_i,x_i,x_j,x_j)} \mid 1\leq i<j\leq 4 \ \text{ and } \ ij\neq 6\}.$$

For the set $TR_1(n):=O_n^{hyp}\cup\{\sigma_{id}\}$ we get

Corollary 22. *Let* $1\leq n\in\mathbb{N}$. *Then the following monoids are the maximal submonoids of* $(TR_1(n);\circ_h,\sigma_{id})$*:* $(A^{hyp}\cup\{\sigma_{id}\};\circ_h,\sigma_{id})$*, where* $(A;\circ)$ *is a maximal subsemigroup of* $(O_n;\circ)$.

The maximal subsemigroups of $(O_n;\circ)$ are listed in [16]. For example, let us consider the case $n=4$.

Example 23. Let $n=4$. Then we have $TR_1(4)=TR_2(4)\cup\{\sigma_{f(x_i,x_i,x_j,x_l)} \mid 1\leq i<j<l\leq 4\}\cup\{\sigma_{f(x_i,x_j,x_j,x_l)} \mid 1\leq i<j<l\leq 4\}\cup\{\sigma_{f(x_i,x_j,x_l,x_l)} \mid 1\leq i<j<l\leq 4\}$. There are ten maximal submonoids of $(TR_1(4);\circ_h,\sigma_{id})$, namely

$$\begin{aligned}
B_{i,j,l} &= TR_1(4)\backslash\{\sigma_{f(x_i,x_i,x_j,x_l)}, \sigma_{f(x_i,x_j,x_j,x_l)}, \sigma_{f(x_i,x_j,x_l,x_l)}\} \text{ for} \\
&\quad 1 \leq i < j < l \leq 4 \\
B_1 &= TR_1(4)\backslash\{\sigma_{f(x_i,x_j,x_j,x_l)} \mid 1 \leq i < j < l \leq 4\} \\
B_2 &= TR_1(4)\backslash\{\sigma_{f(x_1,x_3,x_3,x_4)}, \sigma_{f(x_2,x_3,x_3,x_4)}, \sigma_{f(x_1,x_3,x_4,x_4)}, \sigma_{f(x_2,x_3,x_4,x_4)}\} \\
B_3 &= TR_1(4)\backslash\{\sigma_{f(x_i,x_j,x_4,x_4)} \mid 1 \leq i < j \leq 3\} \\
B_4 &= TR_1(4)\backslash\{\sigma_{f(x_1,x_1,x_i,x_j)} \mid 2 \leq i < j \leq 4\} \\
B_5 &= TR_1(4)\backslash\{\sigma_{f(x_1,x_1,x_2,x_3)}, \sigma_{f(x_1,x_1,x_2,x_4)}, \sigma_{f(x_1,x_2,x_2,x_3)}, \sigma_{f(x_1,x_2,x_2,x_4)}\} \\
B_6 &= TR_1(4)\backslash\{\sigma_{f(x_1,x_1,x_2,x_4)}, \sigma_{f(x_1,x_1,x_3,x_4)}, \sigma_{f(x_1,x_2,x_4,x_4)}, \sigma_{f(x_1,x_3,x_4,x_4)}\}.
\end{aligned}$$

For the set $TR_{mon}(n) := M_n^{hyp} \cup \{\sigma_{id}\}$ we have

Corollary 24. *Let $1 \leq n \in \mathbb{N}$. Then the following monoids are all maximal submonoids of $(TR_{mon}(n); \circ_h, \sigma_{id})$: $(A^{hyp} \cup \{\sigma_{id}\}; \circ_h, \sigma_{id})$, where $(A; \circ)$ is a maximal subsemigroup of $(M_n; \circ)$.*

The maximal subsemigroups of $(M_n; \circ)$ are listed in [13]. For example, let us consider the case $n = 4$.

Example 25. Let $n = 4$. Then we have $TR_{mon}(4) = TR_1(4) \cup \{\sigma_{f(x_i,x_i,x_i,x_j)} \mid 1 \leq j < i \leq 4\} \cup \{\sigma_{f(x_i,x_i,x_j,x_j)} \mid 1 \leq j < i \leq 4\} \cup \{\sigma_{f(x_i,x_j,x_j,x_j)} \mid 1 \leq j < i \leq 4\} \cup \{\sigma_{f(x_i,x_i,x_j,x_l)} \mid 1 \leq l < j < i \leq 4\} \cup \{\sigma_{f(x_i,x_j,x_j,x_l)} \mid 1 \leq l < j < i \leq 4\} \cup \{\sigma_{f(x_i,x_j,x_l,x_l)} \mid 1 \leq l < j < i \leq 4\}$. There are eleven maximal submonoids of $(TR_{mon}(4); \circ_h, \sigma_{id})$, namely

$$C_{i,j,l} = TR_{mon}(4)\backslash\{\sigma_{f(x_i,x_i,x_j,x_l)}, \sigma_{f(x_i,x_j,x_j,x_l)}, \sigma_{f(x_i,x_j,x_l,x_l)}, \sigma_{f(x_l,x_l,x_j,x_i)}, \sigma_{f(x_l,x_j,x_j,x_i)}, \sigma_{f(x_l,x_j,x_i,x_i)}\} \text{ for } 1 \leq i < j < l \leq 4$$

$$C_1 = TR_{mon}(4)\backslash(\{\sigma_{f(x_l,x_l,x_j,x_i)} \mid 1 \leq i < j < l \leq 4\} \cup \{\sigma_{f(x_l,x_j,x_j,x_i)} \mid 1 \leq i < j < l \leq 4\} \cup \{\sigma_{f(x_l,x_j,x_i,x_i)} \mid 1 \leq i < j < l \leq 4\})$$

$$C_2 = TR_{mon}(4)\backslash(\{\sigma_{f(x_i,x_j,x_j,x_l)} \mid 1 \leq i < j < l \leq 4\} \cup \{\sigma_{f(x_l,x_j,x_j,x_i)} \mid 1 \leq i < j < l \leq 4\})$$

$$C_3 = TR_{mon}(4)\backslash\{\sigma_{f(x_1,x_3,x_3,x_4)}, \sigma_{f(x_2,x_3,x_3,x_4)}, \sigma_{f(x_1,x_3,x_4,x_4)}, \sigma_{f(x_2,x_3,x_4,x_4)}, \sigma_{f(x_4,x_3,x_3,x_1)}, \sigma_{f(x_4,x_3,x_3,x_2)}, \sigma_{f(x_4,x_3,x_1,x_1)}, \sigma_{f(x_4,x_3,x_2,x_2)}\}$$

$$C_4 = TR_{mon}(4)\backslash(\{\sigma_{f(x_i,x_j,x_4,x_4)} \mid 1 \leq i < j \leq 3\} \cup \{\sigma_{f(x_4,x_j,x_i,x_i)} \mid 1 \leq i < j \leq 3\})$$

$$C_5 = TR_{mon}(4)\backslash(\{\sigma_{f(x_1,x_1,x_1,x_j)} \mid 2 \leq i < j \leq 4\} \cup \{\sigma_{f(x_j,x_j,x_i,x_1)} \mid 2 \leq i < j \leq 4\})$$

$$C_6 = TR_{mon}(4)\backslash\{\sigma_{f(x_1,x_1,x_2,x_3)}, \sigma_{f(x_1,x_1,x_2,x_4)}, \sigma_{f(x_1,x_2,x_2,x_3)}, \sigma_{f(x_1,x_2,x_2,x_4)}, \sigma_{f(x_3,x_3,x_2,x_1)}, \sigma_{f(x_4,x_4,x_2,x_1)}, \sigma_{f(x_3,x_2,x_2,x_1)}, \sigma_{f(x_4,x_2,x_2,x_1)}\}$$

$$C_7 = TR_{mon}(4)\backslash\{\sigma_{f(x_1,x_1,x_2,x_4)}, \sigma_{f(x_1,x_1,x_3,x_4)}, \sigma_{f(x_1,x_2,x_4,x_4)}, \sigma_{f(x_1,x_3,x_4,x_4)}, \sigma_{f(x_4,x_2,x_1,x_1)}, \sigma_{f(x_4,x_3,x_1,x_1)}, \sigma_{f(x_4,x_4,x_2,x_1)}, \sigma_{f(x_4,x_4,x_3,x_1)}\}.$$

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