

COMPLETION OF PARTIALLY ORDERED SETS*

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Abstract

The paper considers a generalization of the standard completion of a partially ordered set through the collection of all its lower sets.

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1. INTRODUCTION

Let \mathbf{Pos} be the category of partially ordered sets (posets) and order-preserving maps and let \mathbf{JCPos} be its subcategory consisting of complete lattices and join-preserving maps. It is known that the category \mathbf{JCPos} is reflective in \mathbf{Pos} (see, e.g., [1]). The completion of a poset goes through the collection of all its lower-sets.

Given a quantale Q one can consider the category $Q\text{-Mod}$ of modules over Q (see, e.g., [5]). Since the categories $\mathbf{2-Mod}$ and \mathbf{JCPos} are isomorphic one could ask about the generalization of the aforesaid result for an arbitrary quantale Q . We answer the question in two ways using the generalization of the category \mathbf{Pos} in the latter one.

All results from Category Theory used in the paper can be found in [1].

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2. DEFINITION OF THE CATEGORY $Q\text{-Mod}$

In this section we recall basic facts about the category $Q\text{-Mod}$ motivated by the category of modules over a ring [3, 4]. Start by recalling the definition of quantale (see, e.g., [5]).

Definition 2.1. A *quantale* is a triple (Q, \leq, \cdot) such that

- (i) (Q, \leq) is a complete lattice;
- (ii) (Q, \cdot) is a semigroup;
- (iii) $q \cdot (\bigvee S) = \bigvee_{s \in S} (q \cdot s)$ and $(\bigvee S) \cdot q = \bigvee_{s \in S} (s \cdot q)$ for every $q \in Q$ and every $S \subseteq Q$.

Given a quantale Q , denote its top (bottom) element by \top (\perp) respectively.

Definition 2.2. A quantale Q is called *unital* provided that there exists an element $e \in Q$ such that (Q, \cdot, e) is a monoid.

From now on without further references all quantales are supposed to be unital. The following are examples of quantales:

- (i) $(\mathbf{2}, \leq, \wedge, 1)$ where $\mathbf{2} = \{0, 1\}$;
- (ii) $([0, 1], \leq, \wedge, 1)$ where $[0, 1]$ is the unit interval;
- (iii) $([0, 1], \leq, \cdot, 1)$ where \cdot is the usual multiplication;
- (iv) the chain $\mathbf{3}$ with the usual order and the map $\mathbf{3} \times \mathbf{3} \longrightarrow \mathbf{3}$ given by the table:

\cdot	0	1	2
0	0	0	0
1	0	1	2
2	0	2	2

Notice that $\top \neq e$.

As in the last example we *do not* assume that $\top = e$ for every quantale Q we use.

Now define the category $Q\text{-Mod}$ of quantale modules or in other words the category of enriched complete lattices over the category \mathbf{JCPos} .

Definition 2.3. Given a quantale Q , define the category $Q\text{-Mod}$ as follows:

- (i) The objects are triples $(A, \leq, *)$ where (A, \leq) is a complete lattice and $Q \times A \xrightarrow{*} A$ is a map such that
 - (a) $q * (\bigvee S) = \bigvee_{s \in S} (q * s)$ for every $q \in Q, S \subseteq A$;
 - (b) $(\bigvee S) * a = \bigvee_{s \in S} (s * a)$ for every $a \in A, S \subseteq Q$;
 - (c) $q_1 * (q_2 * a) = (q_1 \cdot q_2) * a$ for every $q_1, q_2 \in Q, a \in A$;
 - (d) $e * a = a$ for every $a \in A$.
- (ii) The morphisms are maps $(A, \leq, *) \xrightarrow{f} (B, \leq, *)$ such that
 - (a) $f(\bigvee S) = \bigvee f(S)$ for every $S \subseteq A$;
 - (b) $f(q * a) = q * f(a)$ for every $a \in A, q \in Q$.

We consider the category $Q\text{-Mod}$ as a concrete category over \mathbf{Set} in the following way.

Definition 2.4. Define the forgetful functor $Q\text{-Mod} \xrightarrow{U} \mathbf{Set}$ as follows:

$$U((A, \leq, *) \xrightarrow{f} (B, \leq, *)) = A \xrightarrow{f} B.$$

Definitions 2.3 and 2.4 yield a construct $(Q\text{-Mod}, U)$. One can easily see that $\mathbf{2-Mod}$ is concretely isomorphic to \mathbf{JCPos} (compare with integers \mathbf{Z} in case of the category $R\text{-Mod}$ of modules over the ring R). Thus while considering the category $Q\text{-Mod}$ we study the category \mathbf{JCPos} as well.

Remark 2.1. For $Q = \mathbf{1}$ it follows that $A \in \mathcal{Ob}(Q\text{-Mod})$ iff $A \cong \mathbf{1}$, i.e., $\mathbf{1-Mod}$ is equivalent to the terminal category.

The following theorem states a useful property of the category $Q\text{-Mod}$.

Theorem 2.1. *The category $Q\text{-Mod}$ is a monadic construct.*

Corollary 2.2. *The category $Q\text{-Mod}$ is complete, cocomplete, wellpowered, extremally co-wellpowered, and has regular factorizations.*

3. COMPLETION OF PARTIALLY ORDERED SETS

In this section we generalize the standard method of completion of posets given in the following proposition (see [1]).

Proposition 3.1. *The category \mathbf{JCPos} is reflective in \mathbf{Pos} .*

Proof. Given a poset A one has the complete lattice B_A of all lower-sets of A and the embedding $A \hookrightarrow B_A : a \mapsto \downarrow a$ which is the reflection arrow for A . ■

We are going to generalize the result for the category $Q\text{-Mod}$. The first approach is as follows.

Proposition 3.2. *Let Q be a quantale. Then the category $Q\text{-Mod}$ is reflective in \mathbf{Pos} .*

Proof. Given a poset A , the reflection arrow can be constructed as follows. Define $B_A = \{h \in Q^A \mid a \leq b \text{ implies } h(b) \leq h(a)\}$ and let $A \xrightarrow{r} B_A : a \mapsto \downarrow a$ where

$$\downarrow a : A \longrightarrow Q : b \mapsto \begin{cases} e, & b \leq a \\ \perp, & \text{otherwise.} \end{cases}$$

Given a $Q\text{-Mod}$ -object B and a \mathbf{Pos} -morphism $A \xrightarrow{f} B$, there exists a unique $Q\text{-Mod}$ -morphism $B_A \xrightarrow{\bar{f}} B$ such that $\bar{f} \circ r = f$, i.e., $\bar{f} : B_A \longrightarrow B : h \mapsto \bigvee_{a \in A} h(a) * f(a)$. ■

Another approach is more sophisticated. Start with the following definition.

Definition 3.1. Given a quantale Q , define the category $Q\text{-Pos}$ as follows:

- (i) The objects are triples $(A, \leq, *)$ where (A, \leq) is a poset and $Q \times A \xrightarrow{*} A$ is a map such that
 - (a) the map $A \xrightarrow{q*} A$ is order-preserving for every $q \in Q$;
 - (b) the map $Q \xrightarrow{*a} A$ is order-preserving for every $a \in A$;
 - (c) $q_1 * (q_2 * a) = (q_1 \cdot q_2) * a$ for every $q_1, q_2 \in Q, a \in A$;
 - (d) $e * a = a$ for every $a \in A$.
- (ii) The morphisms are maps $(A, \leq, *) \xrightarrow{f} (B, \leq, *)$ such that
 - (a) f is order-preserving;
 - (b) $q * f(a) \leq f(q * a)$ for every $a \in A, q \in Q$ (notice that we use a rather non-standard definition of morphisms since one would expect " $=$ " instead of " \leq ").

The objects of the category $Q\text{-Pos}$ will be referred to as Q -posets. One can consider the category $Q\text{-Pos}$ as a concrete category over \mathbf{Set} in the following way.

Definition 3.2. Define the forgetful functor $Q\text{-Pos} \xrightarrow{U} \mathbf{Set}$ as follows:

$$U((A, \leq, *) \xrightarrow{f} (B, \leq, *)) = A \xrightarrow{f} B.$$

Definitions 3.1 and 3.2 give a construct $(Q\text{-Pos}, U)$. One can easily see that $\mathbf{2}\text{-Pos}$ is concretely isomorphic to \mathbf{Pos} (for a poset A let $* : \mathbf{2} \times A \longrightarrow A : (q, a) \mapsto a$). Thus while considering the category $Q\text{-Pos}$ we study the category \mathbf{Pos} as well.

Every $Q\text{-Pos}$ -object A has the following map

$$_ \rightarrow _ : A \times A \longrightarrow Q : (a, b) \mapsto \bigvee \{q \in Q \mid q * a \leq b\}.$$

Consider a property of the aforesaid map. Start by recalling some preliminary notions (cf. Chapter 0–3 in [2]).

Definition 3.3. Let \mathbf{C} be an ordered category (i.e., *hom*-sets are partially ordered and composition on both sides is order-preserving). A pair of \mathbf{C} -morphisms $A \begin{smallmatrix} \xrightarrow{g} \\ \xleftarrow{d} \end{smallmatrix} B$ is called *an adjunction* between A and B provided that $id_B \leq g \circ d$ and $d \circ g \leq id_A$.

Notice that for every quantale Q one has the ordered category $Q\text{-Pos}$.

Definition 3.4. Let A be a Q -poset and let $q \in Q$. Define $A \xrightarrow{q_A} A : a \mapsto q * a$.

Given a $Q\text{-Pos}$ -morphism $A \xrightarrow{f} B$, it follows that $q_B \circ f \leq f \circ q_A$ for every $q \in Q$. Moreover, Definition 3.4 gives the following characterization of adjunctions in $Q\text{-Pos}$.

Lemma 3.3. Let $A \begin{smallmatrix} \xrightarrow{g} \\ \xleftarrow{d} \end{smallmatrix} B$ be $Q\text{-Pos}$ -morphisms. The following are equivalent:

- (i) (g, d) is an adjunction between A and B ;
- (ii) $q_B \leq g \circ q_A \circ d$ and $d \circ q_B \circ g \leq q_A$ for every $q \in Q$.

Proof. (i) \Rightarrow (ii) If $q \in Q$, then $q_B \leq q_B \circ (g \circ d) \leq g \circ q_A \circ d$ and $d \circ q_B \circ g \leq (d \circ g) \circ q_A \leq q_A$.

(ii) \Rightarrow (i) Set $q = e$. ■

Theorem 3.4. Let $A \begin{smallmatrix} \xrightarrow{g} \\ \xleftarrow{d} \end{smallmatrix} B$ be maps between Q -posets. The following are equivalent:

- (i) (g, d) is an adjunction between A and B ;
- (ii) (a) d is a $Q\text{-Pos}$ -morphism;
- (b) $g(a) = \max d^{-1}[\downarrow a]$ for every $a \in A$;
- (c) $d \circ q_B \circ g \leq q_A$ for every $q \in Q$.

Proof. (i) \Rightarrow (ii) See the proof of Theorem 0-3.2 in [2] and use Lemma 3.3.

(ii) \Rightarrow (i) By Theorem 0-3.2 in [2] (g, d) is an adjunction in **Pos**. By $q_B \circ g \leq (g \circ d) \circ q_B \circ g = g \circ (d \circ q_B \circ g) \leq g \circ q_A$, g is a Q -**Pos**-morphism. ■

Now the promised property.

Corollary 3.5. *Let A be a Q -poset and let $a \in A$. The following are equivalent:*

- (i) $(a \rightarrow _, _ * a)$ is an adjunction between A and Q (and thus $a \rightarrow (\bigwedge S) = \bigwedge_{s \in S} (a \rightarrow s)$ for every $S \subseteq A$ such that $\bigwedge S$ exists in A);
- (ii) $(a \rightarrow b) * a \leq b$ for every $b \in A$.

Corollary 3.5 gives rise to the following definition which will be useful for us later.

Definition 3.5. Given a Q -poset A , say that it satisfies condition (\mathfrak{A}) provided that $(a \rightarrow b) * a \leq b$ for every $a, b \in A$.

Every Q -module satisfies condition (\mathfrak{A}) . The situation with Q -posets, however, is different as shows the following example. Let $\mathbf{2}$ be ordered by equality. Define $Q \times \mathbf{2} \xrightarrow{*} \mathbf{2} : (q, a) \mapsto a$. Then $\mathbf{2}$ is a Q -poset which does not satisfy (\mathfrak{A}) .

Return to the completion of posets.

Lemma 3.6. *Let A be a Q -**Pos**-object. Then $(b \rightarrow c) \cdot (a \rightarrow b) \leq (a \rightarrow c)$ for every $a, b, c \in A$.*

Proof. Straightforward computations. ■

Now the main proposition.

Proposition 3.7. *Let Q be a quantale. Then the category Q -**Mod** is reflective in Q -**Pos**.*

Proof. Given a Q -poset A , the reflection arrow can be constructed as follows. Define $B_A = \{h \in Q^A \mid h(b) \cdot (a \rightarrow b) \leq h(a) \text{ for every } a, b \in A\}$. Then B_A is a submodule of Q^A (closed under arbitrary meets) and $h \in B_A$ iff $h(a) = \bigvee_{b \in A} h(b) \cdot (a \rightarrow b)$ for every $a \in A$. Let $A \xrightarrow{r} B_A : a \mapsto _ \rightarrow a$. By Lemma 3.6 the map r is correct. For the rest see Proposition 3.2. ■

Below are some properties of the reflection arrow $A \xrightarrow{r} B_A$.

Lemma 3.8. *Let Q be completely distributive. Then r preserves all existing meets.*

Proof. Let $S \subseteq A$ be such that $\bigwedge S$ exists in A and let $a \in A$. Show that $a \rightarrow (\bigwedge S) = \bigwedge_{s \in S} (a \rightarrow s)$. Set $a \rightarrow (\bigwedge S) = \bigvee T$ and $\bigwedge_{s \in S} (a \rightarrow s) = \bigwedge_{s \in S} \bigvee T_s$. It will be enough to show that $\bigwedge_{s \in S} \bigvee T_s \leq \bigvee T$. By the assumption, $\bigwedge_{s \in S} \bigvee T_s = \bigvee_{f \in F} \bigwedge_{s \in S} f(s)$ where F is the set of choice functions defined on S . Since $\bigwedge_{s \in S} f(s) \in T$ for every $f \in F$, the result follows. ■

Lemma 3.9. *Let A satisfy condition (\mathfrak{A}) . Then r is injective and preserves all existing meets.*

Proof. Since the second statement follows from Corollary 3.5 we show that r is injective. Let $a, b \in A$ with $r(a) = r(b)$. Then $e \leq a \rightarrow a = (r(a))(a) = (r(b))(a) = a \rightarrow b$ implies $a = e * a \leq (a \rightarrow b) * a \leq b$. Similarly $b \leq a$. ■

In conclusion let us note that it would be interesting to consider other generalizations of completions, e.g., of the Dedekind-MacNeille completion (see, e.g., [6]).

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