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THE DIMENSION OF A VARIETY

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Abstract

Derived varieties were invented by P. Cohn in [4]. *Derived varieties of a given type* were invented by the authors in [10]. In the paper we deal with the derived variety V_σ of a given variety, by a fixed hypersubstitution σ. We introduce the notion of the *dimension of a variety* as the cardinality κ of the set of all proper derived varieties of V included in V.

We examine dimensions of some varieties in the lattice of all varieties of a given type τ. Dimensions of varieties of lattices and all subvarieties of regular bands are determined.

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1. Notations

By τ we denote a fixed type $\tau : I \to N$, where I is an index set and N is the set of all natural numbers. In the paper we deal only with finite types, i.e., $card(I)$ is finite. We use the definition of an n-ary *term* of type τ from [4, p. 6].

$T(\tau)$ denotes the set of all term symbols of type τ. For a given variety V of type τ, two terms p and q of type τ are called *equivalent* (in V) if the identity $p \approx q$ holds in V.

Definition 1.1. For a given type τ, F denotes the set of all fundamental operations $F = \{f_i : i \in I\}$ of type τ, i.e., $\tau(i)$ is the arity of the operation symbol f_i, for $i \in I$. Let $\sigma = (t_i : i \in I)$ be a fixed choice of terms of type τ with $\tau(t_i) = \tau(f_i)$, for every $i \in I$.

Recall from [10] (cf. [4, p. 13]), that for a given σ, the extension of σ to the map $\overline{\sigma}$ from the set $T(\tau)$ to $T(\tau)$, leaving all the variables unchanged and acting on composed terms as:

$$\overline{\sigma}(f_i(p_0, \ldots, p_{n-1})) = \sigma(f_i)(\overline{\sigma}(p_0), \ldots, \overline{\sigma}(p_{n-1}))$$

is called a *hypersubstitution* of type τ.

In the sequel, we shall use σ instead of $\overline{\sigma}$ for a hypersubstitution.
A hypersubstitution σ will be called *trivial*, if it is the identity mapping.

The set of all hypersubstitutions of type τ will be denoted by $H(\tau)$.

For any algebra $\mathbf{A} = (A, \Omega) = (A, (f_i^{\mathbf{A}} : i \in I)) \in V$, of type τ, the algebra $\mathbf{A}_\sigma = (A, (t_i^{\mathbf{A}} : i \in I))$ or shortly $\mathbf{A}_\sigma = (A, \Omega_\sigma)$, for $\Omega_\sigma = (t_i : i \in I)$ is called a *derived algebra* (of a given type τ) of $\mathbf{A}$, corresponding to σ, for any $\sigma \in H(\tau)$ (cf. [10, 17]).

Definition 1.2. The variety generated by the class of all derived algebras $\mathbf{A}_\sigma$, of algebras $\mathbf{A} \in V$ will be called the *derived variety* of V using σ and it will be denoted by V_σ, for any fixed $\sigma \in H(\tau)$.

For a class K of algebras of a given type τ, $D(K)$ denotes the class of all derived algebras of K for all possible choices of σ of type τ, i.e.:

$$D(K) = \bigcup \{K_\sigma : \sigma \in H(\tau)\}.$$

D is a class operator examined in [10] (cf. [16, 17]).

Let us note, that $V_\sigma = HSP(\sigma(V))$, for a given variety V and σ, where $\sigma(V)$ denotes the class of all derived algebras $\mathbf{A}_\sigma$, for $\mathbf{A} \in V$.

Recall from [12]:

Definition 1.3. For a given set Σ of identities of type τ, $E(\Sigma)$ denotes the set of all consequences of Σ by the rules (1)–(5) of inferences of G. Birkhoff (cf. [1, 12]).

$Mod(\Sigma)$ denotes the variety of algebras determined by Σ.

A variety V is *trivial* if all algebras in V are *trivial* (i.e., one-element). Trivial varieties will be denoted by T. A subclass W of a variety V which is also a variety is called *subvariety* of V.

V is a *minimal* (or *equationally complete*) variety if V is not trivial but the only subvariety of V, which is not equal to V is trivial.

We accept the following definition from [17]:

Definition 1.4. A derived variety V_σ is *proper* if V_σ is not equal to V, i.e., $V_\sigma \neq V$.

Note, that V_σ may be not proper only for nontrivial σ.

Recall from [10]:

Definition 1.5. A variety V of type τ is *solid* if V contains all derived varieties V_σ for every choice of σ of type τ, i.e., $D(V) \subseteq V$.

Definition 1.6. A variety V of type τ is *fluid* if the variety V contains no proper derived varieties V_σ for every choice of σ of type τ.

Fluid varieties appear naturally in many well known examples (cf. [11]). Derived varieties are an important tool for describing the lattice of all subvarieties of a given variety and therefore we expect some practical applications of the invented notion.

Note, that our definition of a *fluid variety* does not coincide with that of [17].

2. The Dimension

Definition 2.1. If V is a variety of type τ, then the dimension of V is the cardinality κ of the set of all proper derived varieties V_σ of V included in V, for $\sigma \in H(\tau)$. We write then that $\kappa = dim(V)$.

From the definitions above it follows that the trivial variety T of a given type is of dimension 0.

Theorem 2.1. *Minimal varieties are of dimension* 0. *Fluid varieties are of dimension* 0.

Later on we shall use the well-known *conjugate property* of [3] (cf. [9, p. 35] and [11]) and quote as:

Theorem 2.2. *Let* $\mathbf{A}$ *be an algebra and* σ *be a hypersubstitution of type* τ. *Then an identity* $p \approx q$ *of type* τ *is satisfied in the derived algebra* $\mathbf{A}_\sigma$ *if and only if the derived identity* $\sigma(p) \approx \sigma(q)$ *holds in* $\mathbf{A}$.

From the theorem above, it immediately follows:

Theorem 2.3. *Let* V *be a variety and two hypersubstitutions* σ_1 *and* σ_2 *of type* τ *be given. If* $\sigma_1(f_i) \approx \sigma_2(f_i)$, *is an identity of* V *for every* $i \in I$, *then the derived varieties* V_{σ_1} *and* V_{σ_2} *are equal.*

Proof. The proof follows by induction on the complexity of terms of type τ. ■

In the proof we use the relation $\sim_V$ on sets of hypersubstitutions which was introduced by J. Płonka in [15] and used in [3] to determine the notion of V-*equivalent hypersubstitutions* in order to simplify the procedure of checking whether an identity is satisfied in a variety V as a hyperidentity.

Recall from [13, p. 221], that an algebra $\mathbf{A}$ is locally finite iff every finitely generated subalgebra of $\mathbf{A}$ is finite. A class of algebras is *locally finite* iff each of its members is a locally finite algebra.

Theorem 2.4. *Assume that a variety* V *(of a finite type) is locally finite. Then* V *is of a finite dimension.*

Proof. As V is locally finite, therefore every finitely generated free algebra in V is finite and therefore for every $n \in N$ there is only a finite number of non-equivalent n-ary terms in V. Moreover, in V there are only finitely many fundamental operations (by the assumption). Therefore in V there is only a finite number of non-equivalent hypersubstitutions of type τ. In cosequence there are only finitely many derived varieties of V and $dim(V)$ is finite. ■

3. Dimensions of varieties of lattices

We present some examples in lattice varieties as an answer to a problem posed by Brian Davey (La Trobe University, Australia) during the Conference on Universal Algebra and Lattice Theory (July 2005) at Szeged University, Szeged (Hungary).

Let $\mathbf{L} = (L, \vee, \wedge)$ be a lattice. A variety L_σ derived from a variety L of lattices must not be a variety of lattices.

This follows from the fact, that there are only four non-equivalent binary terms in lattices, namely x, y, $x \vee y$ and $x \wedge y$. Given a hypersubstitution σ of type (2,2). If σ is trivial, then the derived algebra $\mathbf{L}_\sigma$ is $\mathbf{L}$ itself. If one takes σ generated by $\sigma(x \vee y) = x \wedge y$ and $\sigma(x \wedge y) = x \vee y$, then $\mathbf{L}_\sigma$ is the dual lattice $\mathbf{L}^d = (L, \wedge, \vee)$. Otherwise the derived algebra $\mathbf{L}_\sigma$ is not a lattice at all, as some lattice axioms will be failed, unless $\mathbf{L}$ is trivial (i.e., one-element lattice).

We got immediately:

Example 3.1. Let V be a nontrivial variety of lattices. Then a derived variety V_σ is the dual variety of lattices V^d or a variety which is not a variety of lattices.

Example 3.2. The variety L of all lattices in type (2,2) is fluid and not solid.

The variety L is fluid as it is selfdual, i.e., $L = L^d$. It is not solid, as the commutativity laws for $\vee$ and $\wedge$ are not satisfied as hyperidentities in lattices, for example.

Theorem 3.1. *Every variety of lattices is fluid.*

Proof. Let V be a variety of lattices. Consider the dual variety of V, i.e., the variety V^d of all dual lattices of V. Then there are only two possibilities:

(i) $V^d \subseteq V$ and consequently $V = V^d$
or
(ii) V and V^d are incomparable in the lattice of all varieties of lattices.

Therefore we conclude, that either V is selfdual or V and V^d are incomparable. In consequence V is fluid and $dim(V) = 0$. ∎

4. Dimensions of subvarieties of regular bands

In this section we concentrate on the lattice of all subvarieties of regular bands, described in [6, 7] and [8].

Definition 4.1. *Bands* is the variety B of algebras of type (2), defined by: associativity and idempotency (i.e., a *band* is an idempotent semigroup).

Following [5, p. 11], let us note, that the variety of bands has only six non-equivalent binary terms, therefore only six hypersubstitutions of type (2) in the variety of bands should be checked, namely: $\sigma_1 - \sigma_6$ defined as follows: $\sigma_1(xy) = x$, $\sigma_2(xy) = y$, $\sigma_3(xy) = xy$, $\sigma_4(x,y) = yx$, $\sigma_5(xy) = xyx$ and $\sigma_6(xy) = yxy$ to be considered in order to determine all derived varieties of a given subvariety of regular bands.

Recall Proposition 3.1.5(i) from [4, p. 11, 77]:

Definition 4.2. A variety V of type (2) is called *hyperassociative* if the associativity law is satisfied in V as a hyperidentity.

Proposition 4.1. *A variety of bands is hyperassociative if and only if it is contained in the variety RegB of regular bands.*

The propositions above may be considered as a motivation of our interest in the lattice of all subvarieties of the variety of regular bands.

In order to determine the dimension of all subvarieties of RegB, we shall use the following two theorems of [11]:

Theorem 4.1. *The variety of B of all bands constitutes a not fluid and not solid variety of type* (2).

Theorem 4.2. *A variety V of bands is fluid if and only if it is minimal.*

Remark 4.1. Note, that a nontrivial variety V is of dimension 0 if and only if it is fluid.

Definition 4.3. An identity e of the form $p \approx q$ is called *leftmost* (*rightmost*) if and only if it has the same first (last) variable on each side. An identity which meets both of these conditions is called *outermost*.

First we express three technical lemmas:

Lemma 4.1. *Let Σ be a set of identities of type τ which are leftmost (or rightmost). Then the set $E(\Sigma)$ consists only of leftmost (rightmost) identities.*

Proof. The proof follows from the observation that all rules of inference (1)–(5) preserve the property of being the *leftmost* (or *rightmost*) identity. Therefore the closure of the set of left(right)most identities consists of left(right)most identities. ∎

From [6, 7] and [8] it follows that every subvariety of the variety B of all bands is defined by one additional identity added to two axioms of bands (i.e., associativity and idempotency).

Lemma 4.2. *Assume that V and W are varieties of regular bands, W is defined by a single identity $p \approx q$, i.e., $W = Mod(p \approx q)$ (in the variety of regular bands). Then:*

$$V_\sigma \subseteq W, \quad \textit{for a given } \sigma \in H(\tau),$$

if and only if the derived identity $\sigma(p) \approx \sigma(q)$ is satisfied in V, i.e., $V \models \sigma(p) \approx \sigma(q)$.

Proof. $V_\sigma = HSP(\sigma(V)) \subseteq W$ if and only if $\sigma(V) \models p \approx q$. By Theorem 2.2 we conclude that $\mathbf{A}_\sigma \models p \approx q$, for every algebra $\mathbf{A}_\sigma \in \sigma(V)$, if and only if $\mathbf{A} \models \sigma(p) \approx \sigma(q)$, for every algebra $\mathbf{A} \in V$, i.e., $V \models \sigma(p) \approx \sigma(q)$. ∎

A simple generalization of the above lemma is the following:

Lemma 4.3. *Assume that V and W are varieties of type τ, W is defined by a set Σ of identities of type τ, i.e., $W = Mod(\Sigma)$. Then:*

$$V_\sigma \subseteq W, \textit{ for a given } \sigma \in H(\tau),$$

if and only if the derived identity $\sigma(p) \approx \sigma(q)$ is satisfied in V, i.e.,

$$V \models \sigma(p) \approx \sigma(q), \textit{ for every identity } p \approx q \in \Sigma.$$

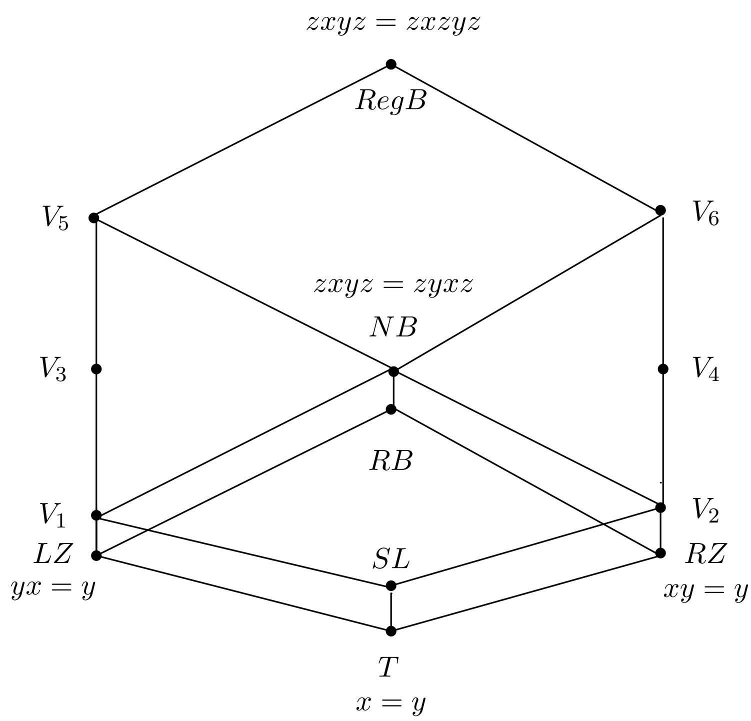
Proof. The proof is similar as that of Lemma 4.2, where $p \approx q$ is any identity of the given axiomatic Σ. ∎

The next three propositions show some regularities in the dimensions of all subvarieties of regular bands described in [7, p. 244] and [8]:

Definition 4.4. The variety RB in the variety B bands is defined by the identity: $y \approx yxy$. It is called the variety of rectangular bands.

The fact that the variety RB is solid was proved in [5, p. 96].

We expressed the situation of theorems above on the diagram, which describes the bottom part of the lattice of all identities of bands, see [10] and [12, p. 244] Proposition 3.1.5 of [4]:



Theorem 4.3. *The variety RB is of dimension* 2.

Proof. The variety of RB of rectangular bands have only two nontrivial subvarieties, namely the variety LZ defined by the identities: $yx \approx y$ (called the variety of left-zero semigroups) and the variety RZ defined by $xy \approx y$

(called the variety of right-zero semigroups), respectively. Both of them are derived varieties of RB by the first and the second projection, respectively. To prove that, let $\mathbf{A}_{\sigma_1} \in (RB)_{\sigma_1}$, for $\mathbf{A} \in RB$. Then the identity $yx \approx y$ is satisfied in $\mathbf{A}_{\sigma_1}$, as: $\sigma_1(yx) \approx y \approx y \approx \sigma_1(y)$ is satisfied in $\mathbf{A}$ and consequently in $(RB)_{\sigma_1}$. Similarly for σ_2. We conclude that $dim(RB) = 2$. ∎

Theorem 4.4. *The varieties V_1 and V_2 of bands defined by the identities:*

$$zxy \approx zyx \tag{1}$$

and

$$yxz \approx xyz, \quad \textit{respectively}, \tag{2}$$

are mutually derived by σ_4. Moreover, $dim(V_1) = dim(V_2) = 1$.

Proof. Note, that the varieties V_1 and V_2 has only two proper nontrivial subvarieties, namely: the variety of left (right) zero-semigroups (respectively) and the variety SL of semilattices. The variety of semilattices, defined (in the variety of bands) by the commutativity law: $xy \approx yx$ is not a derived variety of V_1, neither of V_2. This follows from the fact, that if the variety SL of semilattices would be a derived variety of V_1, then $SL = (V_1)_{\sigma_5}$ or $SL = (V_1)_{\sigma_6}$. This is impossible, via Theorem 1.3, as the derived identity of $xy \approx yx$ by the hypersubstitions σ_5 (or σ_6), i.e., $\sigma_5(xy) \approx \sigma_5(yx)$ (or $\sigma_6(xy) \approx \sigma_6(yx)$) is of the form $xyx \approx yxy$ is neither leftmost nor rightmost and therefore, by Lemma 4.1 is not satisfied in V_1 as every identity satisfied in V_1 is leftmost. Similarly for V_2. The proof follows from the fact that the only proper derived variety of V_1 included in V_1 is the variety LZ of left zero semigroups. Similarly, one can show that the only proper derived subvariety of V_2 by the second projection σ_2 is the variety RZ of right zero semigroups. Finally we conclude that $dim(V_1) = dim(V_2) = 1$. ∎

Definition 4.5. Varieties of dimension 1 will be called *prefluid*.

Theorem 4.5. *The varieties V_3 and V_4 of bands defined by the identities:*

$$yx \approx yxy \tag{3}$$

and

$$xy \approx yxy, \quad \textit{respectively}, \tag{4}$$

are mutually derived by σ_4. Moreover, $dim(V_3) = dim(V_4) = 1$.

Proof. The variety $(V_3)_{\sigma_1}$ is the variety LZ of bands defined by $yx \approx y$. We obtain that: $(V_3)_{\sigma_1}$ is different from V_3, therefore $(V_3)_{\sigma_1}$ is proper and $(V_3)_{\sigma_1} \subseteq V_3$. Note, that the derived variety $(V_3)_{\sigma_2}$ is proper and is the variety RZ of right-zero semigroups but is not included in V_3. Similarly as in the previous theorem we conclude, that the variety SL of semilattices is not a derived variety of V_3, as all the identities of V_3 are left-most. In order to exclude that, the variety V_1 defined by the identity (1) $zxy \approx zyx$ is the derived variety of V_3 by σ_5 consider the derived identity $\sigma_5(zxy) \approx \sigma_5(zyx)$ of (1) by σ_5, i.e., the identity $zxyxz \approx zyxyz$. If this identity would be satisfied in V_3, then the identity $zxyz \approx zyxz$ would be satisfied in V_3, which is not true due to the results of [6]–[8]. Dually, the derived variety of V_3 by σ_6 is not the variety V_1. Therefore we conclude, that $dim(V_3) = 1$. Similarly one can prove that $dim(V_4) = 1$. ■

Theorem 4.6. *The varieties V_5 and V_6 of bands defined by the identities:*

$$zxy \approx zxzy \tag{5}$$

and

$$yxz \approx yzxz, \quad \textit{respectively}, \tag{6}$$

are mutually derived by σ_4. Moreover, $dim(V_5) = dim(V_6) = 3$.

Proof. The proof that $(V_5)_{\sigma_1}$ $((V_5)_{\sigma_2})$ is the variety $LZ(RZ)$ of left (right) zero semigroups follows from the proof of previous observations. Obviously: $\sigma_4(V_5) = V_6$, as the derived identity $\sigma_4(yxz) \approx \sigma_4(yzxz)$ of (6) by σ_4 gives rise to the identity (5) $zxy \approx zxzy$ and vice versa. Therefore V_6 and V_5 are mutually derived by σ_4. We will show that the derived variety of V_5 by the hypersubstitution σ_5 is the variety V_3, i.e., $\sigma_5(V_5) = V_3$. To show this consider the derived identity of (3) by σ_5, i.e., the identity $\sigma_5(yx) \approx \sigma_5(yxy)$. This gives rise to the identity $yxy \approx yxyxy$, which is obviously satisfied in V_5. Moreover, note that the derived identity of (1) by σ_5, i.e., $\sigma_5(zxy) \approx \sigma_5(zyx)$ gives rise to the identity $zxyxz \approx zyxyz$, which can not be satisfied in V_5, as otherwise the identity $zxyz \approx zyxz$ would be satisfied in V_3, which is impossible by the results of [6]–[8] and it has been shown already in the proof of Theorem 4.5. Similarly one can show, that the derived variety of V_5 by σ_6 is the variety V_4. We conclude that $dim(V_5) = 3$. Similarly, $dim(V_6) = 3$. ■

Definition 4.6. The variety NB of normal bands is defined by the identity:

$$zxyz \approx zyxz. \tag{7}$$

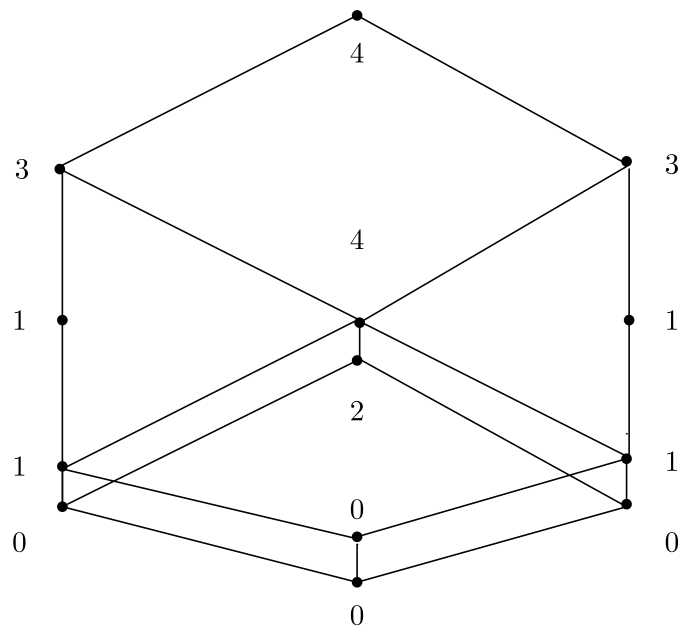
Theorem 4.7. $dim(NB) = 4$.

Proof. For solidity of the variety NB confront [5, p. 96]. It follows, that all derived varieties of the variety NB are included in the variety of NB. Similarly as before we show that $(NB)_{\sigma_1}$ is the variety LZ of left-zero semigroups and $(NB)_{\sigma_2}$ is the variety RZ of right-zero semigroups. Both of them are proper subvarieties of NB. It is obvious that $(NB)_{\sigma_3} = (NB)_{\sigma_4} = NB$. We show only that $(NB)_{\sigma_5} = V_1$, as the derived identity of (1) $zxy \approx zyx$ by σ_5, i.e., $\sigma_5(zxy) \approx \sigma_5(zyx)$ gives rise to the identity $zxyxz \approx zyxyz$ satisfied in NB. In order to exclude that the variety LZ of left zero semigroups, defined by the identity $yx \approx y$ equals to $(NB)_{\sigma_5}$, notice that the derived identity of $yx \approx y$ by σ_5 is the identity $yxy \approx y$, which is not satisfied in NB, as the variety of NB is defined by the set of regular identities (cf. [14]), which has only regular consequences. Similarly $(NB)_{\sigma_6} = V_2$, as the derived identity of (2) $yxz \approx xyz$ by σ_6, i.e., $\sigma_6(yxz) \approx \sigma_6(xyz)$ gives rise to the identity $zxzyzxz \approx zyzxzyz$ satisfied in NB and we conclude that $dim(NB) = 4$. ∎

Theorem 4.8. $dim(RegB) = 4$.

Proof. For solidity of the variety $RegB$ confront [5, p. 96]. Two derived subvarieties of $RegB$ are LZ and RZ, by σ_1 and σ_2, respectively. The derived varieties of $RegB$ via σ_3 and σ_4 are equal to $RegB$. We show that $(RegB)_{\sigma_5} = V_3$. To prove that, consider the derived identity of the identity (3) $yx \approx yxy$ by σ_5, i.e., the identity $\sigma_5(yx) \approx \sigma_5(yxy)$ which gives rise to the identity $yxy \approx yxyxy$ which is satisfied in $RegB$. In order to show that the derived variety of $RegB$ by σ_5 is not equal to the variety V_1, note that the derived identity of (1) $zxy \approx zyx$ by σ_5, i.e., the identity $\sigma_5(zxy) \approx \sigma_5(zyx)$ gives rise to the identity $zxyxz \approx zyxyz$ which is not satisfied in $RegB$, as it was shown in the proof of Theorem 4.6 that this identity is not satisfied in V_5, which is a subvariety of $RegB$. Similarly, one can show, that the derived variety of $RegB$ by σ_6 is the variety V_4. This finishes the proof that $dim(RegB) = 4$. ∎

We expressed the situation of theorems above on the diagram:



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