

ON UNIFORMLY STRONGLY PRIME Γ -SEMIRINGS (II)

TAPAN KUMAR DUTTA AND SUHRID DHARA

Department of Pure Mathematics, University of Calcutta

35, Ballygunge Circular Road, Kolkata-700019, India

e-mail: duttatapankumar@yahoo.co.in

e-mail: suhridd@yahoo.com

Abstract

The Uniformly strongly prime k -radical of a Γ -semiring is a special class which we study via its operator semiring.

Keywords: Γ -semiring, uniformly right strongly prime Γ -semiring, annihilators, essential ideal, essential extension, uniformly strongly prime k -radical, matrix Γ -semiring, special class, super t -system.

2000 Mathematics Subject Classification: 16Y60.

1. INTRODUCTION

In 1987, D.M. Olson [7] introduced the notion of uniformly strongly prime radical in rings. In 1988, G.L. Booth and N.J. Groenwald [1] extended this notion of uniformly strongly prime radical to Γ -ring. In 1977, G.A.P. Heyman and C. Roos [6] introduced the notion of essential extension of rings.

In this paper we introduce the notions of uniformly right strongly prime ideal and uniformly left strongly prime ideal of a Γ -semiring and show that these two notions are equivalent. Also we study uniformly strongly prime k -radical of a Γ -semiring via its operators semirings as G.L. Booth and N.J. Groenwald did in case of Γ -ring. Some earlier works on the operator semiring of a Γ -semiring may be found in [4]. We obtain a relation between uniformly strongly prime k -radical of a Γ -semiring and with that of its matrix Γ -semiring via operator semiring. Lastly, we introduce the notion of super t -system in a Γ -semiring and obtain the relation between the uniformly strongly prime k -radical and super t -system in a Γ -semiring.

2. UNIFORMLY STRONGLY PRIME Γ -SEMIRINGS

Definition 2.1 ([4]). Let S and Γ be two additive commutative semigroups. Then S is called a Γ -semiring if there exists a mapping $S \times \Gamma \times S \rightarrow S$ (image to be denoted by $a\alpha b$, for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions:

- (i) $a\alpha(b + c) = a\alpha b + a\alpha c$
- (ii) $(a + b)\alpha c = a\alpha c + b\alpha c$
- (iii) $a(\alpha + \beta)c = a\alpha c + a\beta c$
- (iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$

for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

Every semiring S is a Γ -semiring with $\Gamma = S$ where $a\alpha b$ denotes the product of elements $a, \alpha, b \in S$.

If S contains an element 0 such that $0+x = x = x+0$ and $0\alpha x = x\alpha 0 = 0$ for all $x \in S$, for all $\alpha \in \Gamma$, then 0 is called the zero of S .

Throughout this paper we assume that a Γ -semiring always contains a zero element.

Definition 2.2 ([4]). Let S be a Γ -semiring and L be the left operator semiring and R be the right operator semiring. If there exists an element

$$\sum_{i=1}^m [e_i, \delta_i] \in L \left(\text{respectively } \sum_{j=1}^n [\nu_j, f_j] \in R \right)$$

such that

$$\sum_{i=1}^m e_i \delta_i a = a \left(\text{respectively } \sum_{j=1}^n a \nu_j f_j = a \right) \text{ for all } a \in S$$

then S is said to have the left unity

$$\sum_{i=1}^m [e_i, \delta_i] \left(\text{respectively the right unity } \sum_{j=1}^n [\nu_j, f_j] \right).$$

Definition 2.3 ([4]). A nonempty subset I of a Γ -semiring S is called an ideal of S if $I + I \subseteq I$, $I \Gamma S \subseteq I$, $S \Gamma I \subseteq I$, where for subsets U, V of S and Δ of Γ ,

$$U \Delta V = \left\{ \sum_{i=1}^n u_i \gamma_i v_i : u_i \in U, v_i \in V, \gamma_i \in \Delta \text{ and } n \text{ is a positive integer} \right\}$$

Definition 2.4. A Γ -semiring S is called uniformly right strongly prime if S and Γ contain finite subsets F and Δ respectively such that for any non zero $x (\neq 0) \in S$, $x \Delta F \Delta y = \{0\}$ implies that $y = 0$ for all $y \in S$. The pair (F, Δ) is called a uniform right insulator for S .

Analogously we can define uniformly left strongly prime Γ -semiring.

Theorem 2.5. A Γ -semiring S is uniformly right strongly prime if and only if there exist finite subsets F of S and Δ of Γ such that for any two nonzero elements x and y of S there exist $f \in F$ and $\alpha, \beta \in \Delta$ such that $x \alpha f \beta y \neq 0$.

Proof. Let S be a uniformly right strongly prime Γ -semiring and (F, Δ) be a uniform right insulator for S . Let x, y be any two nonzero elements of S . Suppose that $x \Delta F \Delta y = \{0\}$. Then $y = 0$, a contradiction. So there exist $f \in S$ and $\alpha, \beta \in \Delta$ such that $x \alpha f \beta y \neq 0$. ■

The converse follows by reversing the above argument.

Corollary 2.6. A Γ -semiring S is uniformly right strongly prime if and only if S is uniformly left strongly prime.

So we ignore the word right from uniformly right strongly prime.

Definition 2.7. A nonzero ideal I of a Γ -semiring S is called an essential ideal of S if for any nonzero ideal J of S , $I \cap J \neq (0)$.

Example 2.8. Let $S = \{r\omega : r \in \mathbf{Z}\}$ and $\Gamma = \{r\omega^2 : r \in \mathbf{Z}\}$, where ω be a cube root of unity and \mathbf{Z} be the set of all integers. Then S is a Γ -semiring with usual addition and multiplication. Let $I = \{2r\omega : r \in \mathbf{Z}\}$. Then I is a nonzero ideal of S . Let J be any nonzero ideal of S . Then $I \cap J \neq (0)$. Hence I is an essential ideal of S .

Definition 2.9. A Γ -semiring T is said to be an essential extension of a Γ -semiring S if S is an essential ideal of T .

Definition 2.10. Let A be a nonempty subset of a Γ -semiring S . Right annihilator of A in S , denoted by $ann_R(A)$, is defined by $ann_R(A) = \{s \in S : A\Gamma s = \{0\}\}$.

Analogously we can define left annihilator $ann_L(A)$ of A in S . Annihilator of a nonempty subset A is denoted by $ann(A)$ which is a left as well as a right annihilator of A .

Remark 2.11. If S is a Γ -semiring then $ann_R(A)$ is a right ideal of S and $ann_L(A)$ is a left ideal of S . If A is an ideal of a Γ -semiring S then both annihilators are ideals of S .

Lemma 2.12. Let S be a Γ -semiring and T be its essential extension. If S is a uniformly strongly prime Γ -semiring then for each nonzero x of T , $x\Delta F = \{0\}$ implies that $x \in ann_R(S)$ and $F\Delta x = \{0\}$ implies that $x \in ann_L(S)$, where (F, Δ) is a uniform insulator for S .

Proof. Let $x\Delta F = \{0\}$. Then $s\alpha x\Delta F\Delta s\alpha x = \{0\}$ for all $s \in S$ and for all $\alpha \in \Gamma$. Since S is an ideal of T and $s \in S$, $s\alpha x \in S$. Again since S is uniformly strongly prime and (F, Δ) is a uniform insulator for S , $s\alpha x = 0$ for all $s \in S$ and for all $\alpha \in \Gamma$ i.e. $S\Gamma x = \{0\}$ i.e. $x \in ann_R(S)$ (By Definition 2.10). ■

Similarly we can prove that $F\Delta x = \{0\}$ implies that $x \in ann_L(S)$.

Lemma 2.13. Let S be a uniformly strongly prime Γ -semiring and T be its essential extension. Then both annihilators of S in T are zero.

Proof. Let (F, Δ) be a uniform insulator for S . If possible let $ann_R(S) \neq (0)$. Then $ann_R(S)$ is a nonzero ideal of T . Since S is an essential ideal of T , $ann_R(S) \cap S \neq (0)$. Let $x(\neq 0) \in ann_R(S) \cap S$. Then $S\Gamma x = \{0\}$. Now $x\Delta F\Delta x \subseteq x\Gamma S\Gamma x = \{0\}$. Since S is a uniformly strongly prime Γ -semiring, $x = 0$, a contradiction. Therefore $ann_R(S) = (0)$. ■

Similarly we can prove that $ann_L(S) = (0)$.

Theorem 2.14. Any essential extension of a uniformly strongly prime Γ -semiring S is a uniformly strongly prime Γ -semiring.

Proof. Let (F, Δ) be a uniform insulator for S and T be an essential extension of S . Let x be a nonzero element of T . Then $x\Delta F$ and $F\Delta x$ both are nonzero. Suppose $x\Delta F = \{0\}$ then by Lemma 2.12, $x \in ann_R(S)$. Also

by Lemma 2.13, $\text{ann}_R(S) = (0)$, which implies that $x = 0$, a contradiction. Therefore $x\Delta F \neq \{0\}$. Similarly $F\Delta x \neq \{0\}$. Let y, z be two nonzero elements of T . Then there exist $f_1, f_2 \in F$ and $\alpha_1, \alpha_2 \in \Delta$ such that $y\alpha_1 f_1 \neq 0$ and $f_2\alpha_2 z \neq 0$. Since S is an ideal of T , so $y\alpha_1 f_1, f_2\alpha_2 z \in S$. Again since S is uniformly strongly prime and (F, Δ) is a uniform insulator for S , there exist $\alpha, \beta \in \Delta$ and $f \in F$ such that $y\alpha_1 f_1 \alpha f \beta f_2 \alpha_2 z \neq 0$. Let $F' = \{f_1 \alpha f \beta f_2 : y\alpha_1 f_1 \alpha f \beta f_2 \alpha_2 z \neq 0; f_1, f, f_2 \in F; \alpha, \beta, \alpha_1, \alpha_2 \in \Delta; y, z \in T\}$. Then $F' \subseteq S \subseteq T$ is a finite subset, since F and Δ are finite subsets. Hence by Theorem 2.5, T is uniformly strongly prime Γ -semiring with insulator (F', Δ) . ■

Definition 2.15. Let S be a Γ -semiring and T be a nonempty subset of S . Then T is said to be a Γ -subsemiring of S if for $t_1, t_2 \in T$ and $\alpha \in \Gamma$, $t_1 + t_2, t_1 \alpha t_2 \in T$.

Remark 2.16. Every ideal of a Γ -semiring S is a Γ -subsemiring of S .

Lemma 2.17. If S is a uniformly strongly prime Γ -semiring and I is an ideal of S , then I is also a uniformly strongly prime Γ -subsemiring.

Proof. Let S be a uniformly strongly prime Γ -semiring and (F, Δ) be a uniform insulator for S . If $I = (0)$ then obviously I is a uniformly strongly prime Γ -subsemiring. Suppose $I \neq (0)$ and r be a fixed nonzero element of I . Let $F' = \{f_1 \alpha r \beta f_2 : f_1, f_2 \in F; \alpha, \beta \in \Delta\}$. Since I is an ideal of S and F, Δ are finite subsets, F' is a finite subset of I . Let $x (\neq 0) \in I$ and $y \in I$. Now $x\Delta F'\Delta y = \{0\}$ implies that $x\gamma f_1 \alpha r \beta f_2 \delta y = 0$ for all $f_1, f_2 \in F$ and for all $\alpha, \beta, \gamma, \delta \in \Delta$ i.e. $x\Delta F\Delta(r\beta f_2 \delta y) = \{0\}$ for all $f_2 \in F$ and for all $\beta, \delta \in \Delta$. Since $r\beta f_2 \delta y \in S$ for all $f_2 \in F$ and for all $\beta, \delta \in \Delta$ and S is a usp Γ -semiring with $x \neq 0$, $r\beta f_2 \delta y = 0$ for all $f_2 \in F$ and for all $\beta, \delta \in \Delta$ i.e. $r\Delta F\Delta y = \{0\}$. By previous argument $y = 0$ as $r \neq 0$. Hence (F', Δ) is a uniform insulator for I . Thus I is a uniformly strongly prime Γ -subsemiring. ■

3. UNIFORMLY STRONGLY PRIME k -RADICALS

Definition 3.1 ([3]). An ideal P of a Γ -semiring S is called a uniformly strongly prime ideal (usp ideal) if S and Γ contain finite subsets F and Δ respectively such that $x\Delta F\Delta y \subseteq P$ implies that $x \in P$ or $y \in P$ for all $x, y \in S$.

Definition 3.2. Let S be a Γ -semiring. The uniformly strongly prime k -radical (usp k -radical) of a Γ -semiring S , denoted by $\tau(S)$ is defined by

$$\tau(S) = \bigcap_{P \in \Lambda_S} P,$$

where Λ_S denote the set of all usp k -ideals of the Γ -semiring S .

Definition 3.3. Let S be a Γ -semiring and L (respectively R) be its left (respectively right) operator semiring, the usp k -radical of L (respectively R), denoted by $\tau(L)$ (respectively $\tau(R)$) is defined by

$$\tau(L) = \bigcap_{A \in \Lambda_L} A,$$

where Λ_L denote the set of all usp k -ideals of the left operator semiring L (respectively $\tau(R) = \bigcap_{B \in \Lambda_R} B$, where Λ_R denote the set of all usp k -ideals of the right operator semiring R).

Theorem 3.4. Let S be a Γ -semiring with left and right unities and L be its left operator semiring then $\tau(L)^+ = \tau(S)$ and $\tau(L) = \tau(S)^+$.

Proof. Let Λ_L and Λ_S denote the set of all usp k -ideals of L and S respectively, then

$$\tau(L) = \bigcap_{A \in \Lambda_L} A \quad \text{and} \quad \tau(S) = \bigcap_{P \in \Lambda_S} P.$$

Hence

$$(\tau(L))^+ = \left(\bigcap_{A \in \Lambda_L} A \right)^+ = \bigcap_{A \in \Lambda_L} A^+.$$

Since for every $A \in \Lambda_L$, $A^+ \in \Lambda_S$ (Cf. Theorem 2.23 of [3]),

$$\bigcap_{A \in \Lambda_S} A \subseteq \bigcap_{A \in \Lambda_L} A^+.$$

Hence

$$(1) \quad \tau(S) \subseteq (\tau(L))^+.$$

Again

$$\tau(S) = \bigcap_{P \in \Lambda_S} P = \bigcap_{P \in \Lambda_S} (P^{+'})^+ = \left(\bigcap_{P \in \Lambda_S} P^{+'} \right)^+.$$

Since for each $P \in \Lambda_S$, $P^{+'} \in \Lambda_L$ (Cf. Theorem 2.23 of [3]),

$$\bigcap_{P \in \Lambda_L} P \subseteq \bigcap_{P \in \Lambda_S} P^{+'}.$$

Hence

$$\left(\bigcap_{P \in \Lambda_L} P \right)^+ \subseteq \left(\bigcap_{P \in \Lambda_S} P^{+'} \right)^+,$$

which implies that

$$(2) \quad (\tau(L))^+ \subseteq \tau(S).$$

Thus from (1) and (2) we get $\tau(S) = (\tau(L))^+$. ■

Similarly we can prove that $\tau(L) = \tau(S)^{+'}$.

Proposition 3.5. *Let S be a Γ -semiring with left and right unities and R be its right operator semiring then $\tau(R)^* = \tau(S)$ and $\tau(R) = \tau(S)^{+'}$.*

Corollary 3.6. *Let S be a Γ -semiring with left and right unities L and R be its left and right operator semirings then $\tau(L)^+ = \tau(S) = \tau(R)^*$.*

Theorem 3.7. *Let S be a semiring with identity 1 and $\tau'(S)$, $\tau(S)$ denote respectively the usp k -radical of the semiring S and the usp k -radical of the Γ -semiring S , where $\Gamma = S$, then $\tau'(S) = \tau(S)$.*

Proof. By Theorem 2.26 of [3] every usp k -ideal of a semiring S is a usp k -ideal of the Γ -semiring S , where $\Gamma = S$. This follows that $\tau'(S) = \tau(S)$. ■

4. USP k -RADICALS IN MATRIX Γ -SEMIRING

Definition 4.1 ([5]). Let S be a Γ -semiring and m, n be positive integers. We denote by S_{mn} and Γ_{nm} respectively the sets of $m \times n$ matrices with entries from S and $n \times m$ matrices with entries from Γ . Let $A, B \in S_{mn}$

and $\Delta \in \Gamma_{nm}$. Then $A\Delta B \in S_{mn}$ and $A + B \in S_{mn}$. Clearly, S_{mn} forms a Γ_{nm} -semiring with these operations. We call it the matrix Γ -semiring S or the matrix Γ_{nm} -semiring S_{mn} or simply the Γ_{nm} -semiring S_{mn} .

We denote the right operator semiring of the matrix Γ_{nm} -semiring S_{mn} by $[\Gamma_{nm}, S_{mn}]$ and the left one by $[S_{mn}, \Gamma_{nm}]$. If $x \in S$, the notation xE_{ij} will be used to denote a matrix in S_{mn} with x in the (i, j) -th entry and zeros elsewhere. The notation αE_{ij} , where $\alpha \in \Gamma$ will have a similar meaning. If $P \subseteq S$, P_{mn} will denote the set of all $m \times n$ matrices with entries from P . If $\Delta \subseteq \Gamma$, Δ_{nm} is similarly defined.

Theorem 4.2. *Let S be a Γ -semiring with left and right unities. Then S is a usp Γ -semiring if and only if S_{mn} is a usp Γ_{nm} -semiring for all positive integers m and n .*

Proof. S is a usp Γ -semiring if and only if L is a usp semiring if and only if L_m is a usp semiring if and only if S_{mn} is a usp Γ_{nm} -semiring (By Theorem 2.18 of [3] and by Lemma 3.13 of [2]). ■

Lemma 4.3. *Let S be a Γ -semiring and m, n be positive integers; then a nonempty subset P of S_{mn} is a usp k -ideal of S_{mn} if and only if $P = Q_{mn}$, for some usp k -ideal Q of S .*

Proof. Let L be the left operator semiring of S and L_m be the left operator semiring of the Γ_{nm} -semiring S_{mn} . Let P be a usp k -ideal of S_{mn} . Then by Theorem 2.23 of [3] (applied to Γ_{nm} -semiring S_{mn}) $P^{+'}$ is a usp k -ideal of the left operator semiring L_m of the Γ_{nm} -semiring S_{mn} . So $P^{+'} = Q_m$ for some usp k -ideal Q of L . This implies that $(P^{+'})^+ = (Q_m)^+$ i.e. $P = (Q^+)_{mn}$ (By Theorem 2.10 and Proposition 3.4 of [5]). Again by Theorem 2.23 of [3], Q^+ is an usp k -ideal of S . So the direct implication follows.

Conversely, suppose Q is a usp k -ideal of S . Then $Q^{+'}$ is a usp k -ideal of L (By Theorem 2.23 of [3]). So $(Q^{+'})_m$ is a usp k -ideal of L_m i.e. $(Q_{mn})^{+'}$ is a usp k -ideal of L_m by previous arguments. Hence $Q_{mn} = ((Q_{mn})^{+'})^+$ is an usp k -ideal of the Γ_{nm} -semiring S_{mn} . ■

Lemma 4.4. *Let S be a Γ -semiring and I be an ideal of S . Then $(S/I)_{mn}$ is isomorphic to S_{mn}/I_{mn} for all positive integers m, n .*

Proof. Let θ be a mapping of the Γ_{nm} -semiring $(S/I)_{mn}$ to the Γ_{nm} -semiring S_{mn}/I_{mn} defined by $\theta((x_{ij}/I)_{mn}) = (x_{ij})_{mn}/I_{mn}$. Let $(x_{ij}/I)_{mn}$,

$(y_{ij}/I)_{mn} \in (S/I)_{mn}$. Now $(x_{ij}/I)_{mn} = (y_{ij}/I)_{mn}$ if and only if $x_{ij}/I = y_{ij}/I$, for all i, j if and only if $x_{ij} + a_{ij} = y_{ij} + b_{ij}$, for some $a_{ij}, b_{ij} \in I$ and for all i, j if and only if $(x_{ij} + a_{ij})_{mn} = (y_{ij} + b_{ij})_{mn}$ if and only if $(x_{ij})_{mn} + (a_{ij})_{mn} = (y_{ij})_{mn} + (b_{ij})_{mn}$ for some matrices $(a_{ij})_{mn}, (b_{ij})_{mn} \in I_{mn}$ if and only if $(x_{ij})_{mn}/I_{mn} = (y_{ij})_{mn}/I_{mn}$ if and only if $\theta((x_{ij}/I)_{mn}) = \theta((y_{ij}/I)_{mn})$. Therefore θ is well defined as well as injective.

Clearly θ is onto and semigroup homomorphism under addition.

Now

$$\begin{aligned} & \theta\left((x_{ij}/I)_{mn}(\alpha_{ij})_{nm}(y_{ij}/I)_{mn}\right) \\ &= \theta\left(\left(\sum_{k=1}^n \sum_{l=1}^m (x_{ik}\alpha_{kl}y_{lj})/I\right)_{mn}\right) \\ &= \left(\sum_{k=1}^n \sum_{l=1}^m (x_{ik}\alpha_{kl}y_{lj})\right)_{mn} / I_{mn} \\ &= ((x_{ij})_{mn}(\alpha_{ij})_{nm}(y_{ij})_{mn}) / I_{mn} \\ &= ((x_{ij})_{mn}/I)(\alpha_{ij})_{nm}((y_{ij})_{mn}/I) \\ &= \theta((x_{ij}/I)_{mn})\iota((\alpha_{ij})_{nm})\theta((y_{ij}/I)_{mn}). \end{aligned}$$

This shows that (θ, ι) is an isomorphism of $(S/I)_{mn}$ onto S_{mn}/I_{mn} , where ι is the identity mapping from Γ_{nm} onto Γ_{nm} . ■

Proposition 4.5. *Let A, B be two ideals of a Γ -semiring S . Then $(A \cap B)_{mn} = A_{mn} \cap B_{mn}$, where A_{mn}, B_{mn} are two ideals of Γ_{nm} -semiring S_{mn} .*

Theorem 4.6. *Let S be a Γ -semiring. Then $\tau(S_{mn}) = (\tau(S))_{mn}$ for all positive integers m, n .*

Proof. By Definition 3.2,

$$\tau(S_{mn}) = \bigcap_{P \in \Lambda_{S_{mn}}} P,$$

where $\Lambda_{S_{mn}}$ denote the set of all usp k -ideals of the Γ -semiring S_{mn} and

$$\tau(S) = \bigcap_{Q \in \Lambda_S} Q,$$

where Λ_S denote the set of all usp k -ideals of the Γ -semiring S .
Therefore

$$(\tau(S))_{mn} = \left(\bigcap_{Q \in \Lambda_S} Q \right)_{mn} = \bigcap_{Q \in \Lambda_S} Q_{mn}$$

(By Proposition 4.5). Hence by Lemma 4.3 we get the result. ■

5. SPECIAL CLASSES

Definition 5.1. A class \wp of Γ -semirings is called hereditary if I is an ideal of a Γ -semiring S and $S \in \wp$ implies that $I \in \wp$.

Definition 5.2. A class \wp of Γ -semirings is called closed under essential extension if I is an essential ideal of a Γ -semiring S and $I \in \wp$ implies that $S \in \wp$.

Definition 5.3. The class \wp of Γ -semirings is called a special class if

- (i) \wp consists of prime Γ -semirings,
- (ii) \wp is hereditary and
- (iii) \wp is closed under essential extension.

Theorem 5.4. A class \mathcal{L} of uniformly strongly prime Γ -semirings is a special class.

Proof. Since uniformly strongly prime implies prime, so \mathcal{L} consists of all prime Γ -semirings. By Lemma 2.17, we get \mathcal{L} is hereditary class. Again by Theorem 2.14, \mathcal{L} is closed under essential extension. Hence \mathcal{L} is a special class. ■

Proposition 5.5. *The following conditions are equivalent for any class ρ of prime Γ -semirings:*

- (i) *if I is an ideal of S , $I \in \rho$ and $\text{ann}(I) = (0)$, then $S \in \rho$*
- (ii) *ρ is closed under essential extension.*

Proof. (i) \Rightarrow (ii) Let I be an essential ideal of S and $I \in \rho$. Let $x \in I \cap \text{ann}(I)$ then $x \in I$ and $x \in \text{ann}(I)$ which implies that $x \in I$ and $I\Gamma x = \{0\} = x\Gamma I$. So $x\Gamma I\Gamma x = \{0\}$ which implies that $\langle x \rangle \Gamma \langle x \rangle = \{0\}$. Since (0) is a prime ideal of S so it follows that $x = 0$. Therefore $\text{ann}(I) = (0)$. Hence by (i), $S \in \rho$ which imply (ii).

(ii) \Rightarrow (i) Let ρ be closed under essential extension. Let $I \in \rho$ be an ideal of S with $\text{ann}(I) = (0)$. Let L be an ideal of S such that $I \cap L = (0)$. Now $I\Gamma L \subseteq I \cap L = (0)$ and $L\Gamma I \subseteq I \cap L = (0)$. So $I\Gamma L = (0) = L\Gamma I$ which implies that $L \subseteq \text{ann}(I) = (0)$, so $L = (0)$. Therefore I is an essential ideal of S . Also $I \in \rho$. Hence by (ii) $S \in \rho$. ■

A Γ -semiring S is called a us(1) prime if it has an insulator of the form $(\{x\}, \{\gamma\})$ where $x \in S$ and $\gamma \in \Gamma$.

As in Theorem 5.4 we can show that the class \mathcal{L}_1 of all us(1) prime Γ -semirings is a special class.

Definition 5.6. A pair of subsets (T, I) of a Γ -semiring S is called a super t -system of S if

- (i) I is an ideal of S
- (ii) $T \cap I \subseteq (0)$ and
- (iii) there exist finite subsets F of S and Δ of Γ such that for all $a, b \in S \setminus I$, $a\Delta F\Delta b \cap T \neq \phi$.

The pair (F, Δ) will be called an insulator of the super t -system. Therefore I is a uniformly strongly prime ideal of S if and only if $(S \setminus I, I)$ is a super t -system.

Theorem 5.7 *For a Γ -semiring S , $\tau(S) = \{a \in S: \text{whenever } (T, I) \text{ is a super } t\text{-system in } S \text{ with } a \in T \text{ then } 0 \in T\}$.*

Proof. Let $H = \{a \in S: \text{whenever } (T, I) \text{ is a super } t\text{-system in } S \text{ with } a \in T \text{ then } 0 \in T\}$. Let $a \in \tau(S)$. Suppose (T, I) is a super t -system in S with $a \in T$ and $0 \notin T$. So $T \cap I = \phi$. Let (F, Δ) be an insulator of the super t -system. By Zorn's lemma there exists a maximal ideal P such that $I \subseteq P$ and $T \cap P = \phi$. We now prove that P is a uniformly strongly prime ideal of S with (F, Δ) be its uniform insulator. If possible let there exist $x, y \notin P$ such that $x\Delta F\Delta y \subseteq P$. Since $x, y \notin P$, then $x, y \notin I$ which implies that $x, y \in S \setminus I$ and $(x\Delta F\Delta y) \cap T \subseteq P \cap T = \phi$, a contradiction, since (T, I) is a super t -system in S . So $x\Delta F\Delta y \subseteq P$ implies that $x \in P$ or $y \in P$ which implies that P is a uniformly strongly prime ideal of S . Hence $a \in \tau(S)$ implies that $a \in P$. Again $a \in T \Rightarrow a \notin P$ as $T \cap P = \phi$, which is a contradiction. So $0 \in T$. Then $a \in H \Rightarrow \tau(S) \subseteq H$.

Conversely, let $a \notin \tau(S)$. Then there exists a uniformly strongly prime k -ideal and hence an ideal say Q of S such that $a \notin Q$. Then $a \in S \setminus Q$, so $(S \setminus Q, Q)$ is a super t -system with $a \in S \setminus Q$ but $0 \notin S \setminus Q$, so $a \notin H$. Hence $H \subseteq \tau(S)$. This completes the proof. ■

Acknowledgement

The authors are thankful to the learned referee for his kind suggestions.

REFERENCES

- [1] G.L. Booth and N.J. Groenwald, *On Uniformly Strongly Prime Γ -rings*, Bull. Austral Math. Soc. **37** (1988), 437–445.
- [2] T.K. Dutta and M.L. Das, *On Uniformly Strongly Prime semirings* (communicated).
- [3] T.K. Dutta and S. Dhara, *On Uniformly Strongly Prime Γ -semirings*, Southeast Asian Bull. of Mathematics **30** (2006), 32–39.
- [4] T.K. Dutta and S.K. Sardar, *On The Operator Semirings of a Γ -semiring*, Southeast Asian Bull. of Mathematics, Springer-Verleg, **26** (2002), 203–213.
- [5] T.K. Dutta and S.K. Sardar, *On Matrix Γ -semirings*, Far East J. Math. Soci. (FJMS) **7** (1) (2002), 17–31.
- [6] G.A.P. Heyman and C. Roos, *Essential extensions in radical theory for rings* J. Austral. Math. Soc. Ser A, **23** (1977), 340–347.

- [7] D.M. Olson, *A Uniformly Strongly Prime Radicals* J. Austral Math. Soc. Ser A **43** (1) (1987), 95–102.
- [8] D.M. Olson and S. Veldsman, *Some remarks on Uniformly Strongly prime Radicals*, Math. Japon **33** (3) (1988), 445–455.

Received 10 July 2006

Revised 13 September 2006